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Numerical Solutions of Fractional Variable Order Differential Equations via Using Shifted Legendre Polynomials

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ABSTRACT

In this manuscript, an algorithm for the computation of numerical solutions to some variable order fractional differential equations (FDEs) subject to the boundary and initial conditions is developed. We use shifted Legendre polynomials for the required numerical algorithm to develop some operational matrices. Further, operational matrices are constructed using variable order differentiation and integration. We are finding the operational matrices of variable order differentiation and integration by omitting the discretization of data. With the help of aforesaid matrices, considered FDEs are converted to algebraic equations of Sylvester type. Finally, the algebraic equations we get are solved with the help of mathematical software like Matlab or Mathematica to compute numerical solutions. Some examples are given to check the proposed method's accuracy and graphical representations. Exact and numerical solutions are also compared in the paper for some examples. The efficiency of the method can be enhanced further by increasing the scale level.

KEYWORDS

Operational matrices; shifted legendre polynomials; FDEs; variable order

1 Introduction

Fractional calculus has been given much recognition during the last few decades. It has been got importance because of its variety of applications in mathematical modeling of real-world problems like biological and physical phenomenons [1]. With the help of the aforesaid calculus, we can comprehensively explain the dynamics of various processes and phenomena in more detailed ways. Keeping these in mind, several researchers have been given attention to investigate FDEs for various results and analysis. Researchers have given much attention to studying the said area from various aspects including theoretical and numerical analysis [2]. For such goals, they have developed various methods and procedures for numerical, analytical and theoretical results. These methods have been

widely applied to study different problems for various results including existence, approximation and stability theory [3].

Since it is a tedious job to solve various problems of FDEs for exact or analytical solutions, therefore like classical differential equations, various tools and methods have been developed in the previous few decades to handle the said area for approximate or analytical results. In this regard, different tools and methods have been established for analytical or semi-analytical solutions like decomposition technique [4], transform method [5], perturbation method [6], etc. On the other hand several numerical procedures have been established for finding the approximate solution to various problems of FDEs. Therefore, in last two decades number of terminologies have been developed for numerical solutions to the aforesaid area. Some of the commonly used methods include Tau method [7,8], collocation method [9] and spectral method [10]. The said methods have been derived by using various orthogonal and non-orthogonal polynomials like Legendre, Jacobi, Chebshev and Bernstein polynomials, etc. Spectral methods based on mentioned polynomials have been introduced for the computation of numerical solutions to various kinds of FDEs. In these methods, researchers have constructed some operational matrices of differentiation and integrations with non-integer order. To form such matrices, we need to utilize some polynomials. For instance shifted Legendre, shifted Jacobi and Bernstein polynomial have been used to construct operational matrices for the numerical solutions of various FDEs in literature (for some detail, see [11–15]).

Here, we state that in aforesaid work, authors have used collocation techniques together with spectral methods to perform numerical analysis of FDEs. Since collocation techniques need discretization of time domain of the function which is time consuming and expensive for memory. To over come this disadvantage, some authors have established operational matrices for fractional differentiation and integration by using shifted Legendre polynomial for the numerical solution of FDEs [16] by omitting the discretization and collocation.

Recently variable order FDEs have gotten some attention from researchers. Because variable order differential and integral operators have various applications in modeling different complex real-world problems. It is less well known part of calculus but has a lot of flexibility in simulating multidisciplinary processes [17]. Recognizing wide applications of the said area, scientists and researchers are increasingly investigating applications of the said area to model systems of engineering and physics. Here, we remark that the first definition has been presented by Samko and Ross in 1993 (see [18]). After that, many researchers have been worked in fractional calculus by discussing the possibility of variable orders derivatives and integrations. Many research articles have been published in recent years about variable order FDEs (see [19]).

Therefore in last few years, variable order problems of integral and differential equations have gotten significant attention. This is due to the fact that these kind problems more properly describe real world phenomenon, (we refer [20]). Applications of variable order problems can also be traced in [21]. Also some results about existence and existence as well as stability analysis have been published recently. Also some authors have established various scheme for numerical solutions. For more information and detail, we refer [22,23]. But to the best of our information, spectral methods in this regard are very rarely used for the variable order problems.

Here we demonstrate that using spectral methods for linear problems are stable always under some specific conditions. For instance, the spectral method based on Lagendre polynomials has been proved stable for linear problem of differential equations (see [24]). Further a numerical scheme is said to be stable if it keeps the control over the numerical solution in such a way that it only depends on the degree of polynomials. For linear problems, the proposed method has been proved stable

and convergent [25]. Since spectral methods are converging exponentially, which demonstrate their significant accuracy than other local methods. Also spectral methods offer a suitable framework to approximate the solution of many problems. Also aforesaid methods and finite difference numerical schemes have close relation because both use same idea. The significant difference between these methods is that spectral methods utilize basis functions which are nonzero over the entire domain, while difference methods using basis functions that are nonzero only on very small sub-domains. Here we remark that as compared to finite difference methods, spectral methods are global techniques. Also numerical spectral methods use the idea of global representations to find greater order approximation. Hence these features make spectral methods more popular in recent times among the researchers (see [26]). The spectral properties of Legendre polynomials have been studied very well in literature. Here we remark some related work on properties of the said method as [27–30].

