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Theoretical Study of the Energies of the Oscillating System with a Well-Distributed Mass of the Spring

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Abstract: The energy of a spring with a well-distributed mass m_s is theoretically studied in this paper. The solution of the wave equation is derived in detail, and then the kinetic energy and potential energy of the spring are studied with the wave equation, as well as the kinetic energy of the oscillating mass *M*. The kinetic energy and potential energy of the spring, and total energy are numerically simulated for different ratios m_s/M with considering the spring's mass, which makes the property of energy of the oscillating system understood easily.

Keywords: Spring; kinetic energy; potential energy of a spring; transcendental equation

1 Introduction

Conventional textbook such as college physics, the mass of a spring m_s is usually ignored in the study of the oscillation, which leads to an ideal model for describing the spring oscillator. For the practical considerations, however, the spring has mass, which is not negligible with respect to the mass M suspended at its end. The topics on an oscillating period and total energy of oscillating system have been studied for many years [1-5]. Worsnop et al have suggested that $m_s/3$ of the spring mass should be added to the mass *M* to calculate the oscillation period [6]. Galloni et al have reported that one may suppose the spring mass is null in which case a mass equal to $m_s/3$ must be added at its end to get the same energy [7,8]. However, the kinetic energy and potential energy of a spring in an oscillation system have not been investigated so far when the mass of a spring is taken into account. The aim of this paper is to analyze the influence of the mass of the spring on the simple harmonic motion of the spring mass system. Specifically, we derive a theoretical description of the kinetic and potential energies, and gain an insight into it with different spring-object mass ratios m_s/M.

The rest part of this paper is organized as follows. In the section II, we briefly describe the solution of the wave equation. The energy of an oscillation system is studied in detail in the Section III. Conclusions are given in the Section IV.

2 The Wave Equation and Dynamic Solution

As illustrated in Fig. 1, we consider a spring-object model, which consists of a spring with a well-distributed mass m_s and an object with the mass *M* at the right end of a spring. The length of the spring is *L* and the elastic constant is *k*. The object attached to at the right end of the spring is released from its equilibrium position. According to the Hooke's law and Newton's second law; we can get a wave equation.

$$u_{tt} - \omega_0^2 L u_{xx} = 0 \tag{1}$$

where $\omega_0^2 = \dfrac{k}{m_s}$. $\omega_0 L$ is the speed of a longitudinal wave. More details about derivation of the wave equation can be found in [1-3,8-11].

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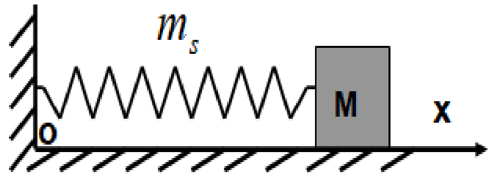

Figure 1: The schematic structure of oscillating system with a spring mass m_s and mass M

In order to get the solution of Eq. (1), the boundary and initial conditions of the wave equation should be considered. First, at the left end of the spring (*x = 0*), the spring is fixed, thus $u(0,t)=0$. Second, both law Hooke's law and Newton's second are applied at the right end of the spring *(x = L*). Then we can obtain an equation $-kLu_x(L,t)=Mu_{tt}(L,t)$, where the minus sign denotes leftward direction of the elastic force. Substituting Eq. (1) into it, we can obtain the other expression $u_x(L,t)=-\frac{ML}{m_s}u_{xx}(L,t)$. Third, as the system is stationary when *t=0*, in other words, the velocity of spring is equal to zero when *t=0*, therefore $u_t(x,0)=0$. In addition, the spring is stretched and the deformation of the spring is l_0 when *t=0*, thus the deformation of the spring at *x* from the left fixed end of the spring can be expressed as $u(x,0)=\frac{x}{L}l_0$.

Based on the above information, the boundary and initial conditions of the wave Eq. (1) are

$$\begin{cases} u(0,t)=0, u_x(L,t)=-\dfrac{ML}{m_s}u_{xx}(L,t) \\ u_t(x,0)=0, u(x,0)=\dfrac{x}{L}l_0 \end{cases} \tag{2}$$

The wave Eq. (1) can be solved using the method of separation of variables. We define $u(x,t)=X(x)T(t)$, and then substitute it into the wave Eq. (1), we can obtain the following equations

$$\begin{cases} X''+\dfrac{1}{\omega_0^2L^2}\omega^2X=0 \\ T''+\omega^2T=0 \end{cases} \tag{3}$$

where ω^2 is the separation constant. Taking the linear combination of them, the general solution of Eq. (2) can be written as

$$u(x,t)=(A_0x+B_0)(C_0t+D)+\sum_{n=1}^{\infty}[A_n\sin(\frac{\omega_nx}{\omega_0L})+B_n\cos(\frac{\omega_nx}{\omega_0L})][C_n\sin(\omega_nt)+D_n\cos(\omega_nt)] \tag{4}$$

where A_0, B_0, C_0, D_0, A_n, B_n, C_n and D_n are the constants and depend on the boundary and initial conditions, *n* is the order number of harmonics.

