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Some Results on Type 2 Degenerate Poly-Fubini Polynomials and Numbers

Ghulam Muhiuddin[1,*], Waseem A. Khan[2], Abdulghani Muhyi[3] and Deena Al-Kadi[4]

[1]Department of Mathematics, University of Tabuk, Tabuk, Saudi Arabia

[2]Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, Al Khobar, Saudi Arabia

[3]Department of Mathematics, Hajjah University, Hajjah, Yemen

[4]Department of Mathematics and Statistic, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

[*]Corresponding Author: Ghulam Muhiuddin. Email: chishtygm@gmail.com

ABSTRACT

In this paper, we introduce type 2 degenerate poly-Fubini polynomials and derive several interesting characteristics and properties. In addition, we define type 2 degenerate unipoly-Fubini polynomials and establish some certain identities. Furthermore, we give some relationships between degenerate unipoly polynomials and special numbers and polynomials. In the last section, certain beautiful zeros and graphical representations of type 2 degenerate poly-Fubini polynomials are shown.

KEYWORDS

Type 2 degenerate poly-Fubini polynomials; modified degenerate polylogarithm function; unipoly functions

1 Introduction

Special functions have gained posses a considerable prominence in various fields of mathematics, physics and applied sciences (see [1–4]) and the references cited therein). Some of the most worthy polynomials in the theory of special polynomials are the Fubini polynomials (see [5–7]), the type 2 poly-Fubini polynomials (see [8]), the degenerate central Fubini polynomials (see [9,10]) and the new type degenerate poly-Bernoulli polynomials (see [11,12]), the degenerate poly-Cauchy polynomials (see [13]), the degenerate poly-Genocchi polynomials (see [14]). Recently, the aforementioned special polynomials and their various extensions have been widely investigated by many researchers (see [1–32]) and see also each of the references cited therein.

The generating functions of the classical Bernoulli $\mathcal{B}_m(u)$, Euler $\mathcal{E}_m(u)$ and Genocchi $\mathcal{G}_m(u)$ polynomials are given as

$$\frac{\xi}{e^\xi - 1} e^{u\xi} = \sum_{m=0}^\infty \mathcal{B}_m(u) \frac{\xi^m}{m!}, |\xi| < 2\pi, \frac{2}{e^\xi + 1} e^{u\xi} = \sum_{m=0}^\infty \mathcal{E}_m(u) \frac{\xi^m}{m!}, |\xi| < \pi$$

and

$$\frac{2\xi}{e^\xi + 1} e^{u\xi} = \sum_{m=0}^\infty \mathcal{G}_m(u) \frac{\xi^m}{m!}, \quad |\xi| < \pi, \quad \text{(see [18–20])}, \tag{1}$$

respectively.

Kargin [29] defined the 2-variable Fubini polynomials by the following generating function:

$$\frac{e^{u\xi}}{1 - v(e^\xi - 1)} = \sum_{m=0}^\infty F_m(u, v) \frac{\xi^m}{m!}. \tag{2}$$

Taking $u = 0$ in Eq. (2), the 2-variable Fubini polynomials $F_m(u, v)$ reduce to the classical Fubini polynomials given by

$$\frac{1}{1 - v(e^\xi - 1)} = \sum_{m=0}^\infty F_m(v) \frac{\xi^m}{m!}. \tag{3}$$

We easily can write

$$F_m\left(u, -\frac{1}{2}\right) = \mathcal{E}_m(u), \quad F_m\left(-\frac{1}{2}\right) = \mathcal{E}_m \tag{4}$$

and

$$F_m(v) = \sum_{\alpha=0}^m S_2(m, \alpha) \alpha! v^\alpha.$$

For $v = 1$ in (3), we get the known Fubini numbers $F_m(1) = F_m$ as follows:

$$\frac{1}{2 - e^\xi} = \sum_{m=0}^\infty F_m \frac{\xi^m}{m!}, \quad \text{(see [5,6,10])}. \tag{5}$$

Some of the applications of Fubini polynomials and numbers can be found in [7–9,20,26,27]. The degenerate form of the exponential function is given as (see [11,12,14,22–29])

$$e_\beta^u(\xi) = (1 + \beta t)^{\frac{u}{\beta}}, \quad e_\beta(\xi) e_\beta^1(\xi) = (1 + \beta\xi)^{\frac{1}{\beta}}, \quad \beta \in \mathbb{R}. \tag{6}$$

The function $e_\beta^u(\xi)$ is defined by the series

$$e_\beta^u(\xi) = \sum_{m=0}^\infty (u)_{m,\beta} \frac{\xi^m}{m!}, \tag{7}$$

where the degenerate Pochhammer symbol $(u)_{m,\beta}$ is specified by

$$(u)_{m,\beta} = u(u-\beta)(u-2\beta)\cdots(u-(m-1)\beta), (u)_{0,\beta} = 1, \quad (m \geq 1).$$

Carlitz et al. [15,16] presented the degenerate Bernoulli polynomials as

$$\frac{\xi}{e_\beta(\xi)-1}e_\beta^u(\xi) = \frac{\xi}{(1+\beta\xi)^{\frac{1}{\beta}}-1}(1+\beta\xi)^{\frac{u}{\beta}} = \sum_{m=0}^{\infty} \mathcal{B}_{m,\beta}(u)\frac{\xi^m}{m!}, \tag{8}$$

where $\mathcal{B}_{m,\beta}(0) := \mathcal{B}_{m,\beta}$ denotes degenerate Bernoulli numbers.

For $\alpha \in \mathbb{Z}$, the modified degenerate polyexponential function [28] is defined by Kim–Kim to be

$$\mathrm{Ei}_{\alpha,\beta}(u) = \sum_{m=1}^{\infty} \frac{(1)_{m,\beta} u^m}{(m-1)! m^\alpha}, \quad (|u| < 1). \tag{9}$$

Note that

$$\mathrm{Ei}_{1,\beta}(u) = \sum_{m=1}^{\infty} \frac{(1)_{m,\beta} u^m}{m!} = e_\beta(u) - 1. \tag{10}$$

Kim et al. [14] presented the generating function of the degenerate poly-Genocchi polynomials as

$$\frac{\mathrm{Ei}_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{e_\beta(\xi)+1}e_\beta^u(\xi) = \sum_{m=0}^{\infty} \mathcal{G}_{m,\beta}^{(\alpha)}(u)\frac{\xi^m}{m!}, \quad (\alpha \in \mathbb{Z}), \tag{11}$$

where $\mathcal{G}_{m,\beta}^{(\alpha)}(0) := \mathcal{G}_{m,\beta}^{(\alpha)}$ denotes the degenerate poly-Genocchi numbers.

The 2-variable degenerate Fubini polynomials $F_{m,\beta}(u; v)$ [26] are defined by

$$\frac{1}{1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)}(1+\beta\xi)^{\frac{u}{\beta}}$$

$$= \sum_{m=0}^{\infty} F_{m,\beta}(u; v)\frac{\xi^m}{m!}, \quad (\text{see } [25,27]), \tag{12}$$

where $F_{m,\beta}(0; 1) := F_{m,\beta}$ denotes the degenerate Fubini numbers.

The degenerate Daehee polynomials $D_{m,\beta}(u)$ [11] are specified by

$$\frac{\log_\beta(1+\xi)}{\xi}(1+\xi)^u = \sum_{m=0}^{\infty} D_{m,\beta}(u)\frac{\xi^m}{m!}, \quad (\text{see } [12,20]), \tag{13}$$

where $D_{m,\beta}(0) := D_{m,\beta}$ denotes the degenerate Daehee numbers.

The degenerate form of the first kind Stirling numbers are specified by

$$\frac{1}{\alpha!}\left(\log_\beta(1+\xi)\right)^\alpha = \sum_{m=\alpha}^{\infty} S_{1,\beta}(m,\alpha)\frac{\xi^m}{m!}, (\alpha \geq 0), \quad (\text{see } [12]). \tag{14}$$

Note here that $\lim_{\beta \to 0} S_{1,\beta}(m, \alpha) = S_1(m, \alpha)$, where $S_1(m,\alpha)$ are the first kind Stirling numbers given by

$$\frac{1}{\alpha!}(\log(1+\xi))^{\alpha} = \sum_{m=\alpha}^{\infty} S_1(m, \alpha) \frac{\xi^m}{m!}, \quad (\alpha \geq 0). \tag{15}$$

The degenerate form of the second kind Stirling numbers are specified by

$$\frac{1}{\alpha!}(e_{\beta}(\xi) - 1)^{\alpha} = \sum_{m=\alpha}^{\infty} S_{2,\beta}(m, \alpha) \frac{\xi^m}{m!}, \quad (\text{see } [23]). \tag{16}$$

Observe here that $\lim_{\beta \to 0} S_{2,\beta}(m, \alpha) = S_2(m, \alpha)$, where $S_2(m, \alpha)$ are the second kind Stirling numbers given by

$$\frac{1}{\alpha!}(e^{\xi} - 1)^{\alpha} = \sum_{m=\alpha}^{\infty} S_2(m, \alpha) \frac{\xi^m}{m!}, \quad (\text{see } [1\text{--}32]). \tag{17}$$

The following paper is as follows. In Section 2, we define type 2 degenerate poly-Fubini polynomials via the modified degenerate polyexponential functions and obtain certain new properties related to these numbers and polynomials. In Section 3, we consider the type 2 degenerate unipoly-Fubini polynomials and discuss some identities of them. In Section 4, we find some values of type 2 poly-Fubini polynomials and draw some beautiful graphs.