Motivated from the above discussion and work, we extend our scheme based on shifted Legendre polynomials for some variable order problems under initial and boundary conditions. We investigate two classes of variable order initial and boundary value problem as

$$\begin{cases} {}_0^C D_t^{\alpha(t)} U(t) + aU(t) = g(t), & a \in R, \ 0 < t \le 1, \\ U(0) = \hat{u}_0, & \hat{u}_0 \in R \end{cases} \tag{1}$$

and

$$\begin{cases} {}_0^C D_t^{\alpha_2(t)} U(t) = l_1 {}_0^C D_t^{\alpha_1(t)} U(t) + l_2 U(t) + g(t), & l_1, l_2 \in R, \ 0 < t \le 1, \\ U(0) = \hat{u}_0, \ U(1) = \hat{u}_1, & \hat{u}_0, \hat{u}_1 \in R, \end{cases} \tag{2}$$

where g is linear continuous function from $[0, 1]) \rightarrow R$ and $1 < \alpha_1(t) \le 2, \ 0 < \alpha_2(t) \le 1$. We establish operational matrices on the basis of shifted Legendre polynomials for our considered problems. Keeping in mind that we avoid descritization and collocation to form some operational matrices of variable order integration and differentiation. Based on these matrices, proposed problems are converted to some algebraic type matrix equations of Sylvester type. Then upon using Matlab, we solve the Sylvester equation to get the required numerical solution. The advantage of our proposed method is that it saves time and extra memory. Further, we testify several examples and present their solutions of variable order at different points graphically.

Our paper is structured as: Section 1 is devoted to introduction. Section 2 is related to basic results and derivation of operational matrices. Section 3 is related to general algorithm. Section 4 is related to numerical examples. Last section is devoted to brief conclusion and discussion.

2 Preliminaries

In this section, we recall definitions of variable order integration and differentiation which can be read in [19,23,31,32].

Definition 2.1. The variable order integration of a function $h \in L[0, 1]$ with $\alpha(t) > 0$ is defined as

$$_0 I_t^{\alpha(t)} h(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t - \tau)^{\alpha(\tau)-1} h(\tau) d\tau, \ t > 0, \tag{3}$$

where $\alpha(t)$ is a continuous and bounded function.

Definition 2.2. Let $\alpha(t) > 0$ is a continuous and bounded function and $h \in C[0, 1]$, then the Caputo differential operator for variable order is defined as

$$_0^C D_t^{\alpha(t)} h(t) = \frac{1}{\Gamma(m - \alpha(t))} \int_0^t (t - \tau)^{m-\alpha(\tau)-1} h^{(m)}(\tau) d\tau, \ m - 1 < \alpha(t) \le m. \tag{4}$$

For more properties of variable order differentiations and integrations, we refer [21,22].

Definition 2.3. [11] The recursive relation for shifted Legendre polynomials over the interval $[-1, 1]$ is defined as

$$L_{m+1}(t) = \frac{2m+1}{m+1} t L_m(t) - \frac{m}{m+1} L_{m-1}(t), where\ i = 1, 2, 3, \ldots. \tag{5}$$

First two polynomials are given as $L_0 = 1$, $L_1(t) = t$.

Next we recall orthogonality condition from [11,16] which are needed in our approximation.

Definition 2.4. The orthogonality condition is defined as

$$\int_0^1 L_r(t) L_s(t) dt = \begin{cases} \dfrac{1}{2r+1}, & \text{if } r = s, \\ 0, & \text{if } r \neq s. \end{cases} \tag{6}$$

Using the orthogonality condition given in Eq. (6), any function h can be approximated in terms of aforementioned polynomial as

$$h(t) \approx \sum_{r=0}^m q_r L_r(t), \quad \text{where} \quad q_r = (2r+1) \int_0^1 h(t) L_r(t) dt. \tag{7}$$

The above Eq. (7) can also be written in vector form as

$$h(t) \approx K_N^T \hat{L}_N(t). \tag{8}$$

Lemma 2.1. Convergence analysis: If $h \in C^{N+1}[0, 1]$, then the best approximation of h given in Eq. (8) over [0, 1] is defined by

$$\|h(t) - K_N^T \hat{L}_N(t)\|_2 \leq \frac{1}{4N^{N+1}} \sum_{t \in [0,1]} |h^{N+1}(t)|. \tag{9}$$

In same line if $h \in \mathscr{S}$, such that $\mathscr{S}$ denotes the spanning set of first N Legendre polynomials. Then, we take $h(t) \approx \sum_{r=0}^m q_r L_r^{\ T}(t)$ and

$$\|h(t) - \sum_{r=0}^m q_r L_r^{\ T}(t)\|^2 \leq \sum_{r=m}^\infty r(r+1) q_r^2,$$

where $q_r = (2r+1) \int_0^1 h(t) L_r^{\ T}(t) dt$.

Proof. The proof is same as given in [33].

Operational matrices corresponding to variable order integration and differentiation are established by following the procedure given in [11,16].