Both $u(0,t)=0$ and $u_x(x,0)=0$ are substituted into Eq. (4), the wave Eq. (4) now reduces to

$$u(x,t)=\sum_{n=1}^{\infty}A_n\sin(\frac{\omega_nx}{\omega_0L})\cos(\omega_nt) \tag{5}$$

By substituting boundary condition $u_x(L,t) = -\frac{ML}{m_s} u_{xx}(L,t)$ into Eq. (5), we get a transcendental equation

$$\cot(\frac{\omega_n}{\omega_0}) = \frac{M}{m_s} \cdot \frac{\omega_n}{\omega_0} \tag{6}$$

Fig. 2 gives the graphical solution of a transcendental equation, the solution of the transcendental equation is the intersection point of the curves $y_1 = \cot(\frac{\omega_n}{\omega_0})$ with the straight line $y_2 = \frac{M}{m_s} \cdot \frac{\omega_n}{\omega_0}$, which is shown in Fig. 2. Tab. 1 gives the solutions of the transcendental equation with different ratios $\frac{m_s}{M}$ for the fundamental oscillating order *n=1*. Tab. 1 shows that the solution of the transcendental equation $(\frac{\omega_1}{\omega_0})$ increases and tends to $\frac{\pi}{2}$ with increasing the ratio $\frac{m_s}{M}$.

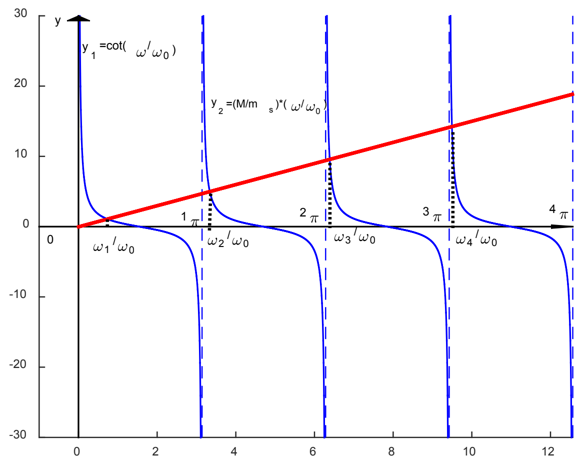

Figure 2: Graphical solution of the transcendental equation $\cot(\frac{\omega_n}{\omega_0}) = \frac{M}{m_s} \cdot \frac{\omega_n}{\omega_0}$

Table 1: Transcendental equation's solution with different ratios $(\frac{m_s}{M})$

$\frac{m_s}{M} =$	0.01	0.10	0.20	0.50	1.00	1.50	2.00	3.00	….	∞
$\frac{\omega_1}{\omega_0} =$	0.099834	0.31105	0.43284	0.65327	0.86033	0.98824	1.0769	1.1925	…	$\frac{\pi}{2}$

Now let us determine the coefficient A_n of Eq. (5). The coefficient A_n of Eq. (5) depends on boundary condition with Fourier series. The function of $\sin(\frac{\omega_n x}{\omega_0 L})$ is not an orthogonal function

$$\begin{cases} \int_0^L \sin^2(\frac{\omega_n x}{\omega_0 L})dx = \frac{L}{2} - \frac{\omega_0 L}{4\omega_n}\sin(\frac{2\omega_n}{\omega_0}) \\ \int_0^L \sin(\frac{\omega_n x}{\omega_0 L})\sin(\frac{\omega_m x}{\omega_0 L})dx = -\frac{ML}{m_s}\sin(\frac{\omega_n}{\omega_0})\sin(\frac{\omega_m}{\omega_0}); (n \neq m) \end{cases} \tag{7}$$

$\cos(\frac{\omega_n x}{\omega_0 L})$ is an orthogonal function but is non-normalized.