2 Type 2 Degenerate Poly-Fubini Polynomials and Numbers

In the present section, we define type 2 degenerate Fubini polynomials by utilizing the modified degenerate polyexponential function and we derive some interesting relations and formulas related to these polynomials and numbers. We start the following definition as follows.

Let $\beta \in \mathbb{C}$ and $\alpha \in \mathbb{Z}$, we consider the type 2 degenerate poly-Fubini polynomials which are defined by

$$\frac{Ei_{\alpha,\beta}\left(\log_{\beta}(1+\xi)\right)}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)}(1+\beta\xi)^{\frac{u}{\beta}} = \sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(u; v) \frac{\xi^m}{m!}, \tag{18}$$

where $F_{m,\beta}^{(\alpha)}(0;1) := F_{m,\beta}^{(\alpha)}$ denotes the type 2 degenerate poly-Fubini numbers, and $\log_{\beta}(\xi) = \frac{1}{\beta}(\xi^{\beta} - 1)$ is the compositional inverse of $e_{\beta}(\xi)$ satisfying the following relation

$\log_{\beta} e_{\beta}(\xi) = e_{\beta}(\log_{\beta}(\xi)) = \xi.$

For $\alpha = 1$ in Eq. (18), we get

$$\frac{1}{1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)}(1+\beta\xi)^{\frac{u}{\beta}} = \sum_{m=0}^{\infty} F_{m,\beta}(u; v) \frac{\xi^m}{m!}, \tag{19}$$

where $F_{m,\beta}(u; v)$ denotes the degenerate Fubini polynomials (see Eq. (12)).

Obviously

$$\lim_{\beta \to 0} \left(\frac{\mathrm{Ei}_{\alpha,\beta} \left(\log_\beta (1+\xi) \right)}{\xi \left(1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right) \right)} (1+\beta\xi)^{\frac{u}{\beta}} \right) = \sum_{m=0}^{\infty} \lim_{\beta \to 0} F_{m,\beta}^{(\alpha)} (u; v) \frac{\xi^m}{m!}$$

$$= \frac{\mathrm{Ei}_\alpha \left(\log (1+\xi) \right)}{\xi \left(1 - v \left(e^\xi - 1 \right) \right)} e^{u\xi} = \sum_{m=0}^{\infty} F_m^{(\alpha)} (u; v) \frac{\xi^m}{m!}. \tag{20}$$

Thus, by (18) and (20), we have

$$\lim_{\beta \to 0} F_{m,\beta}^{(\alpha)} (u; v) = F_m^{(\alpha)} (u; v), \quad (m \geq 0) \tag{20}$$

where $F_m^{(\alpha)} (u; v)$ denotes the type 2 poly-Fubini polynomials (see [28]).

Theorem 2.1. For $m \geq 0$. Then, we have

$$F_{m,\beta}^{(\alpha)} (v) = \sum_{l=0}^{m} \sum_{\rho=0}^{l} \binom{m}{l} F_{m-l,\beta} (v) \frac{(1)_{\beta,\rho+1} S_{1,\beta} (l+1, \rho+1)}{l+1 (\rho+1)^{\alpha-1}}. \tag{22}$$

Proof. By using Eqs. (9), (14) and (18), we observe that

$$\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)} (v) \frac{\xi^m}{m!} = \frac{\mathrm{Ei}_{\alpha,\beta} \left(\log_\beta (1+\xi) \right)}{\xi \left(1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right) \right)}$$

$$= \frac{1}{1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right)} \frac{1}{\xi} \sum_{\rho=1}^{\infty} \frac{(1)_{\rho,\beta} \left(\log_\beta (1+\xi) \right)^\rho}{(\rho-1)! \, \rho^\alpha}$$

$$= \frac{1}{1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right)} \frac{1}{\xi} \sum_{\rho=0}^{\infty} \frac{(1)_{\rho+1,\beta} \left(\log_\beta (1+\xi) \right)^{\rho+1}}{(\rho+1)! \, (\rho+1)^{\alpha-1}}$$

$$= \frac{1}{1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right)} \frac{1}{\xi} \sum_{\rho=0}^{\infty} \frac{(1)_{\rho+1,\beta}}{(\rho+1)^{\alpha-1}} \sum_{l=\rho+1}^{\infty} S_{1,\beta} (l, \rho+1) \frac{\xi^l}{l!}$$

$$= \frac{1}{1 - v \left((1+\beta\xi)^{\frac{1}{\beta}} - 1 \right)} \sum_{\rho=0}^{\infty} \frac{(1)_{\rho+1,\beta}}{(\rho+1)^{\alpha-1}} \sum_{l=\rho}^{\infty} S_{1,\beta} (l+1, \rho+1) \frac{\xi^l}{(l+1)!}$$

$$= \left(\sum_{s=0}^{\infty} F_{m,\beta} (v) \frac{\xi^m}{m!} \right) \left(\sum_{l=0}^{\infty} \sum_{\rho=0}^{l} \frac{(1)_{\rho+1,\beta}}{(\rho+1)^{\alpha-1}} \frac{S_{1,\beta} (l+1, \rho+1)}{l+1} \frac{\xi^l}{l!} \right)$$

$$L.H.S = \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} \sum_{\rho=0}^{l} \binom{m}{l} F_{m-l,\beta} (v) \frac{(1)_{\rho+1,\beta} S_{1,\beta} (l+1, \rho+1)}{l+1 (\rho+1)^{\alpha-1}} \right) \frac{\xi^m}{m!}. \tag{23}$$

Therefore, by (23), we obtain the result.

Corollary 2.1. For $m \geq 0$. Then, we have

$$F_{m,\beta}^{(1)}(v) = \sum_{l=0}^{m} \sum_{\rho=0}^{l} \binom{m}{l} F_{m-l,\beta}(v) \frac{(1)_{\beta,\rho+1} S_{1,\beta}(l+1,\rho+1)}{l+1}. \tag{24}$$

Theorem 2.2. Let $m \geq 0$. Then, we have

$$F_{m,\beta}^{(\alpha)}(u;v) = \sum_{\rho=0}^{m} \binom{m}{\rho} F_{m-\rho,\beta}^{(\alpha)}(v)(u)_{\rho,\beta}. \tag{25}$$

Proof. From (18), we observe that

$$\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(u;v) \frac{\xi^m}{m!} = \frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} (1+\beta\xi)^{\frac{u}{\beta}}$$

$$= \left(\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(v) \frac{\xi^m}{m!}\right) \left(\sum_{\rho=0}^{\infty} (u)_{\rho,\beta} \frac{\xi^\rho}{\rho!}\right)$$

$$= \left(\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(v) \frac{\xi^m}{m!}\right) \left(\sum_{\rho=0}^{\infty} (u)_{\rho,\beta} \frac{\xi^\rho}{\rho!}\right)$$

$$L.H.S = \sum_{m=0}^{\infty} \left(\sum_{\rho=0}^{m} \binom{m}{\rho} F_{m-\rho,\beta}^{(\alpha)}(v)(u)_{\rho,\beta}\right) \frac{\xi^m}{m!}. \tag{26}$$

By comparing the coefficients on both sides of Eq. (26), we reach at the following result (25).

In [12], it is well known that the degenerate second kind Bernoulli polynomials are defined by

$$\frac{\xi}{\log_\beta(1+\xi)} (1+\xi)^u = \sum_{m=0}^{\infty} b_{m,\beta}(u) \frac{\xi^m}{m!}, \tag{27}$$

where $b_{m,\beta}(0) := b_{m,\beta}$ denotes the degenerate second kind Bernoulli numbers.

