Two-Dimensional Interpolation Criterion Using DFT Coefficients

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Abstract: In this paper, we address the frequency estimator for 2-dimensional (2-D) complex sinusoids in the presence of white Gaussian noise. With the use of the sinc function model of the discrete Fourier transform (DFT) coefficients on the input data, a fast and accurate frequency estimator is devised, where only the DFT coefficient with the highest magnitude and its four neighbors are required. Variance analysis is also included to investigate the accuracy of the proposed algorithm. Simulation results are conducted to demonstrate the superiority of the developed scheme, in terms of the estimation performance and computational complexity.

Keywords: 2-D Frequency estimation, parameter estimation, interpolation, discrete Fourier transform.

1 Introduction

Parameter estimation from sinusoids is an of importance research topic in numerous areas of science and engineering [Kay (1993); Marple (1987); Quinn and Hannan (2001)]. It refers to accurately estimating the parameters of interested from a finite number of measurements, which consists of the sinusoidal signal and noise. Since amplitudes and phases of sinusoids are easily obtained by determined frequencies, the frequency estimation is an essential step. Moreover, due to the nonlinearity in the signal model, estimating frequencies is a difficult task and has attracted much attention [Stoica and Moses (2005); So, Chan, Chan et al. (2005); So and Chan (2006)].

In this work, we address a fundamental problem, referred to as estimating the frequency of a 2-dimensional (2-D) single complex-tone in the presence of additive white Gaussian noise. Subspace methods including multiple signal classification (MUSIC) [Schmidt (1986); Huang, Wu, So et al. (2012); Odendaal, Barnard and Pistorius (1994); Li, Razavilar and Liu (1998)] and estimation of signal parameters via rotation invariance techniques (ESPRIT) [Roy, Paulraj and Kailath (1986); Van, Vanderveen and Paulraj (1998); Axmon, Hansson and Sornmo (2005); Rouquette and Najim (2001)] are proposed to obtain the accurate frequency estimation. However, these solvers involve extensive computations because peak search in the full range frequency domain is required. Sun et

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al. [So, Chan, Lau et al. (2010)] suggests principal-singular-vector utilization for modal analysis (PUMA), which utilizes linear prediction of 2-D multiple exponentials to achieve optimal estimation performance in only high signal-to-noise ratio (SNR). Although all these methods can provide the optimal performance, they suffer from the high complexity, especially in the case of big data [Wu (2018)].

Then the class of interpolating on discrete Fourier transform (DFT) coefficients, is devised, which is shown efficient by means of estimation performance and computational requirements. Among this class of methodologies, a coarse estimation and then a fine-tune step are required, where the former one is usually realized by finding the index of the peak magnitude of DFT spectrum, and the latter one refers to interpolation on DFT peak to improve the estimation accuracy. In Quinn et al. [Quinn (1997); Provencher (2010); Candan (2013, 2015)], different interpolation schemes are developed with the use of DFT peak and its neighbor bins. Although these methods can achieve the optimal or nearly optimum estimation performance, they can apply only for the one-dimensional single complex-tone.

In the paper, a 2-D non-iterative and accurate frequency estimator (2-D NIA) is proposed. Here, a new interpolation criterion is devised by utilizing the relationship of ratios of midway magnitudes to the largest one. For the estimation of the first dimensional frequency, the left-and-right neighbors of DFT peak is employed, while the up-and-low neighbors of the peak is used to obtain the second dimensional frequency estimate. The variance analysis is also provided, which indicates the high performance of our method.

The rest of this paper is organized as follows. The NIA method is derived in Section 2, whose variance analysis is also provided in Section 3. Computer simulations in Section 4 are carried out to show that the developed methods can attain Cramer-Rao lower bound (CRLB). Finally, conclusions are drawn in Section 5.

2 Proposed method

Without loss of generality, the 2-D observations are modeled as:

$$y_{m,n} = A\exp(j(\omega m + \nu n)) + q_{m,n} \ , \ m = 0,1,\cdots,M-1, \quad n=0,1,\cdots,N-1 \tag{1}$$

where A is the unknown complex amplitude, $\omega \in (0, 2\pi)$ denotes unknown frequency in the first dimension, while $\nu \in (0, 2\pi)$ is the unknown frequency in the second dimension and $q_{m,n}$ is the independent and identically distributed (IID) noise term following the 2-D zero-mean complex white Gaussian distribution with unknown variance σ^2_{MN}. Without loss of generality, we assume that $M \neq N$. Given the MN samples of $y_{m,n}$, our task is to find the unknown frequency parameters, namely, $\{\omega, \nu\}$.