$$\begin{cases} \int_0^L \cos^2(\frac{\omega_n x}{\omega_0 L})dx = \frac{L}{2} + \frac{\omega_0 L}{4\omega_n}\sin(\frac{2\omega_n}{\omega_0}) \\ \int_0^L \cos(\frac{\omega_n x}{\omega_0 L})\cos(\frac{\omega_m x}{\omega_0 L})dx = 0; (n \neq m) \end{cases} \tag{8}$$

By using the orthogonality between the Fourier series of them, the coefficients A_n of Eq. (5) can be determined. To accomplish it, the initial condition $u(x,0) = \frac{x}{L} l_0$ is differentiated to *x* and substituted into Eq. (5), and then we get the following equation:

$$u_x(x,0) = \frac{l_0}{L} = \sum_{n=1}^{\infty} A_n \frac{\omega_n}{\omega_0 L}\cos(\frac{\omega_n x}{\omega_0 L}) \tag{9}$$

$\cos(\frac{\omega_n x}{\omega_0 L})$ is the orthogonal function but is non-normalized, both sides of Eq. (9) are multiplied by $\cos(\frac{\omega_m x}{\omega_0 L})$, and then is integrated with respect to dx in the domain $0 \le x \le L$. Thus we can get the equation

$$\int_0^L \frac{l_0}{L}\cos(\frac{\omega_m x}{\omega_0 L})dx = \int_0^L [\sum_{n=1}^{\infty} A_m \frac{\omega_m}{\omega_0 L}\cos(\frac{\omega_m x}{\omega_0 L})]\cos(\frac{\omega_m x}{\omega_0 L})dx = \int_0^L A_n \frac{\omega_n}{\omega_0 L}\cos^2(\frac{\omega_n x}{\omega_0 L})dx \tag{10}$$

$$\begin{aligned} A_n &= \frac{\omega_0 l_0}{\omega_n} \frac{\int_0^L \cos(\frac{\omega_n x}{\omega_0 L})dx}{\int_0^L \cos^2(\frac{\omega_n x}{\omega_0 L})dx} = \frac{\omega_0 l_0}{\omega_n} \frac{\frac{\omega_0}{\omega_n}\sin(\frac{\omega_n}{\omega_0})}{\frac{1}{2} + \frac{\omega_0}{4\omega_n}\sin(\frac{2\omega_n}{\omega_0})} = 2l_0(\frac{\omega_0}{\omega_n})^2 \frac{\sin(\frac{\omega_n}{\omega_0})}{1 + \frac{\omega_0}{\omega_n}\sin(\frac{\omega_n}{\omega_0})\cos(\frac{\omega_n}{\omega_0})} \\ &= 2l_0(\frac{\omega_0}{\omega_n})^2 \frac{\sin(\frac{\omega_n}{\omega_0})}{1 + \frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})} \end{aligned} \tag{11}$$

Transcendental Eq. (6) [$\cot(\frac{\omega_n}{\omega_0}) = \frac{M}{m_s} \cdot \frac{\omega_n}{\omega_0}$] is used in Eq. (11).

Initial condition $u(x,0) = \frac{x}{L} l_0$ and Eq. (11) are substituted into Eq. (5), then set *t=0* and *x=L*, we can get

$$u(L,0)=\frac{L}{L}l_0=l_0=\sum_{n=1}^{\infty}[A_n\sin(\frac{\omega_n L}{\omega_0 L})]=\sum_{n=1}^{\infty}[2l_0(\frac{\omega_0}{\omega_n})^2\frac{\sin(\frac{\omega_n}{\omega_0})}{1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})}\sin(\frac{\omega_n}{\omega_0})] \tag{12}$$

Thus we can reach an important conclusion based on Eq. (12).

$$\sum_{n=1}^{\infty}[(\frac{\omega_0}{\omega_n})^2\frac{\sin^2(\frac{\omega_n}{\omega_0})}{1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})}]=\frac{1}{2} \tag{13}$$