Theorem 2.3. For $m \geq 0$, we have

$$F_{m,\beta}^{(\alpha)}(v) = \frac{1}{2} \sum_{\rho=0}^{m} \binom{m}{\rho} \sum_{\rho_1+\cdots+\rho_{\alpha-1}=\rho} \binom{\rho}{\rho_1+\cdots+\rho_{\alpha-1}}$$

$$\times \frac{b_{\rho_1,\beta}(\beta-1)}{\rho_1+1} \frac{b_{\rho_2,\beta}(\beta-1)}{\rho_1+\rho_2+1} \cdots \frac{b_{\rho_{\alpha-1},\beta}(\beta-1)}{\rho_1+\cdots+\rho_{\alpha-1}+1} F_{m-\rho,\beta}(v). \tag{28}$$

Proof. From (9), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}u} \mathrm{E}i_{\alpha,\beta} \left(\log_\beta (1+u)\right) = \frac{\mathrm{d}}{\mathrm{d}u} \sum_{m=1}^{\infty} \frac{(1)_{m,\beta} \left(\log_\beta (1+u)\right)^m}{(m-1)! \, m^\alpha}$$

$$= \frac{(1+u)^{\beta-1}}{\log_\beta (1+u)} \sum_{m=1}^{\infty} \frac{(1)_{m,\beta} \left(\log_\beta (1+u)\right)^m}{(m-1)! \, m^{\alpha-1}} = \frac{(1+u)^{\beta-1}}{\log_\beta (1+u)} \mathrm{E}i_{\alpha-1,\beta} \left(\log_\beta (1+u)\right). \tag{29}$$

Thus, from (18) and (29), we have

$$\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)} (v) \frac{u^m}{m!} = \frac{1}{u\left(1 - v\left((1+\beta u)^{\frac{1}{\beta}} - 1\right)\right)} \mathrm{E}i_{\alpha,\beta} \left(\log_\beta (1+u)\right)$$

$$= \frac{1}{u\left(1 - v\left((1+\beta u)^{\frac{1}{\beta}} - 1\right)\right)} \int_0^u \frac{(1+\xi)^{\beta-1}}{\log_\beta (1+\xi)} \underbrace{\int_0^\xi \cdots \frac{(1+\xi)^{\beta-1}}{\log_\beta (1+\xi)} \int_0^\xi \frac{(1+\xi)^{\beta-1}}{\log_\beta (1+\xi)}}_{(\alpha-2)-\text{times}} t \, dt \cdots dt$$

$$= \frac{1}{u\left(1 - v\left((1+\beta u)^{\frac{1}{\beta}} - 1\right)\right)} \sum_{\rho=0}^{\infty} \sum_{\rho_1 + \cdots + \rho_{\alpha-1} = \rho} \binom{\rho}{\rho_1 + \cdots + \rho_{\alpha-1}}$$

$$\times \frac{b_{\rho_1,\beta} (\beta-1)}{\rho_1 + 1} \frac{b_{\rho_2,\beta} (\beta-1)}{\rho_1 + \rho_2 + 1} \cdots \frac{b_{\rho_{\alpha-1},\beta} (\beta-1)}{\rho_1 + \cdots + \rho_{\alpha-1} + 1} \frac{u^\rho}{\rho!}$$

$$\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)} (v) \frac{u^m}{m!} = \frac{1}{2} \sum_{m=0}^{\infty} \sum_{\rho=0}^{m} \binom{m}{\rho} \sum_{\rho_1 + \cdots + \rho_{\alpha-1} = \rho} \binom{\rho}{\rho_1 + \cdots + \rho_{\alpha-1}}$$

$$\times \frac{b_{\rho_1,\beta} (\beta-1)}{\rho_1 + 1} \frac{b_{\rho_2,\beta} (\beta-1)}{\rho_1 + \rho_2 + 1} \cdots \frac{b_{\rho_{\alpha-1},\beta} (\beta-1)}{\rho_1 + \cdots + \rho_{\alpha-1} + 1} F_{m-\rho,\beta} (v) \frac{u^m}{m!}. \tag{30}$$

Therefore, by (30), we obtain the result.

Corollary 2.2. For $\alpha \geq 2$, we have

$$F_{m,\beta}^{(2)} (v) = \frac{1}{2} \sum_{\rho=0}^{m} \binom{m}{\rho} \frac{b_{\rho,\beta} (\beta-1)}{\rho+1} F_{m-\rho,\beta} (v).$$

Theorem 2.4. Let $\alpha \geq 1$ and $\alpha \in \mathbb{N} \cup \{0\}$, $s \in \mathbb{C}$, we have

$$\eta_{\alpha,\beta} (-\rho) = (-1)^\rho F_{\rho,\beta}^{(\alpha)} (v).$$

Proof. Let $\alpha \geq 1$, be an integer. For $s \in \mathbb{C}$, we define the function $\eta_{\alpha,\beta}(s)$ as

$$\eta_{\alpha,\beta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{\xi^{s-1}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right) dt$$

$$= \frac{1}{\Gamma(s)} \int_0^1 \frac{\xi^{s-1}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right) dt$$

$$+ \frac{1}{\Gamma(s)} \int_1^\infty \frac{\xi^{s-1}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right) dt. \tag{31}$$

Here, we note that the second integral converges absolutely for any $s \in \mathbb{C}$, then the second term on the r.h.s. vanishes at non-positive integers. Hence,

$$\lim_{s \to -\rho} \left| \frac{1}{\Gamma(s)} \int_1^\infty \frac{\xi^{s-1}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right) dt \right| \leq \frac{1}{\Gamma(-\rho)} M = 0. \tag{32}$$

Also, for $\Re(s) > 0$, the first integral in (32) can be written as

$$\frac{1}{\Gamma(s)} \sum_{l=0}^\infty \frac{F_{l,\beta}^{(\alpha)}(v)}{l!} \frac{1}{s+l}.$$

Using (31) and (32), we see that

$$\eta_{\alpha,\beta}(-\rho) = \lim_{s \to -\rho} \frac{1}{\Gamma(s)} \int_0^1 \frac{\xi^{s-1}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right) dt$$

$$= \lim_{s \to -\rho} \frac{1}{\Gamma(s)} \int_0^1 \xi^{s-1} \sum_{l=0}^\infty \frac{F_{l,\beta}^{(\alpha)}(v)\xi^l}{l!} d\xi = \lim_{s \to -\rho} \frac{1}{\Gamma(s)} \sum_{l=0}^\infty \frac{F_{l,\beta}^{(\alpha)}(v)}{s+l} \frac{1}{l!}$$

$$= \cdots + 0 + \cdots + 0 + \lim_{s \to -\rho} \frac{1}{\Gamma(s)} \frac{1}{s+\rho} \frac{F_{\rho,\beta}^{(\alpha)}(v)}{\rho!} + 0 + 0 + \cdots$$

$$= \lim_{s \to -\rho} \frac{\left(\frac{\Gamma(1-s)\sin\pi s}{\pi}\right) F_{\rho,\beta}^{(\alpha)}(v)}{s+\rho} \frac{F_{\rho,\beta}^{(\alpha)}(v)}{\rho!} = \Gamma(1+\rho)\cos(\pi\rho) \frac{F_{\rho,\beta}^{(\alpha)}(v)}{\rho!}$$

$$= (-1)^\rho F_{\rho,\beta}^{(\alpha)}(v). \tag{33}$$

Therefore, by (33), the result is obtained.

Theorem 2.5. Let $\alpha \geq 1$ and $\rho \in \mathbb{N} \cup \{0\}$, $s \in \mathbb{C}$, we have

$$F_{m,\beta}^{(\alpha)}(v) = \frac{1}{1+v}\left[v\sum_{\rho=0}^{m}\binom{m}{\rho}(1)_{\rho,\beta}F_{m-\rho,\beta}^{(\alpha)}(v) + \sum_{\rho=0}^{m}\frac{1}{(\rho+1)^{\alpha-1}}\frac{(1)_{\rho+1,\beta}S_{1,\beta}(m+1,\rho+1)}{m+1}\right].$$

Proof. From (18), we note that

$$\frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi} = \left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)\sum_{m=0}^{\infty}F_{m,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!}$$

$$= \sum_{m=0}^{\infty}F_{m,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!} - y\sum_{m=0}^{\infty}\sum_{\rho=0}^{m}\binom{m}{\rho}(1)_{\rho,\beta}F_{m-\rho,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!} + y\sum_{m=0}^{\infty}F_{m,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!}$$

$$= (1+v)\sum_{m=0}^{\infty}F_{m,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!} - y\sum_{m=0}^{\infty}\sum_{\rho=0}^{m}\binom{m}{\rho}(1)_{\rho,\beta}F_{m-\rho,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!}. \tag{34}$$

On the other hand,

$$\frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi} = \frac{1}{\xi}\sum_{\rho=1}^{\infty}\frac{(1)_{\rho,\beta}\left(\log_\beta(1+\xi)\right)^\rho}{(\rho-1)!\,\rho^\alpha}$$

$$= \frac{1}{\xi}\sum_{\rho=0}^{\infty}\frac{(1)_{\rho+1,\beta}\left(\log_\beta(1+\xi)\right)^{\rho+1}}{\rho!\,(\rho+1)^\alpha}\frac{(\rho+1)!}{(\rho+1)!}$$

$$= \frac{1}{\xi}\sum_{\rho=0}^{\infty}\frac{(1)_{\rho+1,\beta}}{(\rho+1)^{\alpha-1}}\sum_{m=\rho+1}^{\infty}S_{1,\beta}(m,\rho+1)\frac{\xi^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(\sum_{\rho=0}^{m}\frac{1}{(\rho+1)^{\alpha-1}}\frac{(1)_{\rho+1,\beta}S_{1,\beta}(m+1,\rho+1)}{m+1}\right)\frac{\xi^m}{m!}. \tag{35}$$

Therefore, by (34) and (35), we reach at the desired result.

For $\alpha = 1$ in Theorem 2.5., the following corollary is obtained.