Consider the 2D-DFT on the observed data $\mathbf{y}$, while (L_1, L_2) are the coordinate of the largest peak of the spectrum, then the true frequencies can be described by

$$\omega = 2\pi \frac{L_1 + \delta}{M} \tag{2}$$

$$\nu = 2\pi \frac{L_2 + \mu}{N} \tag{3}$$

where $\delta, \mu \in [-0.5, 0.5] \times [-0.5, 0.5]$ is the residual. The estimates of ω and ν, namely, $\hat{\omega}$ and $\hat{\nu}$, can be obtained by estimating δ and μ, respectively.

The $(k_1, k_2)_{\text{th}}$ DFT coefficients can be expressed by:

$$Y_{k_1, k_2} = \sum_{n=0}^{N-1} \left\{ \sum_{m=0}^{M-1} y_{m,n} \exp(-j\theta_{k_1} m) \right\} \exp(-j\theta_{k_2} n)$$
$$= X_{k_1, k_2} + Q_{k_1, k_2} \tag{4}$$

where $\theta_{k_1} = 2\pi \dfrac{k_1}{M}, \theta_{k_2} = 2\pi \dfrac{k_2}{N}$, Q_{k_1, k_2} is the Fourier coefficient of the noise part and

$$X_{k_1, k_2} = A \exp(j\phi(k_1, k_2)) \frac{\sin(\pi(L_1 - \delta - k_1))}{\sin(\frac{\pi}{N}(L_1 - \delta - k_1))} \times \frac{\sin(\pi(L_2 - \mu - k_2))}{\sin(\frac{\pi}{N}(L_2 - \mu - k_2))} \tag{5}$$

denotes the DFT of noise-free terms with

$$\phi(k_1, k_2) = \frac{\pi(M-1)}{M}(L_1 + \delta - k_1) + \frac{\pi(N-1)}{N}(L_2 + \mu - k_2) \tag{6}$$

Ignoring the noise term, the midway of the $(L_1, L_2)_{\text{th}}$ DFT coefficient and $(L_1 - 1, L_2)_{\text{th}}$ as well as $(L_1 + 1, L_2)_{\text{th}}$, can be expressed as:

$$Y_{L_1, L_2} = A \exp(j\phi_{L_1, L_2}) \frac{\sin(\pi\delta)}{\sin(\frac{\pi}{N}\delta)} \cdot \frac{\sin(\pi\mu)}{\sin(\frac{\pi}{N}\mu)} \tag{7}$$

$$Y_{L_1 - 0.5, L_2} = A \exp(j\phi_{L_1 - 0.5, L_2}) \frac{\sin(\pi(\delta + 0.5, L_2))}{\sin(\frac{\pi}{N}(\delta + 0.5, L_2))} \cdot \frac{\sin(\pi\mu)}{\sin(\frac{\pi}{N}\mu)} \tag{8}$$

$$Y_{L_1 + 0.5, L_2} = A \exp(j\phi_{L_1 + 0.5, L_2}) \frac{\sin(\pi(\delta - 0.5, L_2))}{\sin(\frac{\pi}{N}(\delta - 0.5, L_2))} \cdot \frac{\sin(\pi\mu)}{\sin(\frac{\pi}{N}\mu)} \tag{9}$$

Let $r_1 = \dfrac{\left| Y_{L_1 + 0.5, L_2} \right|}{\left| Y_{L_1, L_2} \right|}$ and $r_2 = \dfrac{\left| Y_{L_1 - 0.5, L_2} \right|}{\left| Y_{L_1, L_2} \right|}$ with $|x|$ being the magnitude of the complex value x.