3 The Energy of an Oscillating System

Now we study the energy of an oscillating system, including kinetic energy and elastic potential energy of a spring, as well as the kinetic energy of an oscillating mass *M* [12]. For the kinetic energy of a spring E_{km}, each element Δx with mass $\frac{m_s}{L}\Delta x$ and velocity $u_t(x,t)$ has energy $\frac{1}{2}\frac{m_s}{L}u_t^2(x,t)\Delta x$. Take it for example, when $\Delta x = x_b - x_a$, the kinetic energy of the element Δx in the range $x_a \le x \le x_b$ can be expressed as

$$\begin{aligned}
\Delta E_{km} &= \int_{x_a}^{x_b}[\frac{1}{2}\frac{m_s}{L}u_t^2(x,t)dx] = \int_{x_a}^{x_b}\frac{1}{2}\frac{m_s}{L}[-\sum_{n=1}^{\infty}A_n\omega_n\sin(\frac{\omega_n x}{\omega_0 L})\sin(\omega_n t)]^2dx \\
&= \frac{1}{2}\frac{m_s}{L}\int_{x_a}^{x_b}[\sum_{n=1}^{\infty}A_n^2\omega_n^2\sin^2(\frac{\omega_n x}{\omega_0 L})\sin^2(\omega_n t)]dx \\
&+ \frac{1}{2}\frac{m_s}{L}\int_{x_a}^{x_b}[\sum_{n\neq m}^{\infty}A_nA_m\omega_n\omega_m\sin(\frac{\omega_n x}{\omega_0 L})\sin(\frac{\omega_m x}{\omega_0 L})\sin(\omega_n t)\sin(\omega_m t)]dx \\
&= \frac{1}{2}\frac{m_s}{L}\sum_{n=1}^{\infty}A_n^2\omega_n^2\sin^2(\omega_n t)\int_{x_a}^{x_b}[\sin^2(\frac{\omega_n x}{\omega_0 L})]dx \\
&+ \frac{1}{2}\frac{m_s}{L}\sum_{n\neq m}^{\infty}A_nA_m\omega_n\omega_m\sin(\omega_n t)\sin(\omega_m t)\int_{x_a}^{x_b}[\sin(\frac{\omega_n x}{\omega_0 L})\sin(\frac{\omega_m x}{\omega_0 L})]dx
\end{aligned} \tag{14}$$

In order to show the each element of kinetic energy of a spring (E_{km}) clearly, we set the elastic constant $k = 3.0$ *N/m* (which is measured in our experiment lab) and the mass of a spring $m_s = 0.03$ *kg*, $\Delta x = 0.1L$. The kinetic energy of each element Δx with different ratios m_s/M ($m_s/M = 0.1$, $m_s/M = 0.5$, $m_s/M = 1.0$, $m_s/M = 3.0$) and different times ($t = 0$, $t = T/8$, $t = 2T/8$, $t=3T/8$ and $t = 4T/8$) are given in Fig. 3, where the oscillating period is $T = 2\pi\sqrt{\frac{M + m_s/3}{k}}$ [5,6]. Fig. 3 shows that E_{km} enhances gradually from $t = 0$ to $t = 2T/8$ and reduces from $t = 2T/8$ to $t = 4T/8$. This can be explained as the oscillating system reaches its position of equilibrium when $t = 2T/8$. In addition, E_{km} is much stronger at the position $x = L$ than any other position, and E_{km} is stronger with increasing the spring mass m_s when $m_s \le M$. It is interesting to find that the biggest energy of E_{km} is at about $x = 0.4L$ when $m_s/M = 3.0$, in other words, the biggest energy appears in the other position when $m_s > M$ or m_s is bigger enough.