Corollary 2.3. For $\rho \in \mathbb{N} \cup \{0\}$, we have

$$F_{m,\beta}(v) = \frac{1}{1+v}\left[v\sum_{\rho=0}^{m}\binom{m}{\rho}(1)_{\rho,\beta}F_{m-\rho,\beta}(v) + \sum_{\rho=0}^{m}\frac{(1)_{\rho+1,\beta}S_{1,\beta}(m+1,\rho+1)}{m+1}\right].$$

Theorem 2.6. Let $m \geq 0$. Then, we have

$$vF_{m,\beta}^{(\alpha)}(u+1;v) = (v+1)F_{m,\beta}^{(\alpha)}(u;v) - \sum_{l=0}^{m}\sum_{\rho=0}^{l}\binom{m}{l}\frac{1}{(\rho+1)^{\alpha-1}}\frac{S_{1,\beta}(l+1,\rho+1)}{l+1}(u)_{m-l,\beta}.$$

Proof. From (18), we note that

$$\sum_{m=0}^{\infty} \left(F_{m,\beta}^{(\alpha)}(u+1;v) - F_{m,\beta}^{(\alpha)}(u;v) \right) \frac{\xi^m}{m!}$$

$$= \frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} (1+\beta\xi)^{\frac{u}{\beta}} \left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)$$

$$= \frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi} \left(\frac{1}{v} \left(\left(\frac{(1+\beta\xi)^{\frac{u}{\beta}}}{1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)} \right) - (1+\beta\xi)^{\frac{u}{\beta}} \right) \right)$$

$$= \frac{1}{v} \left(\frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)(1+\beta\xi)^{\frac{u}{\beta}}}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} - \frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)(1+\beta\xi)^{\frac{u}{\beta}}}{\xi} \right)$$

$$= \frac{1}{v} \left(\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(u;v) \frac{\xi^m}{m!} - \sum_{l=0}^{\infty}\sum_{\rho=0}^{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{S_{1,\beta}(l+1,\rho+1)}{l+1} \frac{\xi^l}{l!} \sum_{m=0}^{\infty} (u)_{m,\beta} \frac{\xi^m}{m!} \right)$$

$$= \frac{1}{v} \sum_{m=0}^{\infty} \left(F_{m,\beta}^{(\alpha)}(u;v) - \sum_{l=0}^{m}\sum_{\rho=0}^{l} \binom{m}{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{S_{1,\beta}(l+1,\rho+1)}{l+1} (u)_{m-l,\beta} \right) \frac{\xi^m}{m!}. \tag{36}$$

Upon comparing the coefficients of $\frac{\xi^m}{m!}$ of the above equation, we get the result.

When $u = 0$ and $u = -1$ in Theorem (2.6), we get

$$v F_{m,\beta}^{(\alpha)}(1;v) = (v+1) F_{m,\beta}^{(\alpha)}(v) - \sum_{l=0}^{m}\sum_{v=0}^{l} \binom{m}{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{S_{1,\beta}(l+1,\rho+1)}{l+1}, \quad (m \geq 0).$$

and

$$v F_{m,\beta}^{(\alpha)}(v) = (v+1) F_{m,\beta}^{(\alpha)}(-1;v) - \sum_{l=0}^{m}\sum_{\rho=0}^{l} \binom{m}{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{S_{1,\beta}(l+1,\rho+1)}{l+1} (-1)_{m-l,\beta}, \quad (m \geq 0).$$

Theorem 2.7. Let $m \geq 0$. Then, we have

$$\sum_{\rho=0}^{m} \binom{m}{\rho} F_{m-\rho,\beta}^{(\alpha)}(u_1;v_1) F_{\rho,\beta}^{(\alpha)}(u_2;v_2)$$

$$= \frac{v_2 F_{m,\beta}^{(\alpha)}(u_1+u_2;v_2) - v_1 F_{m,\beta}^{(\alpha)}(u_1+u_2;v_1)}{v_2 - v_1}.$$

Proof. Now, we observe that

$$
\left(\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi\left(1-v_1\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u_1}{\beta}}\right)\left(\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{t\left(1-v_2\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u_2}{\beta}}\right)
$$

$$
=\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi}\left(\frac{v_2}{v_2-v_1}\frac{(1+\beta\xi)^{\frac{u_1+u_2}{\beta}}}{1-v_2\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)}-\frac{v_1}{v_2-v_1}\frac{(1+\beta\xi)^{\frac{u_1+u_2}{\beta}}}{1-v_2\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)}\right)
$$

$$
=\sum_{m=0}^\infty\left(\frac{v_2F_{m,\beta}^{(\alpha)}(u_1+u_2;v_2)-v_1F_{m,\beta}^{(\alpha)}(u_1+u_2;v_1)}{v_2-v_1}\right)\frac{\xi^m}{m!}. \tag{37}
$$

On the other hand,

$$
\left(\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi\left(1-v_1\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u_1}{\beta}}\right)\left(\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{t\left(1-v_2\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u_2}{\beta}}\right)
$$

$$
=\left(\sum_{m=0}^\infty F_{m,\beta}^{(\alpha)}(u_1;v_1)\frac{\xi^m}{m!}\right)\left(\sum_{\rho=0}^\infty F_{\rho,\beta}^{(\alpha)}(u_2;v_2)\frac{\xi^\rho}{\rho!}\right)
$$

$$
=\sum_{m=0}^\infty\left(\sum_{\rho=0}^m\binom{m}{\rho}F_{m-\rho,\beta}^{(\alpha)}(u_1;v_1)F_{\rho,\beta}^{(\alpha)}(u_2;v_2)\right)\frac{\xi^m}{m!}. \tag{38}
$$

Therefore, by (37) and (38), we get the result.

For $u_1=u_2=0$ in Theorem 2.7, we have

Corollary 2.4. Let $\alpha\in\mathbb{Z}$ and $m\geq 0$, we have

$$
\sum_{\rho=0}^m\binom{m}{\rho}F_{m-\rho,\beta}^{(\alpha)}(v_1)F_{\rho,\beta}^{(\alpha)}(v_2)=\frac{v_2F_{m,\beta}^{(\alpha)}(v_2)-v_1F_{m,\beta}^{(\alpha)}(v_1)}{v_2-v_1}.
$$

Theorem 2.8. Let $m\geq 0$. Then, we have

$$
F_{m,\beta}^{(\alpha)}(u;v-1)=(-1)^m F_{m,-\beta}^{(\alpha)}(1-u;-v).
$$

Proof. From (18), we note that

$$
\sum_{m=0}^\infty F_{m,\beta}^{(\alpha)}(u;v-1)\frac{\xi^m}{m!}=\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi\left(1-(v-1)\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u}{\beta}}
$$

$$
=\frac{\mathrm{E}i_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)(1+\beta\xi)^{\frac{u}{\beta}}}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)+(1+\beta\xi)^{\frac{1}{\beta}}-1\right)}
$$

$$= \frac{Ei_{\alpha,\beta}\left(\log_\beta (1+\xi)\right)(1+\beta\xi)^{\frac{u-1}{\beta}}}{\xi\left(1 - v(1-\beta\xi)^{\frac{1}{\beta}}\right)}$$

$$= \frac{Ei_{\alpha,\beta}\left(\log_\beta (1+\xi)\right)(1-\beta(-\xi))^{-\frac{1-u}{\beta}}}{\xi\left(1 + v\left((1-\beta(-\xi))^{-\frac{1}{\beta}} - 1\right)\right)}$$

$$= \sum_{m=0}^{\infty} F_{m,-\beta}^{(\alpha)}(1-u;-v)\frac{(-1)^m \xi^m}{m!}. \tag{39}$$

Comparing the coefficients of $\frac{\xi^m}{m!}$ of the above equation, we obtain the result.

Adjusting $u = 0$ in Theorem 2.8, we get

Corollary 2.5. Let $\alpha \in \mathbb{Z}$ and $m \geq 0$, we have

$$F_{m,\beta}^{(\alpha)}(v-1) = (-1)^m F_{m,-\beta}^{(\alpha)}(1;-v).$$

Theorem 2.9. Let $\alpha \in \mathbb{Z}$ and $m \geq 0$, we have

$$F_{m,\beta}^{(\alpha)}(v) = \sum_{l=0}^{m}\sum_{r=0}^{m-l}\sum_{\rho=0}^{l} \binom{m}{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{(1)_{\rho+1,\beta} S_{1,\beta}(l+1,\rho+1)}{l+1} v^r S_{2,\beta}(m-l,r).$$

Proof. From (18), we see that

$$\sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(v)\frac{\xi^m}{m!} = \frac{Ei_{\alpha,\beta}\left(\log_\beta (1+\xi)\right)}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)}$$

$$= \left(\frac{Ei_{\alpha,\beta}\left(\log_\beta (1+\xi)\right)}{\xi}\right)\left(\sum_{r=0}^{\infty} v^r\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)^r\right)$$

$$= \frac{1}{\xi}\left(\sum_{\rho=1}^{\infty} \frac{(1)_{\rho,\beta}\left(\log_\beta (1+\xi)\right)^\rho}{(\rho-1)!\,\rho^\alpha}\right)\left(\sum_{m=0}^{\infty}\sum_{r=0}^{m} v^r S_{2,\beta}(m,r)\frac{\xi^m}{m!}\right)$$

$$\left(\sum_{l=0}^{\infty}\sum_{\rho=0}^{\infty} \frac{1}{(\rho+1)^{\alpha-1}} \frac{(1)_{\rho+1,\beta} S_{1,\beta}(l+1,\rho+1)}{l+1}\frac{\xi^l}{l!}\right)\left(\sum_{m=0}^{\infty}\sum_{r=0}^{m} v^r S_{2,\beta}(m,r)\frac{\xi^m}{m!}\right)$$

$$= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{r=0}^{m-l}\sum_{\rho=0}^{l} \binom{m}{l} \frac{1}{(\rho+1)^{\alpha-1}} \frac{(1)_{\rho+1,\beta} S_{1,\beta}(l+1,\rho+1)}{l+1} v^r S_{2,\beta}(m-l,r)\right)\frac{\xi^m}{m!}. \tag{40}$$

Therefore, utilizing Eqs. (40), the result is obtained.