Lemma 2.2. If $\hat{L_N}(t)$ be the vector function of Legendre polynomials, then the variable order integration is defined as

$${}_0I_t^{\alpha(t)}(L_N(t)) \approx B_{N \times N}^{\alpha(t)} \hat{L_N}(t),$$

where $B_{N \times N}^{\alpha(t)}$ is the $\alpha(t)$ order operational matrix of integration as follows:

$$
B_{N \times N}^{\alpha(t)} = \begin{bmatrix}
\sum_{e=0}^{0} \rho_{0,0,e} & \sum_{e=0}^{0} \rho_{0,1,e} & \cdots & \sum_{e=0}^{0} \rho_{0,b,e} & \cdots & \sum_{e=0}^{0} \rho_{0,n,b} \\
\sum_{e=0}^{1} \rho_{1,0,e} & \sum_{e=0}^{1} \rho_{1,1,e} & \cdots & \sum_{e=0}^{1} \rho_{1,b,e} & \cdots & \sum_{e=0}^{1} \rho_{1,n,b} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{e=0}^{a} \rho_{a,0,e} & \sum_{e=0}^{a} \rho_{a,1,e} & \cdots & \sum_{e=0}^{a} \rho_{a,b,e} & \cdots & \sum_{e=0}^{a} \rho_{a,n,e} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{e=0}^{n} \rho_{n,0,e} & \sum_{e=0}^{n} \rho_{n,1,e} & \cdots & \sum_{e=0}^{n} \rho_{n,b,e} & \cdots & \sum_{e=0}^{n} \rho_{n,n,e}
\end{bmatrix}, \tag{10}
$$

where

$$
\rho_{a,b,e} = (2b+1) \frac{\sum_{p=0}^{b} (-1)^{a+b+e+p} \Gamma(a+e+1)\Gamma(p+b+1)}{\Gamma(a-e+1)\Gamma(e+1)\Gamma(e+\alpha(t)+1)\Gamma(b-p+1)\Gamma^2(p+1)(e+p+\alpha(t)+1)},
$$

where $b = 0, 1, 2, \ldots$.

Corollary 1. With the help of operational matrix given in Eq. (10), the error

$$
|\hat{E}_N| = |_0 I_t^{\alpha(t)} \hat{L_N}(t) - K_N^T L_{N \times N}^{\alpha(t)} \hat{L}_N(t)|
$$

is bounded. The concerned error bound is computed as

$$
|\hat{E}_N| \leq |\sum_{e=n+1}^{\infty} d_k \sum_{a=0}^{n} \Omega_{a,b,e}|,
$$

where the constants d_k are denoting the spectral coefficients of $\hat{L_N}(t)$.

Lemma 2.3. Let $\hat{L}_N(t)$ be the vector function, then fractional variable order differentiation of $\hat{L}_N(t)$ is given by $_0^C D_t^{\alpha(t)} \hat{L}_N(t) \approx H_{N \times N}^{\alpha(t)} \hat{L}_N(t)$, where $H_{N \times N}^{\alpha(t)}$ is operational matrix of variable order $\alpha(t)$ differentiation given by

$$
H_{N \times N}^{\alpha(t)} = \begin{bmatrix}
\sum_{e=0}^{1} \phi_{1,1,e} & \sum_{e=0}^{1} \phi_{1,2,e} & \cdots & \sum_{e=0}^{1} \phi_{1,b,e} & \cdots & \sum_{e=0}^{1} \phi_{1,n,b} \\
\sum_{e=0}^{2} \phi_{2,1,e} & \sum_{e=0}^{2} \phi_{2,2,e} & \cdots & \sum_{e=0}^{2} \phi_{2,b,e} & \cdots & \sum_{e=0}^{2} \phi_{2,n,b} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{e=0}^{a} \phi_{a,1,e} & \sum_{e=0}^{a} \phi_{a,2,e} & \cdots & \sum_{e=0}^{a} \phi_{a,b,e} & \cdots & \sum_{e=0}^{a} \phi_{a,n,e} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_{e=0}^{n} \phi_{n,1,e} & \sum_{e=0}^{n} \phi_{n,2,e} & \cdots & \sum_{e=0}^{n} \phi_{n,b,e} & \cdots & \sum_{e=0}^{n} \phi_{n,n,e}
\end{bmatrix}, \tag{11}
$$

where

$$\phi_{a,b,e} = (2b+1)\frac{\sum\limits_{l=0}^{b}(-1)^{a+b+e+l}\Gamma(a+e+1)\Gamma(l+b+1)}{\Gamma(a-e+1)\Gamma(e+1)\Gamma(e-\alpha(t)+1)\Gamma(b-l+1)\Gamma^2(l+1)(e+l-\alpha(t)+1)},$$

with $b = 0, 1, 2, \ldots$.

Corollary 2. The error in computation of variable order differentiation of a function U is given by

$$|\hat{E}_N| = |D^{\alpha(t)}\hat{L}_N(t) - K_N^T H_{N\times N}^{\alpha(t)}\hat{L}_N(t)|.$$

Further the said error is bounded by

$$|\hat{E}_N| \leq |\sum_{c=n+1}^{\infty} d_k \sum_{a=[\alpha(t)]}^{n} \phi_{a,b,\tau}|.$$