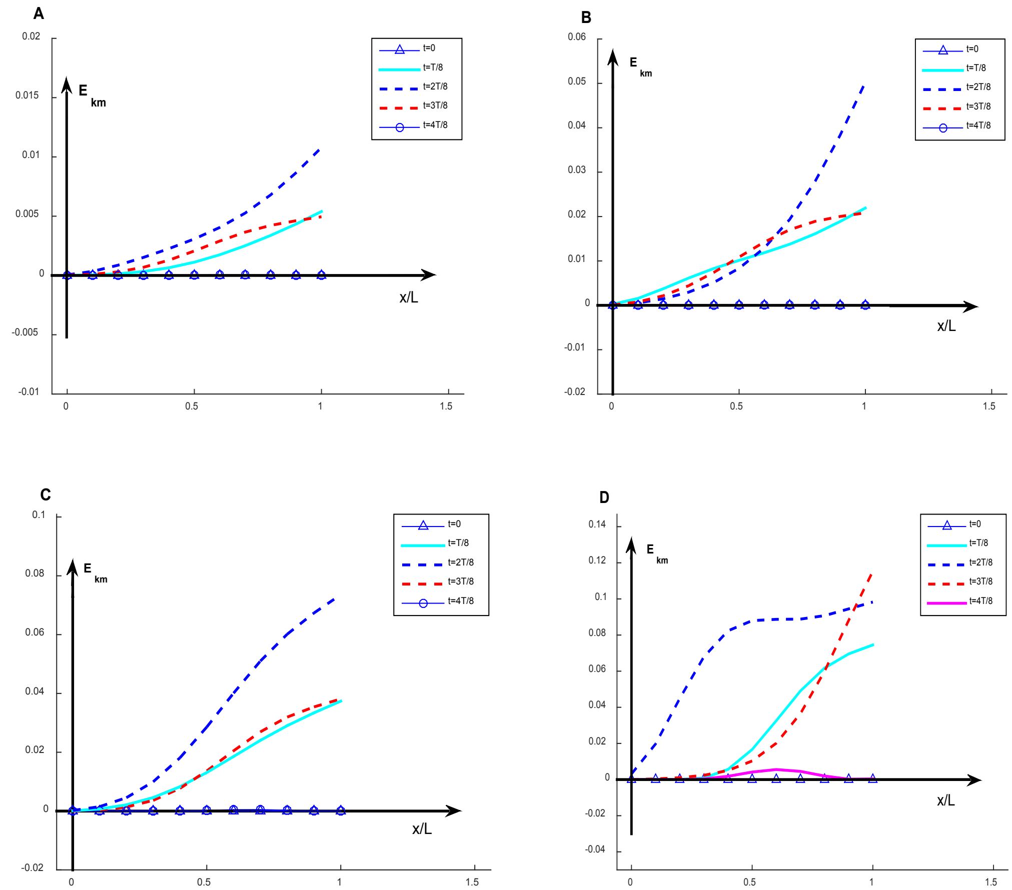

Figure 3: The kinetic energy of a spring E_{km} with different ratios m_s/M. A. $m_s/M = 0.1$, B. $m_s/M = 0.5$, C. $m_s/M = 1.0$, D. $m_s/M = 3.0$

In addition, potential energy of a spring of each element Δx is also studied, each element Δx has a potential energy $\Delta E_p = F(average)du = [\frac{F(x)}{2}]du = \frac{kL}{2}\frac{\partial u(x,t)}{\partial x}du = \frac{kL}{2}u_x^2(x,t)\Delta x$ [8], and then the potential energy of element Δx in the range $x_a \le x \le x_b$ is

$$
\begin{aligned}
\Delta E_p &= \int_{x_a}^{x_b} dE_p = \int_{x_a}^{x_b} \frac{kL}{2}u_x^2(x,t)dx = \int_{x_a}^{x_b}\frac{kL}{2}[\sum_{n=1}^{\infty} A_n \frac{\omega_n}{\omega_0 L}\cos(\frac{\omega_n x}{\omega_0 L})\cos(\omega_n t)]^2 dx \\
&= \frac{kL}{2}\int_{x_a}^{x_b}[\sum_{n=1}^{\infty} A_n^2 \frac{\omega_n^2}{\omega_0^2 L^2}\cos^2(\frac{\omega_n x}{\omega_0 L})\cos^2(\omega_n t)]dx + \\
&\quad \frac{kL}{2}\int_{x_a}^{x_b}[\sum_{n\neq m}^{\infty} A_n A_m \frac{\omega_n}{\omega_0 L}\frac{\omega_m}{\omega_0 L}\cos(\frac{\omega_n x}{\omega_0 L})\cos(\frac{\omega_m x}{\omega_0 L})\cos(\omega_n t)\cos(\omega_m t)]dx \\
&= \frac{m_s}{2L}\sum_{n=1}^{\infty} A_n^2\omega_n^2\cos^2(\omega_n t)\int_{x_a}^{x_b}[\cos^2(\frac{\omega_n x}{\omega_0 L})]dx + \\
&\quad \frac{m_s}{2L}\sum_{n\neq m}^{\infty} A_n A_m \omega_n \omega_m \cos(\omega_n t)\cos(\omega_m t)\int_{x_a}^{x_b}[\cos(\frac{\omega_n x}{\omega_0 L})\cos(\frac{\omega_m x}{\omega_0 L})]dx
\end{aligned}
\tag{15}
$$

Fig. 4 shows the potential energy of each element Δx with different ratios m_s/M ($m_s/M = 0.1$, $m_s/M = 0.5$, $m_s/M = 1.0$, $m_s/M = 3.0$) and times ($t = 0$, $t = T/8$, $t = 2T/8$, $t = 3T/8$ and $t = 4T/8$), where $\Delta x = x_b - x_a = 0.1L$. Compared with the kinetic energy of each element shown in Fig. 3, potential energy E_p of Fig. 4 exhibits the opposite property to kinetic energy E_{km}. Specifically, E_p gradually reduces from $t = 0$ to $t = 2T/8$ and enhances from $t = 2T/8$ to $t = 4T/8$. $t = 2T/8$ indicates the oscillating system at the equilibrium position, and the potential energy of a spring $E_p = 0$ [13]. In addition, E_p is bigger than E_{km} when $m_s \leq M$, in other words, the potential energy E_p plays more dominant role in total energy than kinetic energy E_{km}. However, kinetic energy E_{km} is equal to potential energy E_p when m_s is big enough, for example $m_s / M = 3.0$.