Corollary 2.6. Let $\alpha \in \mathbb{Z}$ and $m \geq 0$, we have

$$F_{m,\beta}(v) = \sum_{l=0}^{m} \sum_{r=0}^{m-l} \sum_{\rho=0}^{l} \binom{m}{l} \frac{(1)_{\rho+1,\beta} S_{1,\beta}(l+1,\rho+1)}{l+1} v^r S_{2,\beta}(m-l,r).$$

In the next section, the type 2 degenerate unipoly-Fubini polynomials are established and, certain explicit expressions and properties are obtained.

3 Type 2 Degenerate Unipoly-Fubini Polynomials and Numbers

Recently, Kim et al. [22] defined the unipoly function by

$$u_k(u \mid p) = \sum_{m=1}^{\infty} \frac{p(m)}{m^{\alpha}} u^m, \quad (\alpha \in \mathbb{Z}), \tag{41}$$

where p denotes any real or complex valued arithmetic function which is a defined on $\mathbb{N}$ (set of positive integers).

Moreover,

$$u_k(u \mid 1) = \sum_{m=1}^{\infty} \frac{u^m}{m^{\alpha}} = \mathrm{Li}_{\alpha}(u), \quad (\text{see [17]}), \tag{42}$$

denotes the ordinary polylogarithm function.

Dolgy et al. [13] introduced the degenerate unipoly function attached to polynomials $p(u)$ as follows:

$$u_{\alpha,\beta}(u \mid p) = \sum_{i=1}^{\infty} p(i) \frac{(1)_{i,\beta} u^i}{i^{\alpha}}. \tag{43}$$

We note that

$$u_{\alpha,\beta}\left(u \mid \frac{1}{\Gamma}\right) = \mathrm{E}i_{\alpha,\beta}(u) \tag{44}$$

is the modified degenerate polyexponential function.

By using (43), we introduce the degenerate unipoly-Fubini polynomials as follows:

$$\frac{u_{\alpha,\beta}\left(\log_{\beta}(1+\xi) \mid p\right)}{\xi\left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} (1+\beta\xi)^{\frac{u}{\beta}} = \sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(u;v) \frac{\xi^m}{m!}. \tag{45}$$

In the case when $u = 0$ and $y = 1$, $F_{m,\beta,p}^{(\alpha)} = F_{m,\beta,p}^{(\alpha)}(0;1)$ are called the type 2 degenerate unipoly-Fubini numbers.

Theorem 3.1. Let $m \geq 0$. Then, we have

$$F_{m,\beta,\frac{1}{\Gamma}}^{(\alpha)}(u;v) = F_{m,\beta}^{(\alpha)}(u;v), \quad (k \in \mathbb{Z}).$$

Proof. Let us take $p(m) = \frac{1}{\Gamma\beta}$. Then we have

$$\sum_{m=0}^{\infty} F_{m,\beta,\frac{1}{\Gamma}}^{(\alpha)}(u;v) \frac{\xi^m}{m!} = \frac{u_{\alpha,\beta}\left(\log_\beta(1+\xi)\mid\frac{1}{\Gamma}p\right)}{\xi\left(1 - v((1+\beta\xi)^{\frac{1}{\beta}} - 1)\right)} (1+\beta\xi)^{\frac{u}{\beta}}$$

$$= \frac{1}{\xi\left(1 - v((1+\beta\xi)^{\frac{1}{\beta}} - 1)\right)} \sum_{\rho=1}^{\infty} \frac{(1)_{\rho,\beta}\left(\log_\beta(1+\xi)\right)^\rho}{\rho^\alpha(\rho-1)!} (1+\beta\xi)^{\frac{u}{\beta}}$$

$$= \frac{Ei_{\alpha,\beta}\left(\log_\beta(1+\xi)\right)}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} (1+\beta\xi)^{\frac{u}{\beta}}$$

$$= \sum_{m=0}^{\infty} F_{m,\beta}^{(\alpha)}(u;v) \frac{\xi^m}{m!}. \tag{46}$$

Thus, by (46), the result is obtained.

Theorem 3.2. For $m \in \mathbb{N}$ and $\alpha \in \mathbb{Z}$, we have

$$F_{m,\beta,p}^{(\alpha)}(v) = \sum_{l=0}^{\infty} \sum_{\rho=0}^{l} \binom{m}{l} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)! S_{1,\beta}(\rho+1,l+1) F_{m-l,\beta}(v)}{(\rho+1)^\alpha(l+1)}. \tag{47}$$

Proof. From (45), we get

$$\sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(v) \frac{\xi^m}{m!} = \frac{u_{\alpha,\beta}\left(\log_\beta(1+\xi)\mid p\right)}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))}$$

$$= \frac{1}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} \sum_{\rho=1}^{\infty} \frac{p(\rho)(1)_{\rho,\beta}}{\rho^\alpha}\left(\log_\beta(1+\xi)\right)^\rho$$

$$= \frac{1}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} \sum_{\rho=0}^{\infty} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^\alpha} \sum_{l=\rho+1}^{\infty} S_{1,\beta}(\rho+1,l) \frac{\xi^l}{l!}$$

$$= \left(\sum_{j=0}^{\infty} F_{j,\beta}(v) \frac{\xi^j}{j!}\right)\left(\sum_{\rho=0}^{\infty} \sum_{l=0}^{\rho} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)! S_{1,\beta}(\rho+1,l+1)}{(\rho+1)^\alpha} \frac{\xi^l}{l+1} \frac{\xi^l}{l!}\right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{\infty} \sum_{\rho=0}^{l} \binom{m}{l} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)! S_{1,\beta}(\rho+1,l+1) F_{m-l,\beta}(v)}{(\rho+1)^\alpha(l+1)}\right) \frac{\xi^m}{m!}. \tag{48}$$

Thus, we complete the proof of this theorem.

In particular,

$$F_{m,\beta,\frac{1}{\Gamma}}^{(\alpha)}(v) = F_{m,\beta}^{(\alpha)}(v) = \sum_{l=0}^{\infty}\sum_{\rho=0}^{l}\binom{m}{l}\frac{S_{1,\beta}(\rho+1,l+1)F_{m-l,\beta}(v)}{(\rho+1)^{\alpha-1}(l+1)}. \tag{49}$$

Theorem 3.3. Let $m \geq 0$ and $\alpha \in \mathbb{Z}$. Then we have

$$F_{m,\beta,p}^{(\alpha)}(u;v) = \sum_{l=0}^{m}\sum_{\rho=0}^{l}\binom{m}{l}F_{m-l,\beta,p}^{(\alpha)}(v)(u)^{(\rho)}S_{2,\beta}(l,\rho;-\rho). \tag{50}$$

Proof. From (45), we observe that

$$\sum_{m=0}^{\infty}F_{m,\beta}^{(\alpha,p)}(u;v)\frac{\xi^m}{m!} = \frac{u_{\alpha,\beta}\left(\log_{\beta}(1+\xi)|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}\left(e_{\beta}^{-1}(\xi)-1+1\right)^{u}$$

$$= \frac{u_{\alpha,\beta}\left(\log_{\beta}(1+\xi)|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}\sum_{\rho=0}^{\infty}\binom{u+\rho-1}{\rho}\left(1-e_{\beta}^{-1}(\xi)\right)^{\rho}$$

$$= \left(\sum_{m=0}^{\infty}F_{m,\beta,p}^{(\alpha)}(v)\frac{\xi^m}{m!}\right)\left(\sum_{\rho=0}^{\infty}(u)^{(\rho)}\sum_{l=\rho}^{\infty}S_{2,\beta}(l,\rho;-\rho)\frac{\xi^l}{l!}\right)$$

$$= \left(\sum_{m=0}^{\infty}F_{m,\beta,p}^{(\alpha)}(v)\frac{\xi^m}{m!}\right)\left(\sum_{l=0}^{\infty}\sum_{\rho=0}^{l}(u)^{(\rho)}S_{2,\beta}(l,\rho;-\rho)\frac{\xi^l}{l!}\right)$$

$$L.H.S = \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\sum_{\rho=0}^{l}\binom{m}{l}F_{m-l,\beta,p}^{(\alpha)}(v)(u)^{(\rho)}S_{2,\beta}(l,\rho;-\rho)\right)\frac{\xi^m}{m!}. \tag{51}$$

By (51), we obtain the result.