We have

$$\sin\left(\frac{\pi(1 - 2\delta)}{2M}\right) r_1 - \sin\left(\frac{\pi(2\delta + 1)}{2M}\right) r_2 = \frac{\cos(\pi\delta)}{\left| Y_{L_1, L_2} \right|} - \frac{\cos(\pi\delta)}{\left| Y_{L_1, L_2} \right|} = 0 \tag{10}$$

Utilizing the product-to-sum identities, (10) is rewritten as

$$\tan\left(\frac{\pi\delta}{M}\right) = \tan\left(\frac{\pi}{2M}\right)\frac{\left|Y_{L_1+0.5,L_2}\right| - \left|Y_{L_1-0.5,L_2}\right|}{\left|Y_{L_1+0.5,L_2}\right| + \left|Y_{L_1-0.5,L_2}\right|} \tag{11}$$

According to (2) and (11), the estimate of ω, denoted by $\hat{\omega}$, is

$$\hat{\omega} = 2\tan^{-1}\left(\tan\left(\frac{\pi}{2M}\right)\frac{\left|Y_{L_1+0.5,L_2}\right| - \left|Y_{L_1-0.5,L_2}\right|}{\left|Y_{L_1+0.5,L_2}\right| + \left|Y_{L_1-0.5,L_2}\right|}\right) + \theta_{L_1} \tag{12}$$

where $\tan^{-1}(\cdot)$ denotes arc-tangent operator.

Similarly, the midway the midway of the $(L_1, L_2)_{th}$ DFT coefficient and $(L_1, L_2-1)_{th}$ as well as $(L_1, L_2+1)_{th}$ are

$$Y_{L_1,L_{2-0.5}} = A\exp(j\phi_{L_1,L_{2-0.5}})\frac{\sin(\pi\delta)}{\sin(\frac{\pi}{N}\delta)} \cdot \frac{\sin(\pi(\mu+0.5, L_2))}{\sin(\frac{\pi}{N}(\mu+0.5, L_2))} \tag{13}$$

$$Y_{L_1,L_{2+0.5}} = A\exp(j\phi_{L_1,L_{2+0.5}})\frac{\sin(\pi\delta)}{\sin(\frac{\pi}{N}\delta)} \cdot \frac{\sin(\pi(\mu-0.5, L_2))}{\sin(\frac{\pi}{N}(\mu-0.5, L_2))} \tag{14}$$

Let $l_1 = \dfrac{\left|Y_{L_1,L_{2+0.5}}\right|}{\left|Y_{L_1,L_2}\right|}$ and $l_2 = \dfrac{\left|Y_{L_1,L_{2-0.5}}\right|}{\left|Y_{L_1,L_2}\right|}$, using (7), we still can obtain

$$\sin\left(\frac{\pi(1-2\mu)}{2N}\right)l_1 - \sin\left(\frac{\pi(2\mu+1)}{2N}\right)l_2 = 0 \tag{15}$$

According to (2) and (15), the estimate of ν, denoted by $\hat{\nu}$, is

$$\hat{\nu} = 2\tan^{-1}\left(\tan\left(\frac{\pi}{2N}\right)\frac{\left|Y_{L_1,L_{2+0.5}}\right| - \left|Y_{L_1,L_{2-0.5}}\right|}{\left|Y_{L_1,L_{2+0.5}}\right| + \left|Y_{L_1,L_{2-0.5}}\right|}\right) + \theta_{L_2} \tag{16}$$

Then in this method, the estimates of ω and ν can be easily obtained by (12) and (16).

3 Variance analysis

As discussed in Aboutanios et al. [Aboutanios and Mulgrew (2005)], the Fourier coefficients of additive white Gaussian noise are IID zero-mean Gaussian distributed variables with variance $MN\sigma^2$. Furthermore, it is proved that the noise term Q_k is $O(\sqrt{MN\ln(MN)})$, where $O(\cdot)$ denotes the asymptotic notation [Chen and Hannan (2010)].