Lemma 2.4. Let $\Theta(t)$ be defined over the interval [0, 1] and $V(t) = K_N^T \hat{L}_N(t)$, then

$$\Theta(t)[I^{\alpha(t)}V(t)] = K_N^T Q_{N\times N}^{\alpha(t),\Theta}\hat{L}_N(t),$$

where $Q_{N,N}^{\alpha(t),\Theta}$ is the operational matrix given by

$$Q_{N\times N}^{\alpha(t),\Theta} = \begin{bmatrix} \wp_{0,0} & \wp_{0,1} & \cdots & \wp_{0,r} & \cdots & \wp_{0,n} \\ \wp_{1,0} & \wp_{1,1} & \cdots & \wp_{1,r} & \cdots & \wp_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \wp_{s,0} & \wp_{s,1} & \cdots & \wp_{s,r} & \cdots & \wp_{s,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \wp_{n,0} & \wp_{n,1} & \cdots & \wp_{n,r} & \cdots & \wp_{n,n} \end{bmatrix}, \tag{12}$$

where

$$\wp_{s,r} = (2j+1)\Delta_a \int_0^1 \Theta(t)L_r(t)dt, \ \Delta_a = \sum_{p=0}^{s} \frac{(-1)^{s+p}\Gamma(s+p+1)}{\Gamma(s-p+1)\Gamma(p+1)\Gamma(\alpha(t)+p)}.$$

Proof. The proof is same as done in [16].

3 General Algorithms for Variable Orders Problems

Here, we establish the required scheme for initial and boundary value problems of variable order FDEs in two sub-sections.

3.1 Construction of Algorithm for Variable Initial Value Problem

Consider the following case, when $\alpha(t) \in (0, 1]$ as

$$\begin{cases} {}_0^C D_t^{\alpha(t)}U(t) + aU(t) = g(t), & a \in R, \ 0 < t \leq 1, \\ U(0) = \hat{u}_0, & \hat{u}_0 \in R. \end{cases} \tag{13}$$

Consider

$${}_0^C D_t^{\alpha(t)}U(t) = K_N^T \hat{L}_N(t). \tag{14}$$

By the application of fractional variable integration with order $\alpha(t)$, Eq. (14) implies that

$$U(t) = K_N^T B_{N \times N}^{\alpha(t)} \hat{L}_N(t) + a_0. \tag{15}$$

By using initial condition, one has $a_0 = \hat{u}_0$. Insert value of a_0 in Eq. (15) yields

$$U(t) = K_N^T B_{N \times N}^{\alpha(t)} \hat{L}_N(t) + \hat{u}_0. \tag{16}$$

We write the approximation as $\hat{u}_0 = Q_N^T \hat{L}_N(t)$ and $g(t) = H_N^T \hat{L}_N(t)$. After simplification, Eq. (13) gives

$$K_N^T \hat{L}_N(t) + a \left[K_N^T B_{N \times N}^{\alpha(t)} \hat{L}_N(t) + Q_N^T \hat{L}_N(t) \right] - H_N^T \hat{L}_N(t) = 0. \tag{17}$$

Further simplification implies that

$$\left[K_N^T + \left(a K_N^T B_{N \times N}^{\alpha(t)} + a Q_N^T - H_N^T \right) \right] \hat{L}_N(t) = 0. \tag{18}$$

$$K_N^T + a \left(K_N^T B_{N \times N}^{\alpha(t)} + Q_N^T \right) - H_N^T = 0. \tag{19}$$

This is a simple Sylvester type algebraic equation. Upon using Matlab, we solve it to compute the coefficient matrix K_N^T to receive the required numerical solution of (13).

3.2 Algorithm for Variable Order Boundary Value Problem

Here, we construct the general scheme for variable order problems, when $1 < \alpha_2(t) \leq 2$, $0 < \alpha_1(t) \leq 1$ as

$$\begin{cases} {}_0^C D_t^{\alpha_2(t)} U(t) = l_1 \, {}_0^C D_t^{\alpha_1(t)} U(t) + l_2 U(t) + g(t), & l_1, l_2 \in R, \ 0 < t \leq 1, \\ U(0) = \hat{u}_0, \ U(1) = \hat{u}_1, \ \hat{u}_0, \hat{u}_1 \in R. \end{cases} \tag{20}$$

Assume that

$${}_0^C D_t^{\alpha_2(t)} U(t) = K_N^T \hat{L}_N(t). \tag{21}$$

By applying the fractional order integration of order $\alpha_2(t)$, Eq. (21) implies that

$${}_0^C I_t^{\alpha_2(t)} \left[{}_0^C D_t^{\alpha_2(t)} U(t) \right] = {}_0^C I_t^{\alpha_2(t)} \left[K_N^T \hat{L}_N(t) \right]. \tag{22}$$

Eq. (22) can be written as

$$U(t) = K_N^T B_{N \times N}^{\alpha_2(t)} \hat{L}_N(t) + c_1 + c_2 t. \tag{23}$$

By using the initial and boundary conditions, we can easily get $c_1 = \hat{u}_0$ and for c_2 use $U(1) = \hat{u}_1$ in Eq. (23) to get

$$c_2 = \hat{u}_1 - \hat{u}_0 - K_N^T B_{N \times N}^{\alpha_2(t)} \hat{L}_N(1). \tag{24}$$

By inserting the values of constants c_1, c_2 in Eq. (23), we have

$$U(t) = K_N^T B_{N \times N}^{\alpha_2(t)} \hat{L}_N(t) + \hat{u}_0 + (\hat{u}_1 - \hat{u}_0)t - t K_T^N B_{N \times N}^{\alpha_2(t)} \hat{L}_N(1). \tag{25}$$