Theorem 3.4. For $m \geq 0$ and $\alpha \in \mathbb{Z}$, we have

$$F_{m,\beta,p}^{(\alpha)}(v) = \sum_{l=0}^{m}\sum_{a=0}^{m-l}\sum_{\rho=0}^{l}\binom{m}{l}\binom{m-l}{a}D_{m-l-a,\beta}F_{a,\beta}(v)\frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^{\alpha}}S_{1,\beta}(l,\rho+1). \tag{52}$$

Proof. From (45), we observe that

$$\sum_{m=0}^{\infty}F_{m,\beta,p}^{(\alpha)}(v)\frac{\xi^m}{m!} = \frac{u_{\alpha,\beta}\left(\log_{\beta}(1+\xi)|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}$$

$$= \frac{1}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}\sum_{\rho=0}^{\infty}\frac{p(\rho+1)(1)_{\rho+1,\beta}\,\rho!}{(\rho+1)^{\alpha}\,\rho!}\left(\log_{\beta}(1+\xi)\right)^{\rho+1}$$

$$= \frac{\log_\beta (1+\xi)}{\xi \left(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)\right)} \sum_{\rho=0}^{\infty} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^\alpha (\rho+1)!} \left(\log_\beta (1+\xi)\right)^{\rho+1}$$

$$= \frac{\log_\beta (1+\xi)}{\xi} \frac{1}{1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right)} \sum_{\rho=0}^{\infty} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^\alpha} \sum_{l=\rho}^{\infty} S_{1,\beta}(l,\rho+1) \frac{\xi^l}{l!}$$

$$= \left(\sum_{s=0}^{\infty} D_{s,\beta} \frac{\xi^s}{s!}\right) \left(\sum_{a=0}^{\infty} F_{a,\beta}(v) \frac{\xi^a}{a!}\right) \left(\sum_{l=0}^{\infty} \sum_{\rho=0}^{m} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^\alpha} S_{1,\beta}(l,\rho+1) \frac{\xi^l}{l!}\right)$$

$$= \left(\sum_{b=0}^{\infty} \sum_{a=0}^{b} \binom{b}{a} D_{b-a,\beta} F_{a,\beta}(v) \frac{\xi^b}{b!}\right) \left(\sum_{l=0}^{\infty} \sum_{\rho=0}^{m} \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho!+1)}{(\rho+1)^\alpha} S_{1,\beta}(l,\rho+1) \frac{\xi^l}{l!}\right)$$

$$= \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} \sum_{a=0}^{m-l} \sum_{\rho=0}^{l} \binom{m}{l} \binom{m-l}{a} D_{m-l-a,\beta} F_{a,\beta}(v) \frac{p(\rho+1)(1)_{\rho+1,\beta}(\rho+1)!}{(\rho+1)^\alpha} S_{1,\beta}(l,\rho+1)\right) \frac{\xi^m}{m!}. \quad (53)$$

Comparing the coefficients of $\frac{\xi^m}{m!}$ of the above equation, the result is established.

Theorem 3.5. For $m \geq 0$ and $\alpha \in \mathbb{Z}$, we have

$$F_{m,\beta,p}^{(\alpha)}(u;v) = mF_{m-1,\beta,p}^{(\alpha)}(u;v). \tag{54}$$

Proof. Applying the difference operator $\triangle_\beta$ on both sides of Eq. (45), gives

$$\triangle_\beta \left(\sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(u;v) \frac{\xi^m}{m!}\right) = \triangle_\beta \left(\frac{u_{\alpha,\beta}\left(\log_\beta (1+\xi)|p\right)}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} (1+\beta\xi)^{\frac{u}{\beta}}\right)$$

and then we have

$$\sum_{m=0}^{\infty} \triangle_\beta F_{m,\beta,p}^{(\alpha)}(v) \frac{\xi^m}{m!} = \frac{u_{\alpha,\beta}\left(\log_\beta (1+\xi)|p\right)}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} \triangle_\beta e_\beta^u(\xi)$$

$$= \frac{u_{\alpha,\beta}\left(\log_\beta (1+\xi)|p\right)}{\xi(1 - v\left((1+\beta\xi)^{\frac{1}{\beta}} - 1\right))} e_\beta^u(\xi) \xi$$

$$= \sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(u;v) \frac{\xi^{m+1}}{m!}. \tag{55}$$

Therefore, by (55), we obtain the result.

Theorem 3.6. Let $m \geq 0$ and $\alpha \in \mathbb{Z}$. Then we have

$$\frac{\partial}{\partial u} F_{m,\beta,p}^{(\alpha)}(u;v) = \sum_{\rho=0}^{m} \binom{m}{\rho} F_{m-\rho,\beta,p}^{(\alpha)}(u;v) (1)_{\rho,\beta}.$$

Proof. Applying the operator $\frac{\delta}{\delta u}$ in both sides of (45), we have

$$
\frac{\partial}{\partial u}\left(\sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(u;v)\frac{\xi^m}{m!}\right) = \frac{\partial}{\partial u}\left(\frac{u_{\alpha,\beta}\left(\log_\beta(1+\xi)\,|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u}{\beta}}\right)
$$

$$
\sum_{m=0}^{\infty}\frac{\partial}{\partial u}F_{m,\beta,p}^{(\alpha)}(u;v)\frac{\xi^m}{m!}
$$

$$
= \frac{u_{\alpha,\beta}\left(\log_\beta(1+\xi)\,|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}\frac{\partial}{\partial u}(1+\beta\xi)^{\frac{u}{\beta}}
$$

$$
= \frac{u_{\alpha,\beta}\left(\log_\beta(1+\xi)\,|p\right)}{\xi\left(1-v\left((1+\beta\xi)^{\frac{1}{\beta}}-1\right)\right)}(1+\beta\xi)^{\frac{u}{\beta}}(1+\beta\xi)^{\frac{1}{\beta}}
$$

$$
= \left(\sum_{m=0}^{\infty} F_{m,\beta,p}^{(\alpha)}(u;v)\frac{\xi^m}{m!}\right)\left(\sum_{\rho=0}^{\infty}(1)_{\rho,\beta}\frac{\xi^\rho}{\rho!}\right)
$$

$$
= \sum_{m=0}^{\infty}\left(\sum_{\rho=0}^{m}\binom{m}{\rho}F_{m-\rho,\beta,p}^{(\alpha)}(u;v)(1)_{\rho,\beta}\right)\frac{\xi^m}{m!}. \tag{56}
$$

In view of (56), we complete the proof.

4 Computation of Zeros and Graphical Representation

To understand the structure of the type 2 degenerate poly-Fubini polynomials $F_{m,\beta}^{(\alpha)}(u;v)$, we employ some numerical investigations (using *Mathematica*) to examine properties of figures, look and to discover some interesting patterns. First, we give few $F_{m,\beta}^{(\alpha)}(u;v)$ (for $\alpha = 5$, $\beta = 4$) as follows:

$$
F_{0,\beta}^{(\alpha)}(u;v) = \frac{1}{\log(4)},
$$

$$
F_{1,\beta}^{(\alpha)}(u;v) = -\frac{3}{32\log(4)^2} - \frac{1}{2\log(4)} + \frac{u}{\log(4)} + \frac{v}{\log(4)},
$$

$$
F_{2,\beta}^{(\alpha)}(u;v) = \frac{7}{81\log(4)^3} + \frac{3}{16\log(4)^2} - \frac{3u}{16\log(4)^2} - \frac{3v}{16\log(4)^2} + \frac{2}{3\log(4)} - \frac{5u}{\log(4)} + \frac{u^2}{\log(4)}
$$

$$
-\frac{4v}{\log(4)} + \frac{2uv}{\log(4)} + \frac{2v^2}{\log(4)},
$$

$$
F_{3,\beta}^{(\alpha)}(u;v) = -\frac{231}{1024\log(4)^4} - \frac{7}{18\log(4)^3} + \frac{7u}{27\log(4)^3} + \frac{7v}{27\log(4)^3} - \frac{33}{64\log(4)^2} + \frac{27u}{16\log(4)^2}
$$

$$-\frac{9u^2}{32\log(4)^2}+\frac{45v}{32\log(4)^2}-\frac{9uv}{16\log(4)^2}-\frac{9v^2}{16\log(4)^2}-\frac{3}{2\log(4)}+\frac{40u}{\log(4)}-\frac{27u^2}{2\log(4)}$$

$$+\frac{u^3}{\log(4)}+\frac{55v}{2\log(4)}-\frac{24uv}{\log(4)}+\frac{3u^2v}{\log(4)}-\frac{21v^2}{\log(4)}+\frac{6uv^2}{\log(4)}+\frac{6v^3}{\log(4)},$$

$$F_{4,\beta}^{(\alpha)}(u;v)=\frac{693}{625\log(4)^5}+\frac{231}{128\log(4)^4}-\frac{231u}{256\log(4)^4}-\frac{231v}{256\log(4)^4}+\frac{49}{27\log(4)^3}-\frac{98u}{27\log(4)^3}$$

$$+\frac{14u^2}{27\log(4)^3}-\frac{28v}{9\log(4)^3}+\frac{28uv}{27\log(4)^3}+\frac{28v^2}{27\log(4)^3}+\frac{15}{8\log(4)^2}-\frac{297u}{16\log(4)^2}$$