In the following, we first discuss the variance analysis for the first dimension frequency

estimation. Let $Y_{L_1 \pm 0.5, L_2} = X_{L_1 \pm 0.5, L_2} + Q_{L_1 \pm 0.5, L_2}$. Then we introduce a new variable δ_1, which is defined as

$$\delta_1 = \frac{\left| Y_{L_1+0.5,L_2} \right| - \left| Y_{L_1-0.5,L_2} \right|}{\left| Y_{L_1+0.5,L_2} \right| + \left| Y_{L_1-0.5,L_2} \right|} \tag{17}$$

From (12) and (17), we can devise the relationship between δ and δ_1, which is

$$\delta = 2\tan^{-1}\left(\tan\left(\frac{\pi}{2M}\right)\delta_1 \right) \tag{18}$$

Therefore, the variance analysis of δ can be devised easily form that of δ_1. As $N \to \infty$, we have from (13)

$$\hat{\delta}_1 = \frac{\left| X_{L_1+0.5,L_2} \right|\left| 1 + \frac{Q_{L_1+0.5,L_2}}{X_{L1+0.5,L_2}} \right| - \left| X_{L_1-0.5,L_2} \right|\left| 1 + \frac{Q_{L_1-0.5,L_2}}{X_{L_1-0.5,L_2}} \right|}{\left| X_{L_1+0.5,L_2} \right|\left| 1 + \frac{Q_{L_1+0.5,L_2}}{X_{L_1+0.5,L_2}} \right| + \left| X_{L_1-0.5,L_2} \right|\left| 1 + \frac{Q_{L_1-0.5,L_2}}{X_{L_1-0.5,L_2}} \right|} \tag{19}$$

Then according to findings in [18], when $\left| \dfrac{Q_{L1\pm0.5,L_2}}{X_{L1\pm0.5,L_2}} \right| \ll 1$, it can be expanded as

$$\left| 1 + \frac{Q_{L_1\pm0.5,L_2}}{X_{L_1\pm0.5,L_2}} \right| = 1 - \Re\left\{ \left| \frac{Q_{L_1\pm0.5,L_2}}{X_{L_1\pm0.5,L_2}} \right| \right\} + O(1) \tag{20}$$

where $\Re\{x\}$ is the real part of x. For large data size MN, (19) is simplified as

$$\hat{\delta}_1 = \frac{\left| X_{L_1+0.5,L_2} \right| - \left| X_{L_1-0.5,L_2} \right| - \left(Z_{L_1+0.5,L_2} - Z_{L_1-0.5,L_2} \right)}{\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right| - \left(Z_{L_1+0.5,L_2} + Z_{L_1-0.5,L_2} \right)}$$

$$= \frac{2\delta_1 - \dfrac{1}{\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right|}\left(Z_{L_1+0.5,L_2} - Z_{L_1-0.5,L_2} \right)}{1 - \dfrac{1}{\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right|}\left(Z_{L_1+0.5,L_2} + Z_{L_1-0.5,L_2} \right)} \tag{21}$$

where

$$Z_{L_1+0.5,L_2} = \left| X_{L_1+0.5,L_2} \right| \Re\left\{ \frac{Q_{L1+0.5,L_2}}{X_{L1+0.5,L_2}} \right\} = \Re\left\{ Q_{L_1+0.5,L_2} \exp(-j\phi_{0.5,L_2}) \right\} \qquad (22)$$

$$Z_{L_1-0.5,L_2} = \left| X_{L_1-0.5,L_2} \right| \Re\left\{ \frac{Q_{L1-0.5,L_2}}{X_{L_1-0.5,L_2}} \right\} = \Re\left\{ Q_{L_1+0.5,L_2} \exp(-j\phi_{0.5,L_2}) \right\} \qquad (23)$$

With the use of the discussion in Aboutanios et al. [Aboutanios and Mulgrew (2005)], (21) is further rewritten as

$$\hat{\delta}_1 = \left[\delta_1 - \frac{1}{\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right|} \left(Z_{L_1+0.5,L_2} - Z_{L_1-0.5,L_2} \right) \right]$$

$$\times \left[1 + \frac{1}{\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right|} \left(Z_{L_1+0.5,L_2} + Z_{L_1-0.5,L_2} \right) + O(N^{-1}\ln(N)) \right] \qquad (24)$$

Expanding and simplifying (21) yields

$$\hat{\delta}_1 = \delta_1 + \frac{(\delta-1)Z_{L_1+0.5,L_2}}{\left(\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right| \right)} + \frac{(1+\delta)Z_{L_1-0.5,L_2}}{\left(\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right| \right)} \qquad (25)$$