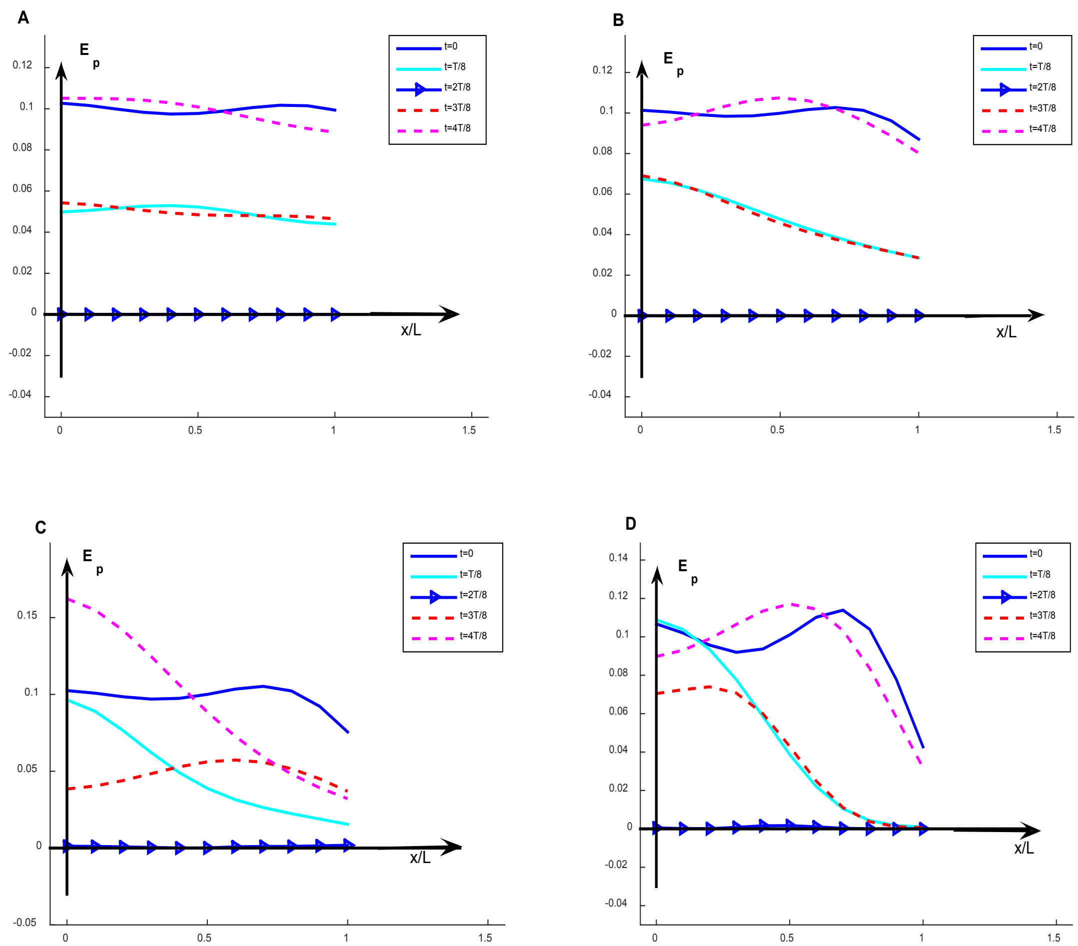

Figure 4: The potential energy of a spring E_p with different ratios m_s/M. A. $m_s/M = 0.1$, B. $m_s/M = 0.5$, C. $m_s/M = 1.0$, D. $m_s/M = 3.0$

Now let us consider the total kinetic energy of a spring, each element dx with mass $\frac{m_s}{L}dx$ and velocity $u_t(x,t)$ has an energy $\frac{1}{2}\frac{m_s}{L}u_t^2(x,t)dx$, and then the total kinetic energy of a spring E_{km} is integrated with respect to dx in the range $0 \leq x \leq L$.