$$+\frac{45u^2}{8\log(4)^2}-\frac{3u^3}{8\log(4)^2}-\frac{213v}{16\log(4)^2}+\frac{81uv}{8\log(4)^2}-\frac{9u^2v}{8\log(4)^2}+\frac{9v^2}{\log(4)^2}$$

$$-\frac{9uv^2}{4\log(4)^2}-\frac{9v^3}{4\log(4)^2}+\frac{24}{5\log(4)}-\frac{470u}{\log(4)}+\frac{204u^2}{\log(4)}-\frac{26u^3}{\log(4)}+\frac{u^4}{\log(4)}$$

$$-\frac{291y}{\log(4)}+\frac{334uv}{\log(4)}-\frac{72u^2v}{\log(4)}+\frac{4u^3v}{\log(4)}+\frac{266v^2}{\log(4)}-\frac{132uv^2}{\log(4)}+\frac{12u^2v^2}{\log(4)}$$

$$-\frac{120v^3}{\log(4)}+\frac{24uv^3}{\log(4)}+\frac{24v^4}{\log(4)}.$$

Next, we display the shape of $F_{m,\beta}^{(\alpha)}(u;v)$ for $v = 4$, $-100 \le u \le 100$ in Fig. 1, for $m = 1, 2, \ldots, 8$.
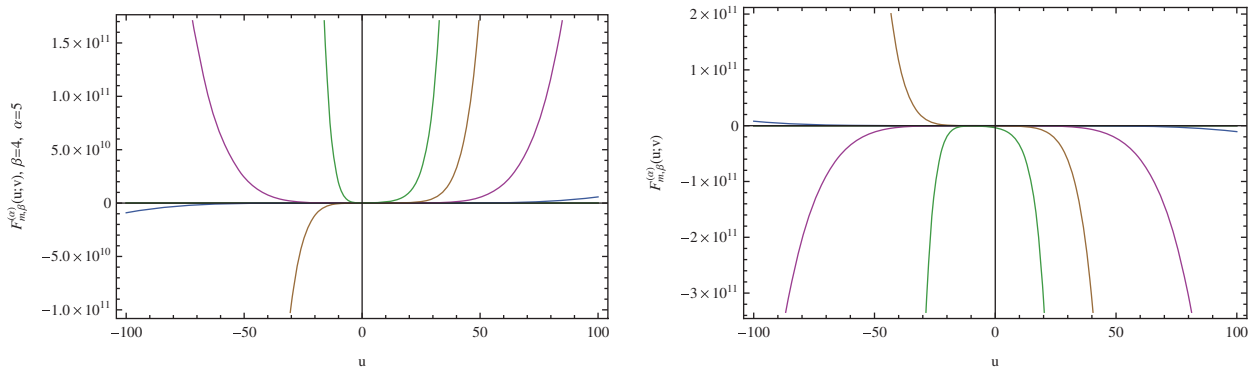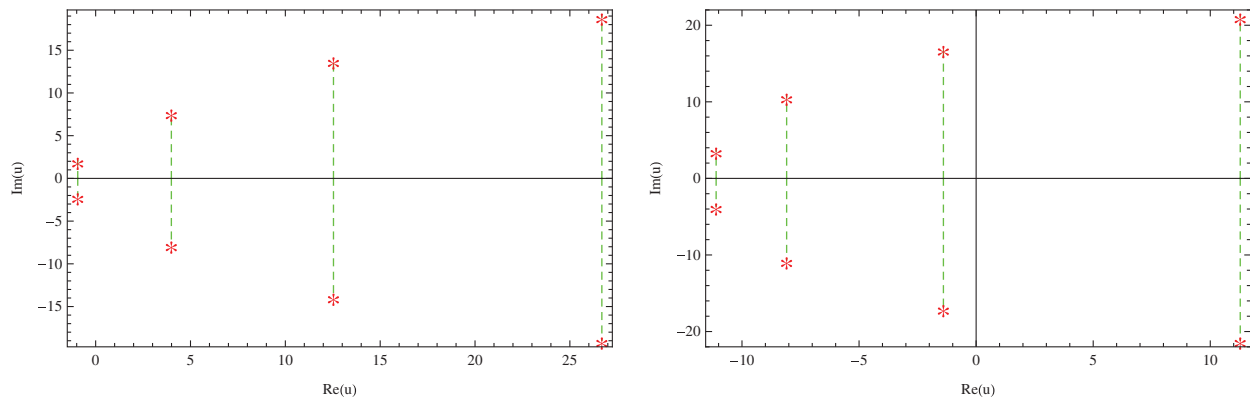

Figure 1: Graphs of $F_{m,\beta}^{(\alpha)}(u;v)$ (for $\alpha = 5$, $\beta = 4$ (left) and $\alpha = 7$, $\beta = \frac{1}{3}$ (right))

The real and complex zeros of $F_{m,\beta}^{(\alpha)}(u;v) = 0$ for $v = 4$, $\alpha = 5$, $\beta = 4$ are listed in Tab. 1.

The zeros of $F_{8,\beta}^{(\alpha)}(u;v) = 0$ for $u \in \mathbb{C}, v = 4$ are plotted in Fig. 2.

From Fig. 2, we note that $F_{8,\beta}^{(\alpha)}(u;v)$ has $Im\,(u) = 0$ reflection symmetry.

Table 1: Approximate solution of $F_{m,\beta}^{(\alpha)}(u;v)$

m	Real zeros	Complex zeros
1	-3.43237	$-$
2	$-$	$-1.43237 - 3.77547i, -1.43237 + 3.77547i$
3	-2.90029	$2.30158 - 6.91248i, 2.30158 + 6.91248i$
4	$-$	$-1.5333 - 2.93783i, -1.5333 + 2.93783i,$ $6.66855 - 9.72262i, 6.66855 + 9.72262i$
5	-2.40918	$1.23681 - 5.7923i, 1.23681 + 5.7923i,$ $11.3869 - 12.2852i, 11.3869 + 12.2852i$
6	$-$	$-1.27333 - 2.44972i, -1.27333 + 2.44972i, 4.64133 - 8.58081i,$ $4.64133 - 8.58081i, 16.3349 - 14.6551i, 16.3349 + 14.6551i$
7	-1.97775	$1.08051 - 5.05226i, 1.08051 + 5.05226i, 8.44738 - 11.2601i,$ $8.44738 + 11.2601i, 21.4477 - 16.8698i, 21.4477 + 16.8698i$
8	$-$	$-0.943915 - 2.07556i, -0.943915 + 2.07556i, 3.99085 - 7.7234i,$ $3.99085 + 7.7234i, 12.5375 - 13.8303i, 12.5375 + 13.8303i,$ $26.686 - 18.9561i, 26.686 + 18.9561i$