Since Q_m is zero-mean Gaussian distributed, the bias of $\hat{\delta}_1$, denoted by $Bias(\hat{\delta}_1)$, is

$$Bias(\hat{\delta}_1) = E(\hat{\delta}_1) - \delta_1$$

$$= \frac{(\delta-1)\Re\left\{ E\left\{ Q_{L1+0.5,L_2} \right\} \exp(-j\phi_{0.5,L_2}) \right\}}{2\left(\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right| \right)}$$

$$+ \frac{(1+\delta)\Re\left\{ E\left\{ Q_{L1-0.5,L_2} \right\} \exp(-j\phi_{-0.5,L_2}) \right\}}{2\left(\left| X_{L_1+0.5,L_2} \right| + \left| X_{L_1-0.5,L_2} \right| \right)} = 0 \qquad (26)$$

which indicates the unbiasedness of the proposed estimator.

Since $\mathrm{var}\left[\Re\left\{ Q_{L1\pm0.5,L_2} \exp(-j\phi_{\pm0.5,L_2}) \right\} \right] = \frac{MN\sigma^2}{2}$ [Kay (1993)], the variance of δ,

$\mathrm{var}(\hat{\delta}_1)$ is

$$\text{var}(\hat{\delta}_1) = \frac{(4\delta_1^2 + 1)\sigma^2}{4A^2MN\left(\text{sinc}\left(\frac{2\delta_1 + 1}{2}\right) + \text{sinc}\left(\frac{1 - 2\delta_1}{2}\right)\right)^2} \tag{27}$$

The variance of ω, $\text{var}(\hat{\omega})$, has the form of

$$\text{var}(\hat{\omega}) = \frac{M(1 + 4\delta^2)\sin^2\left(\frac{\pi(1 - 2\delta)}{2M}\right)\sin^2\left(\frac{\pi(1 + 2\delta)}{2M}\right)\cos^2\left(\frac{\pi\delta}{M}\right)}{4\pi^2\rho\cos^2(\pi\delta)\cos^2\left(\frac{\pi}{2M}\right)} \tag{28}$$

Similarly, the variance of ν, $\text{var}(\hat{\nu})$, has the form of

$$\text{var}(\hat{\nu}) = \frac{N(1 + 4\mu^2)\sin^2\left(\frac{\pi(1 - 2\mu)}{2N}\right)\sin^2\left(\frac{\pi(1 + 2\mu)}{2N}\right)\cos^2\left(\frac{\pi\mu}{N}\right)}{4\pi^2\rho\cos^2(\pi\mu)\cos^2\left(\frac{\pi}{2N}\right)} \tag{29}$$