On using Lemma 2.4 and after simplification Eq. (25), one has

$$U(t) = K_N^T \left(B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t), \Theta} \right) H_{N \times N}^{\alpha_1(t)} \hat{L}_N(t) + G_{2_N}^T \hat{L}_N(t), \tag{26}$$

where

$$G_{2_N}^T L_N\hat{}(t) = \hat{u}_0 + (\hat{u}_1 - \hat{u}_0)\, t.$$

Now by the use of Lemma 2.4, one has

$$_0^C D_t^{\alpha_1(t)} U(t) = K_N^T (B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t),\Theta}) H_{N \times N}^{\alpha_1(t)} L_N\hat{}(t) + G_{2_N}^T \hat{L}_N(t). \tag{27}$$

Insert Eqs. (21), (25) and (27) in (20), one has

$$\begin{aligned}
K_N^T L_N\hat{}(t) &= l_1 K_N^T \left(B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t),\Theta}\right) H_{N \times N}^{\alpha_1(t)} L_N\hat{}(t) + l_2 K_N^T \left(B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t),\Theta}\right) L_N\hat{}(t) \\
&\quad + l_1 G_{2_N}^T H_{N \times N}^{\alpha_1}(t) L_N\hat{}(t) + l_2 G_{2_N}^T + F_{1_N}^T \hat{L}_N(t),
\end{aligned} \tag{28}$$

where $F_{1_N}^T \hat{L}_N^T = g(t)$. By further simplification and rearranging terms, Eq. (28) can be written as

$$\left(K_N^T - K_N^T \left(B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t),\Theta} \left(l_1 H_{N \times N}^{\alpha_1(t)} + l_2 I_{N \times N}\right) - D_{N \times N}\right) L_N\hat{}(t) = 0. \tag{29}$$

After simplification, we get

$$K_N^T - K_N^T \left(B_{N \times N}^{\alpha_2(t)} - Q_{N \times N}^{\alpha_2(t),\Theta}\right) \left(l_1 H_{N \times N}^{\alpha_1(t)} + l_2 I_{N \times N}\right) - D_{N \times N} = 0. \tag{30}$$

Eq. (30) is simple algebraic equation of Sylvester type and can be solved through the Matlab software for required numerical solution.

4 Illustrative Examples

Here we give examples for both kinds of problems in two sub-sections.

4.1 Numerical Examples for Variable Initial Value Problems

Consider the following example as

Example 1.

$$\begin{cases} _0^C D_t^{\alpha(t)} U(t) + 2U(t) = g(t), \ 0 < \alpha(t) \le 1, \ t \in [0, 1], \\ U(0) = 2. \end{cases} \tag{31}$$

For $\alpha(t) = e^{-t}$, the exact solution is given by

$$U(t) = 2(1 - t)^2,$$

where

$$g(t) = \frac{4t^{2-e^{-t}}}{\Gamma(3 - e^{-t})} - \frac{4t^{1-e^{-t}}}{\Gamma(2 - e^{-t})} + \frac{2t^{2-e^{-t}}}{\Gamma(1 - e^{-t})} + 2(1 - t)^2.$$

Here, we give graphical presentation of approximate solution for different values of variable order at t and the corresponding absolute error using scale level equal to 5 in Figs. 1 and 2, respectively.
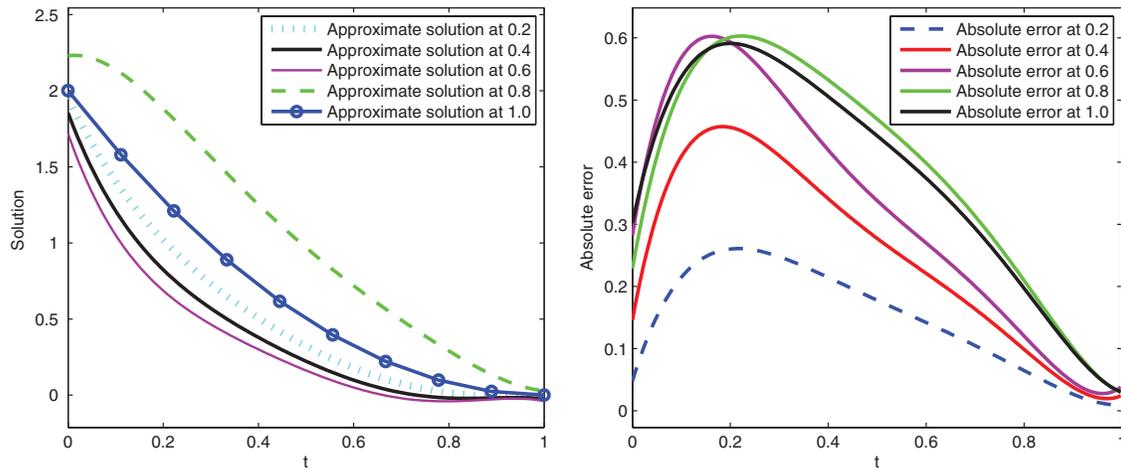
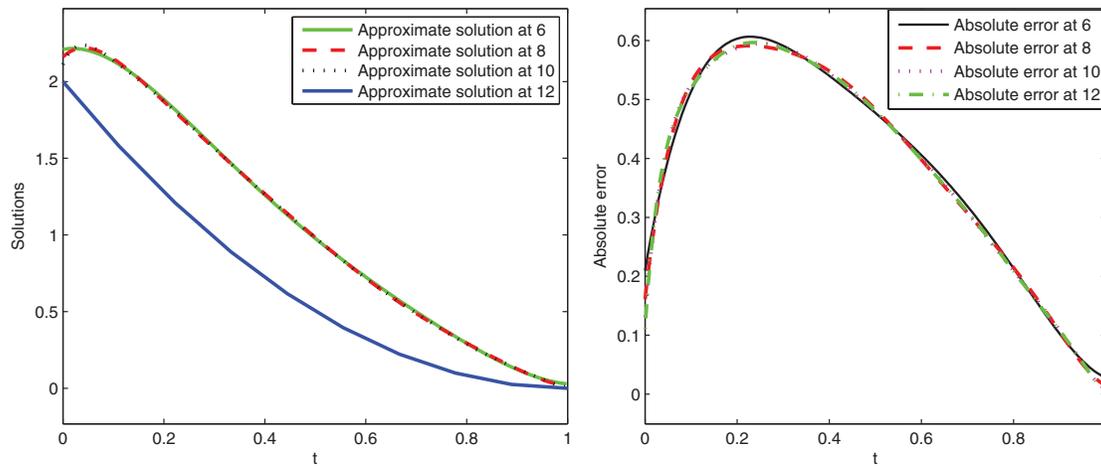**Figure 1:** Graphical presentation of approximate solutions and absolute error of Example 1