$$
\begin{aligned}
E_{km} &= \int_0^L [\frac{1}{2}\frac{m_s}{L}u_t^2(x,t)dx] = \int_0^L \frac{1}{2}\frac{m_s}{L}[-\sum_{n=1}^{\infty} A_n\omega_n \sin(\frac{\omega_n x}{\omega_0 L})\sin(\omega_n t)]^2 dx \\
&= \frac{1}{2}\frac{m_s}{L}\int_0^L [\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n x}{\omega_0 L})\sin^2(\omega_n t)]dx \\
&+ \frac{1}{2}\frac{m_s}{L}\int_0^L [\sum_{n\neq m}^{\infty} A_n A_m\omega_n\omega_m \sin(\frac{\omega_n x}{\omega_0 L})\sin(\frac{\omega_m x}{\omega_0 L})\sin(\omega_n t)\sin(\omega_m t)]dx \\
&= \frac{1}{2}\frac{m_s}{L}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\omega_n t)\int_0^L \sin^2(\frac{\omega_n x}{\omega_0 L})dx \\
&+ \frac{1}{2}\frac{m_s}{L}\sum_{n\neq m}^{\infty} A_n A_m\omega_n\omega_m \sin(\omega_n t)\sin(\omega_m t)\int_0^L \sin(\frac{\omega_n x}{\omega_0 L})\sin(\frac{\omega_m x}{\omega_0 L})dx \\
&= \frac{m_s}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\omega_n t)[1-\frac{\omega_0}{2\omega_n}\cdot\sin(\frac{2\omega_n}{\omega_0})] \\
&- \frac{M}{2}\sum_{n\neq m}^{\infty} A_n A_m\omega_n\omega_m \sin(\omega_n t)\sin(\omega_m t)\sin(\frac{\omega_n}{\omega_0})\sin(\frac{\omega_m}{\omega_0})
\end{aligned}
\tag{16}
$$

Eq. (16) means that the total kinetic energy of a spring E_{km} is varied with the time, which can be seen in Fig. 5.

As for total potential energy of a spring, each element dx has an energy $dE_p = F(average)du = [\frac{F(x)}{2}]du = \frac{kL}{2}\frac{\partial u(x,t)}{\partial x}du = \frac{kL}{2}u_x^2(x,t)dx$, and then the total potential energy of spring E_p is integrated with respect to dx in the domain $0 \le x \le L$. The elastic potential energy of a spring E_p is

$$
\begin{aligned}
E_p &= \int_0^L dE_p = \int_0^L \frac{kL}{2}u_x^2(x,t)dx = \int_0^L \frac{kL}{2}[\sum_{n=1}^{\infty} A_n \frac{\omega_n}{\omega_0 L}\cos(\frac{\omega_n x}{\omega_0 L})\cos(\omega_n t)]^2 dx \\
&= \frac{kL}{2}\int_0^L [\sum_{n=1}^{\infty} A_n^2 \frac{\omega_n^2}{\omega_0^2 L^2}\cos^2(\frac{\omega_n x}{\omega_0 L})\cos^2(\omega_n t)]dx + \\
&\frac{kL}{2}\int_0^L [\sum_{n\neq m}^{\infty} A_n A_m \frac{\omega_n}{\omega_0 L}\frac{\omega_m}{\omega_0 L}\cos(\frac{\omega_n x}{\omega_0 L})\cos(\frac{\omega_m x}{\omega_0 L})\cos(\omega_n t)\cos(\omega_m t)]dx \\
&= \frac{m_s}{2L}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \cos^2(\omega_n t)\int_0^L [\cos^2(\frac{\omega_n x}{\omega_0 L})]dx + \\
&\frac{m_s}{2L}\sum_{n\neq m}^{\infty} A_n A_m\omega_n\omega_m \cos(\omega_n t)\cos(\omega_m t)\int_0^L [\cos(\frac{\omega_n x}{\omega_0 L})\cos(\frac{\omega_m x}{\omega_0 L})]dx \\
&= \frac{m_s}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \cos^2(\omega_n t)[1+\frac{\omega_0}{2\omega_n}\cdot\sin(\frac{2\omega_n}{\omega_0})]
\end{aligned}
\tag{17}
$$

Eq. (17) shows that the total potential energy of a spring E_p is related with the time, seen in Fig. 5.