4 Simulations

To verify the correctness of the interpolation formulas, computer simulations have been conducted. We employ the mean square error (MSE) of $\hat{\omega}$ and $\hat{\nu}$ as the performance metrics, which are defined as $E\{(\omega - \hat{\omega})^2\}$ and $E\{(\nu - \hat{\nu})^2\}$, respectively. The 2-D complex sinusoid is generated according to (1), while the corresponding parameters are $A = \sqrt{2}\exp(j0.5)$, $\omega = 0.967$ and $\nu = 2.48$. The Cramer-Rao lower bound (CRLB) [So and Chan (2006)] is included as the benchmark while comparisons with PUMA [So, Chan, Lau et al. (2010)] and ESPRIT [Sun and So (2004)] methods are also provided.
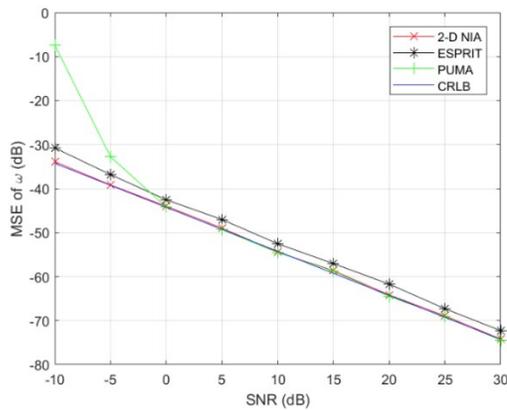
Figure 1: The MSE of $\hat{\omega}$ *vs.* SNR   **Figure 2:** The MSE of $\hat{\nu}$ *vs.* SNR
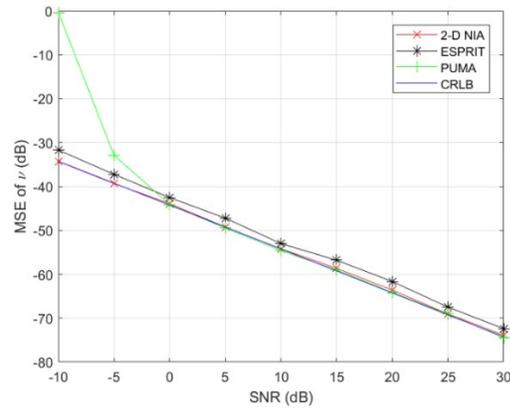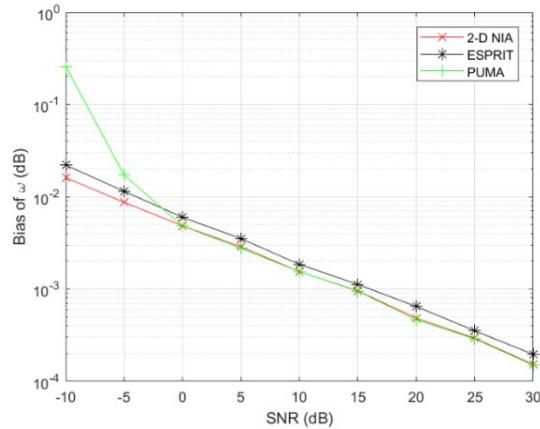
Figure 3: The bias of $\hat{\omega}$ *vs.* SNR

The stopping criterion of the PUMA algorithm is a fixed three iterations. All results are simulated using Inter(R)Xeon(R)CPUE5-1603 v3@2.8 GHz and based on 5000 Monte Carlo trials with a data length of $M = 20$ and $N = 20$.

First of all, we investigate the performance of the proposed methods in different noise conditions. The MSEs and biases of $\hat{\omega}$ and $\hat{\nu}$ versus SNR are plotted in Figs. 1-4. It is observed in Fig. 1 and Fig. 2 that the proposed method is superior to the other two estimators since it can attain CRLB fastest. Fig. 3 and Fig. 4 also verifies this result since our approach can provide stable estimates when SNR>-5 dB, but those of the other methods are SNR>0 dB.

Then the estimation performance and the computational cost versus the data length M are studied. Here all parameters are set to as the same with the previous experiment. The stopwatch timer is utilized to measure the operation times of all methods. It is indicated in Fig. 5 and Fig. 6 that our method can still provide a high estimation accuracy. Furthermore, it can be seen in Fig. 7 that in the case of nearly optimal estimation performance, the complexity of our approach is significantly lower than those of the PUMA and ESPRIT methods. It is worth to point out that in the case of varying N, the results are similar.