Figure 2: Graphical presentation of approximate solutions and absolute error of Example 1 at various scale level and at $t = 0.75$

Example 2. Consider another problem as

$$\begin{cases} {}_0^C D_t^{\alpha(t)} U(t) + U(t) = g(t), \ 0 < \alpha(t) \leq 1, \\ U(0) = 1. \end{cases} \tag{32}$$

For $\alpha(t) = \frac{t+1}{2}$, the exact solution is given by

$$U(t) = t^2 + t + 1,$$

where the source function g is given as

$$g(t) = \frac{2t^{2-\frac{t+1}{2}}}{\Gamma(3 - \frac{t+1}{2})} + \frac{t^{1-\frac{t+1}{2}}}{\Gamma(2 - \frac{t+1}{2})} + \frac{t^{-\frac{t+1}{2}}}{\Gamma(1 - \frac{t+1}{2})} + t^2 + t + 1.$$

Here, we give graphical presentation of approximate solution for various values of variable order at t and the corresponding absolute error using scale level equal to 6 in Figs. 3 and 4, respectively.
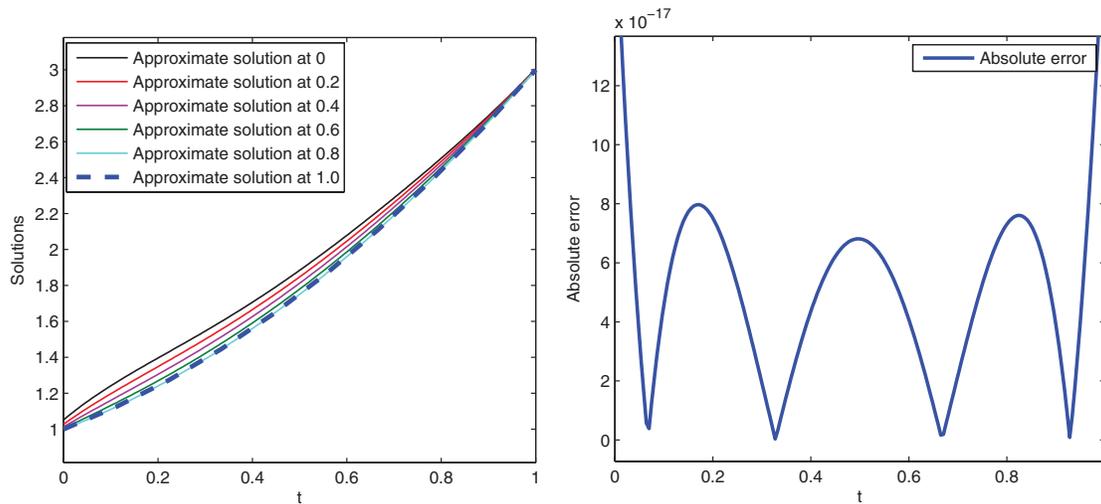
Figure 3: Graphical presentation of approximate solutions and absolute error of Example 2
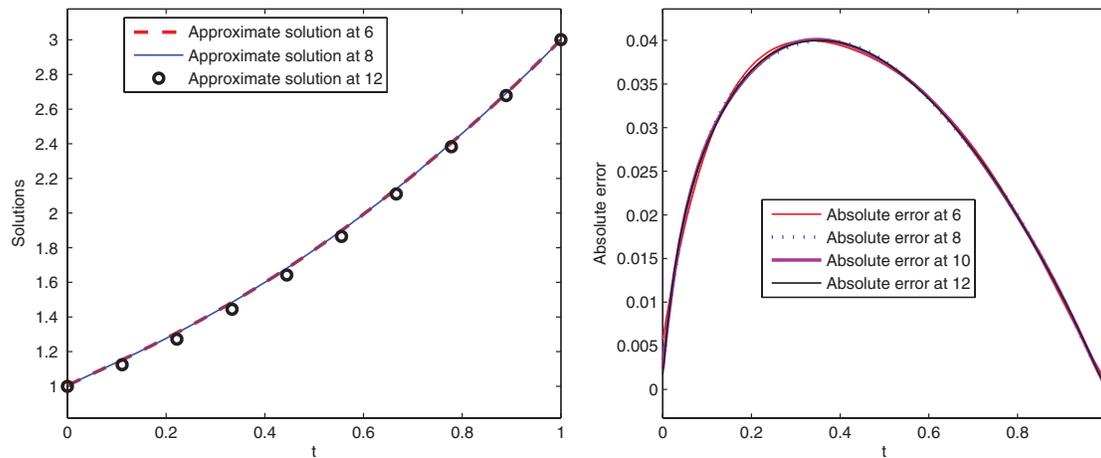