Figure 2: Zeros of $F_{8,\beta}^{(\alpha)}(u;v)$ (for $\alpha = 5$, $\beta = 4$ (left) and $\alpha = 7, \beta = \frac{1}{3}$ (right))
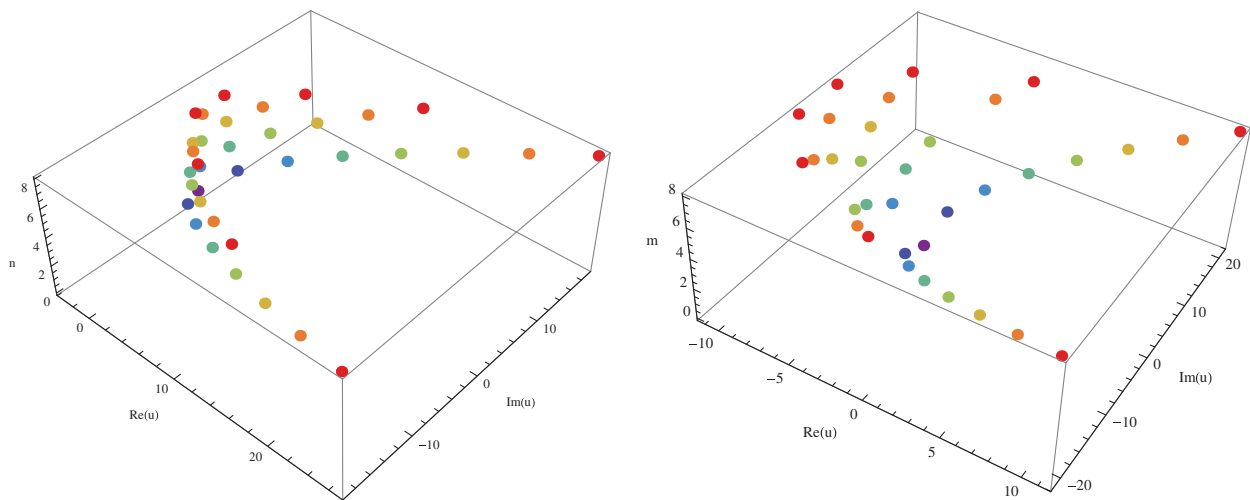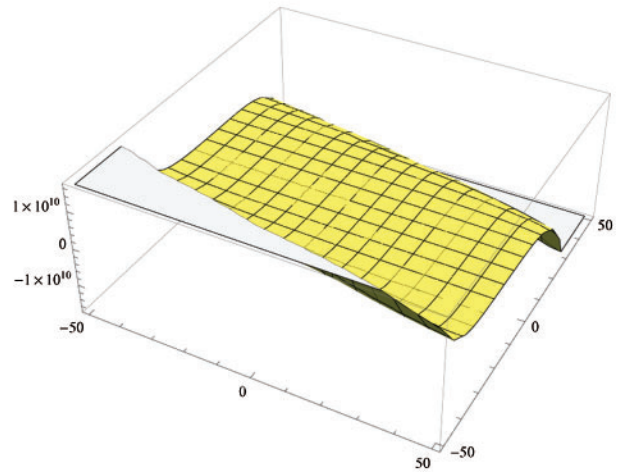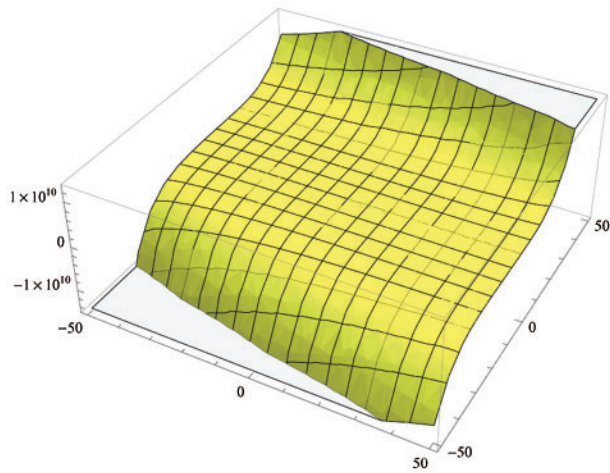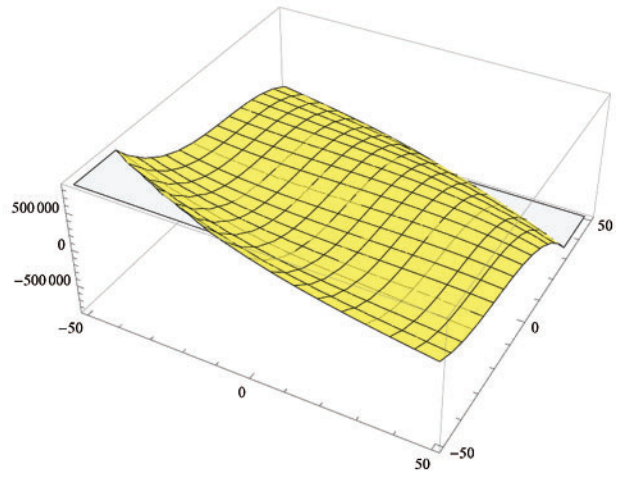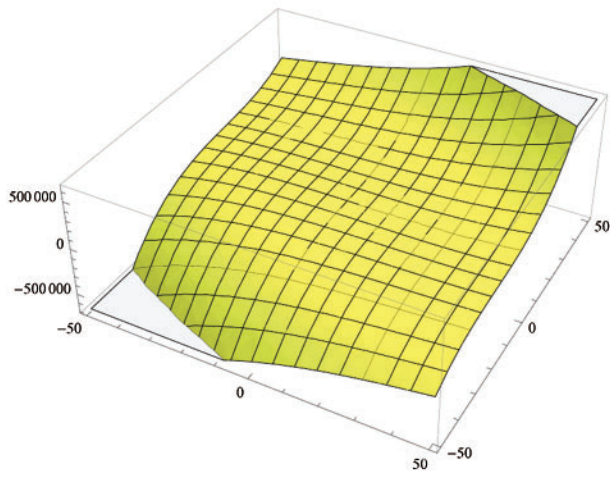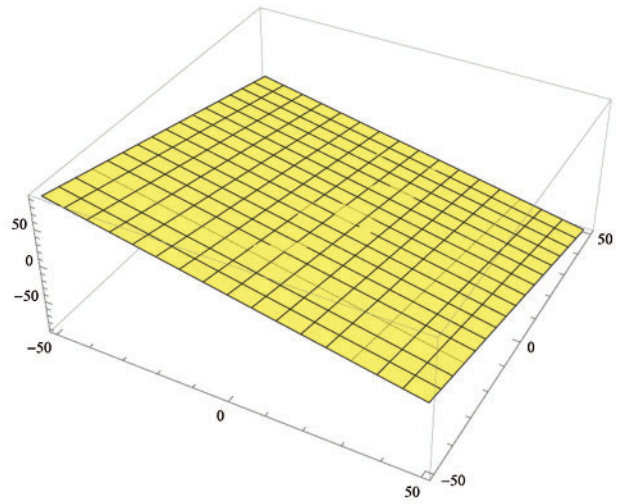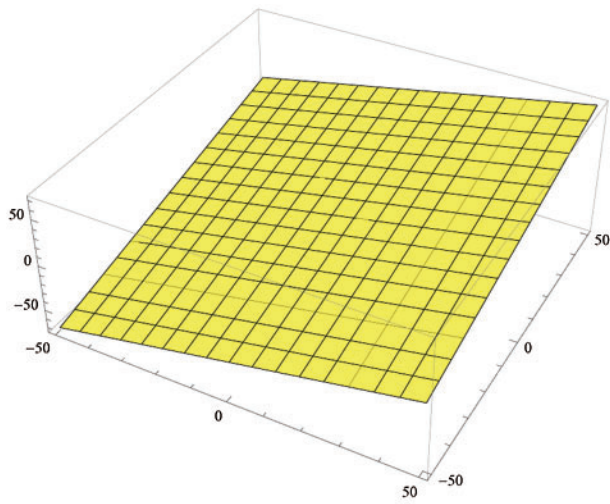

Figure 3: Stacking structure of approximate roots of $F_{m,\beta}^{(\alpha)}(u;v)$ (for $\alpha = 5$, $\beta = 4$ (left) and $\alpha = 7, \beta = \frac{1}{3}$ (right))
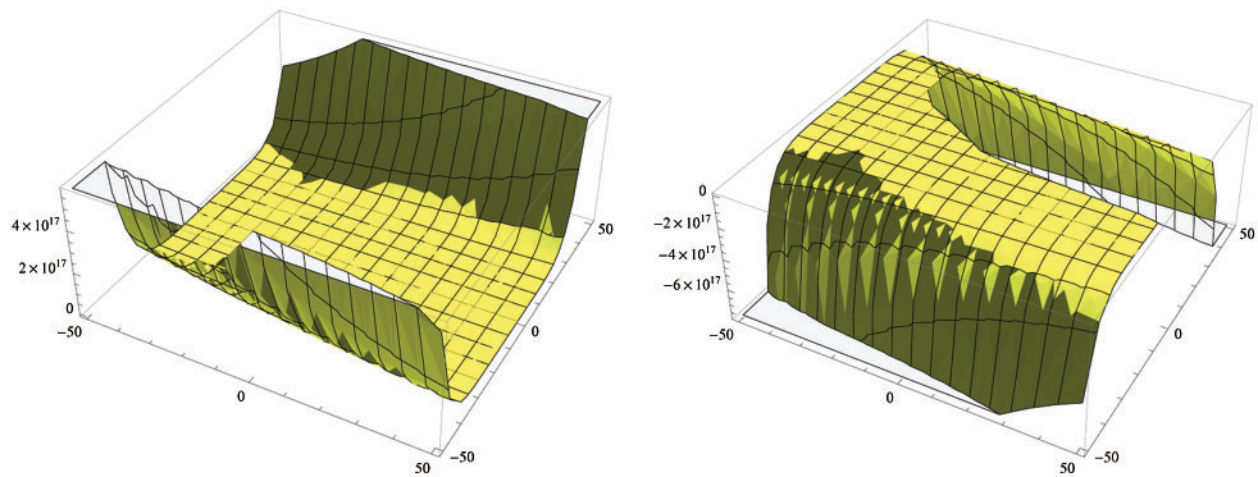
Figure 4: Surface plot of $F_{m,\beta}^{(\alpha)}(u;v)$, $m = 1, 3, 5, 8$ (for $\alpha = 5$, $\beta = 4$ (left) and $\alpha = 7, \beta = \frac{1}{3}$ (right))

The stacking structure of approximate roots of $F_{m,\beta}^{(\alpha)}(u;v) = 0$ for $v = 4$, $m = 1, 2, \ldots, 8$ is given in Fig. 3.

To provide more clear explanation of the structure of the 2 degenerate poly-Fubini polynomials $F_{m,\beta}^{(\alpha)}(u;v)$, we present the 3D surface for some values of m, these surfaces are displayed in Fig. 4.

5 Conclusions

In this article, we have introduced type 2 degenerate poly-Fubini polynomials and established certain identities of these numbers and polynomials. Further, we defined the degenerate unipoly-Fubini polynomials and derived certain properties of these numbers and polynomials and given multifarious relations including derivative and integral properties. In addition, we have provided relationships between degenerate unipoly-Fubini polynomials and degenerate special polynomials. In the last section, we have these polynomials with the help of Mathematica.

Conflicts of Interest: The authors declare that they have no conflict of interest to report regarding the present study.

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