Third, the estimation performance for different δ and μ is examined with $L_1 = 3$, $L_2 = 8$ and the SNR is 10 dB. We vary δ when μ is fixed to -0.1, while in the case of varying $\mu, \delta = 0.08$ is selected. It is shown in Fig. 8 and Fig. 9 that the gap between the MSE of the proposed method and CRLB is smallest than the other two estimators, in all values of δ and μ.
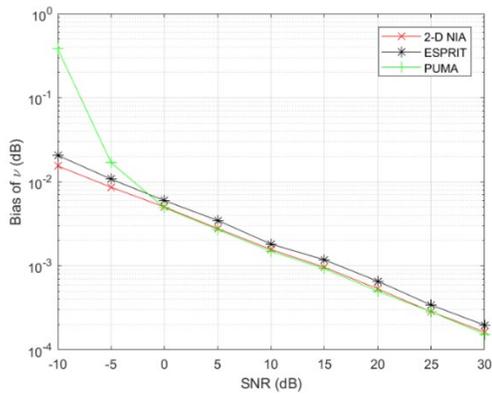
Figure 4: The bias of $\hat{\nu}$ *vs.* SNR
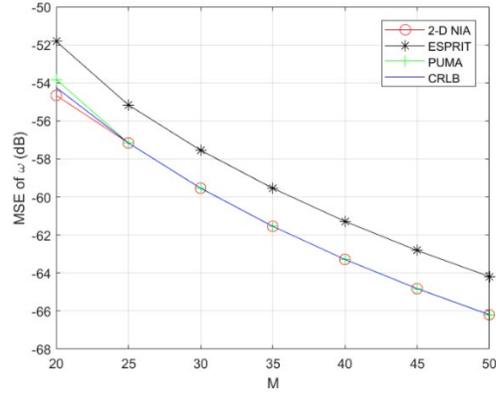
Figure 5: The bias of $\hat{\omega}$ *vs.* M
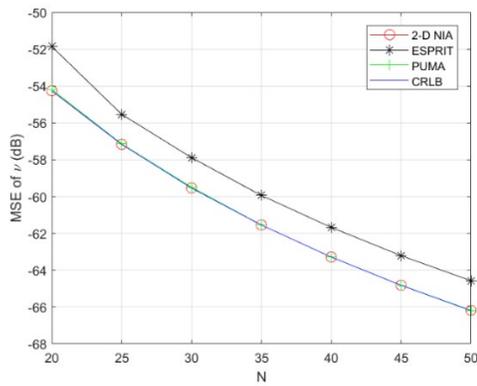
Figure 6: The bias of $\hat{\nu}$ *vs.* M
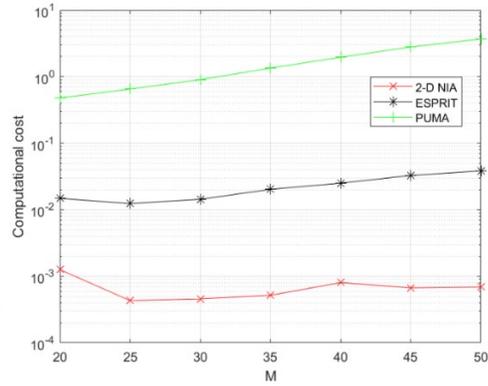
Figure 7: The computational cost *vs.* M

In summary, in the scenarios of different SNR and offsets, the MSEs of the proposed method attain CRLB, indicating the optimal performance. Meanwhile, compared with other estimators, our methods have the lowest complexity.
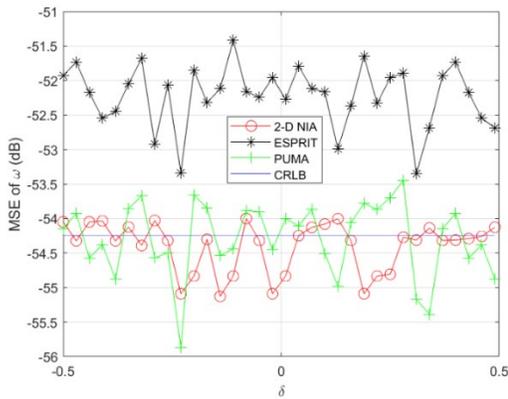
Figure 8: The MSE *vs.* δ
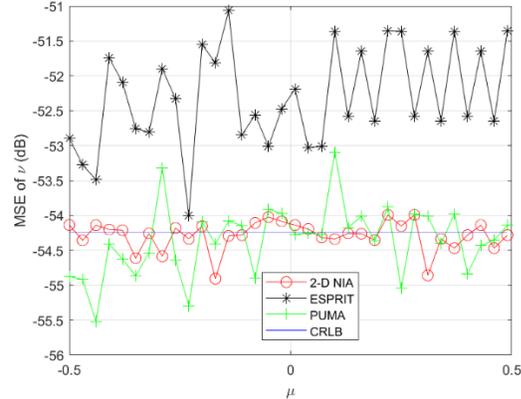
Figure 9: The MSE *vs.* μ

5 Conclusion

In this paper, an accurate frequency estimators of 2-D measurements using Fourier coefficients interpolation are proposed, which can attain lower complexity than that of existing methods. Computer simulations show that the proposed algorithms perform superior to PUMA and ESPRIT methods in terms of high accuracy and low complexity. Moreover, it is indicated that with the increasing observation data set, the computational complexity of our methods has the smaller rate of complexity than that of other methods, which can be applied in big data.

Conflicts of Interest: The authors declare that they have no conflicts of interest to report regarding the present study.

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