Figure 4: Graphical presentation of approximate solutions and absolute error of Example 2 at various scale level and at $t = 0.75$

4.2 Numerical Examples for Variable Order Boundary Value Problems

To demonstrate the second scheme for boundary value problems, we give some examples here.

Example 3. Consider the problem

$$\begin{cases} D^{\alpha(t)} U(t) + U(t) = g(t), \ 1 < \alpha(t) \leq 2, \\ U(0) = 0, U(1) = 1. \end{cases} \tag{33}$$

For $\alpha(t) = \sin(t) + 1$, the exact solution is given by $U(t) = t^4$. Further, the source function is given as

$$g(t) = \frac{120 t^{4-\alpha(t)}}{\Gamma(5 - \alpha(t))} + t^4.$$

Here, we give graphical presentation of approximate solution for various values of variable order at t and the corresponding absolute error using scale level equal to 6 in Figs. 5 and 6, respectively.
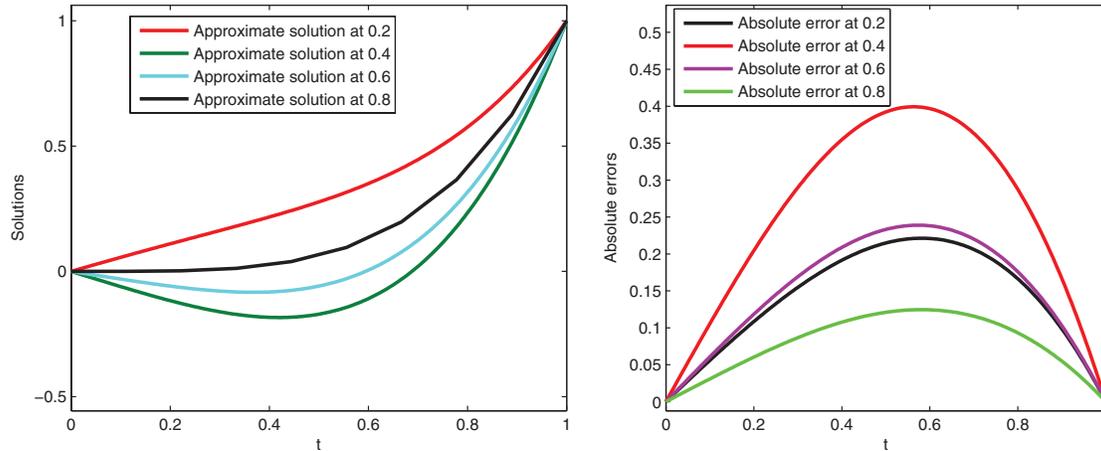

Figure 5: Graphical presentation of approximate solutions and absolute error of Example 3 at various values of t and scale level 6
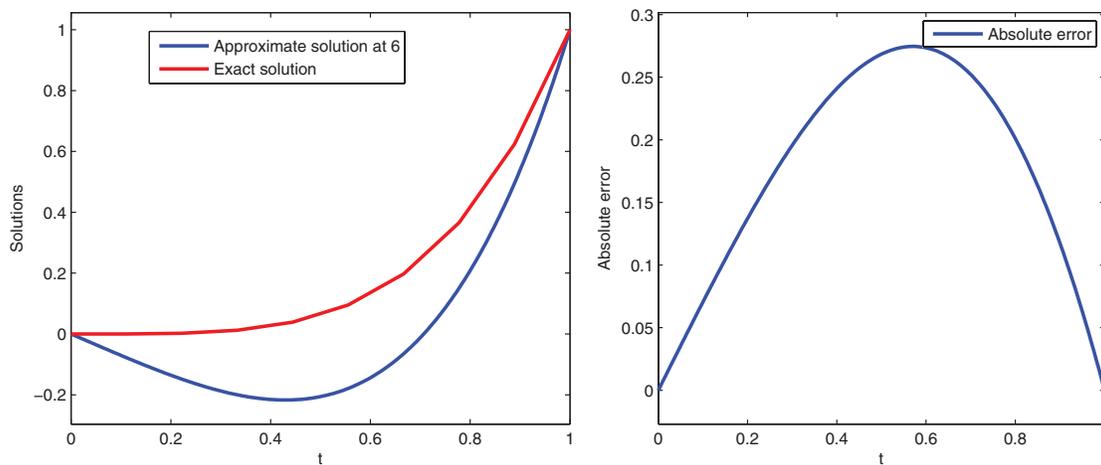

Figure 6: Graphical presentation of approximate solutions and absolute error of Example 3

Example 4. Consider the problem as

$$
\begin{cases}
D^{\alpha(t)} U(t) + U(t) = g(t), \ 1 < \alpha(t) \leq 2, \ t \in (0, 1], \\
U(0) = 0, U(1) = 2.
\end{cases}
\tag{34}
$$

For $\alpha(t) = \frac{e^t + 1}{2}$, the exact solution is given as $U(t) = t^6 + t^4$ and the source function as

$$
g(t) = \frac{720 t^{6-\alpha(t)}}{\Gamma(6 - \alpha(t))} + \frac{24 t^{4-\alpha(t)}}{\Gamma(5 - \alpha(t))} + t^6 + t^4.
$$

Here, we give graphical presentation of approximate solution for various values of variable order at t and the corresponding absolute error using scale level equal to 6 in Figs. 7 and 8, respectively.
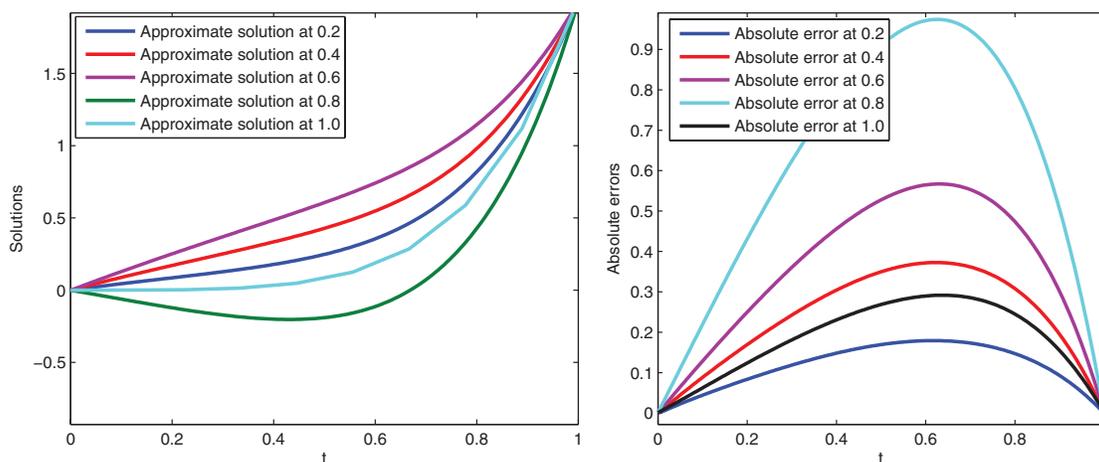
Figure 7: Graphical presentation of approximate solutions and absolute error of Example 4 at various values of t and scale level 6
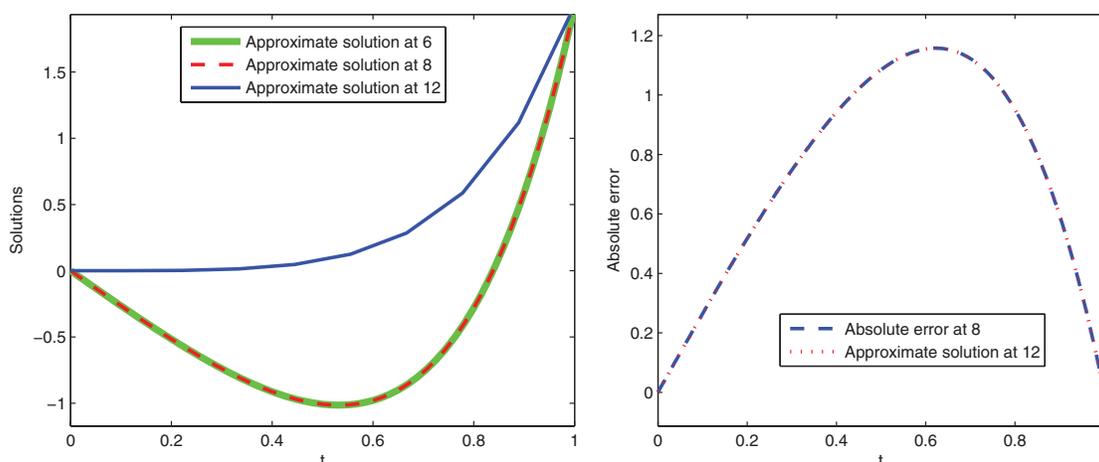

Figure 8: Graphical presentation of approximate solutions and absolute error of Example 4

5 Conclusion and Discussion

In this research paper, various classes of variable order FDEs have been studied under boundary and initial conditions for numerical solutions. Properties of shifted Legendre polynomials have been used to develop operational matrices for fractional variable order derivative and integration. Based on these operational matrices, considered problems with initial/boundary conditions have been reduced to algebraic equations of the Sylvester type. By using mathematical softwares like Matlab, we have solved the obtained algebraic equation for the required numerical solution. In this regard, various examples have been solved based on the proposed method. We have omitted the collocations and sub-division of the time domain in small intervals. Also, the proposed spectral method has been proved stable. Moreover, some error analysis has been recorded which shows that the greater the scale level higher be the accuracy and vice versa. From our numerical experiments, we have observed that the accuracy of the proposed method is excellent and can be improved further by enlarging the scale level. Because of using a higher scale level, the efficiency of the proposed method can be improved very well. An absolute error has also been computed at different scale levels. We have also examined various problems

using different fractional variable order to check the results. Hence, from numerical examples, we have concluded that the operational matrices method based on shifted Legendre polynomials can also be used as a powerful tool to handle different classes of variable order FDEs as well as integral equations for their numerical solutions.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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