The kinetic energy of mass M, E_{kM} is

$$E_{kM} = \frac{1}{2}M[u_t(L,t)]^2 = \frac{1}{2}M[A_n\omega_n \sin(\frac{\omega_n}{\omega_0})\sin(\omega_n t)]^2$$
$$= \frac{1}{2}M\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n}{\omega_0})\sin^2(\omega_n t) \tag{18}$$
$$+\frac{M}{2}\sum_{n\neq m}^{\infty} A_n A_m \omega_n \omega_m \sin(\frac{\omega_n}{\omega_0})\sin(\frac{\omega_m}{\omega_0})\sin(\omega_n t)\sin(\omega_m t)$$

And then the total energy of an oscillating system is

$$E = E_{km} + E_p + E_{kM}$$
$$= \frac{m_s}{4}\{\sum_{n=1}^{\infty} A_n^2\omega_n^2 + \sum_{n=1}^{\infty} A_n^2\omega_0\omega_n \sin(\frac{\omega_n}{\omega_0})\cos(\frac{\omega_n}{\omega_0})[\cos^2(\omega_n t) - \sin^2(\omega_n t)]\}$$
$$+\frac{1}{2}M\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n}{\omega_0})\sin^2(\omega_n t)$$
$$= \frac{m_s}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 + \frac{M}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n}{\omega_0})[\cos^2(\omega_n t) - \sin^2(\omega_n t)]$$
$$+\frac{1}{2}M\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n}{\omega_0})\sin^2(\omega_n t)$$
$$= \frac{m_s}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 + \frac{M}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2 \sin^2(\frac{\omega_n}{\omega_0}) = \frac{m_s}{4}\sum_{n=1}^{\infty} A_n^2\omega_n^2[1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})]$$
$$= \frac{m_s}{4}\sum_{n=1}^{\infty} 4l_0^2(\frac{\omega_0}{\omega_n})^4[\frac{\sin(\frac{\omega_n}{\omega_0})}{1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})}]^2\omega_n^2[1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})]$$
$$= kl_0^2\sum_{n=1}^{\infty}(\frac{\omega_0}{\omega_n})^2[\frac{\sin(\frac{\omega_n}{\omega_0})}{1+\frac{M}{m_s}\sin^2(\frac{\omega_n}{\omega_0})}]^2 = \frac{1}{2}kl_0^2 \tag{19}$$

Eq. (19) means the total energy of an oscillating system is constant.

In order to understand the energy of oscillating system fully, the kinetic energy of a spring (E_{km}) and potential energy of a spring E_p, kinetic energy of mass M (E_{kM}) and the total kinetic energy (E) are plotted with time for different ratios m_s/M in Fig. 5. The parameters of the elastic constant k and spring's mass m_s are the same as the above model, i.e., the elastic constant is $k = 3.0\ N/m$ and the mass of a spring is $m_s = 0.03\ kg$. Fig. 5(A) shows that the kinetic energy of the oscillator E_{kM} and the elastic potential energy E_p versus time without considering the mass of the spring ($m_s = 0$) [13]. We can find that kinetic energy E_{kM} and potential energy E_p can transform each other, whereas the total energy of the system E is conserved. That is, the elastic potential energy of the spring E_p becomes zero while the kinetic energy of mass E_{kM} reaches its maximum value. Fig. 5(B) shows the energy distribution of the spring system with the mass $M = 0.3\ kg$. When the elastic potential of the spring E_p reaches its maximum and equals to the total energy E, the kinetic energy of the oscillator E_{kM} and the spring E_{km} are both at their minimum, i.e., zero. In addition, the time-varying characteristic of the spring's kinetic energy E_{km} is the same as that of the oscillator E_{kM}. That is, they reach their maxima/minima at the same time. Although the kinetic energy of the spring E_{km}

is much smaller than the kinetic energy of the oscillator E_{kM}, the total energy of the system E is still conserved. The elastic potential energy E_p changes in Figs. 5(C) and 5(D) are the same as that of Figs. 5(A) and 5(B). The kinetic energies of the spring E_{km} and the oscillator E_{kM} reach their minimum at the same time. However, the kinetic energy of the oscillator E_{kM} gets its maximum before the spring E_{km}. From Figs. 5(E) and 5(F), we can find that the kinetic energies of the oscillator E_{kM} and the spring E_{km} increase periodically as the mass of the oscillator M decreases (or increases the ratio m_s/M), and the maximum of the spring kinetic energy E_{km} is greater than that of the oscillator E_{kM}, whereas the total energy of the system E is still conserved.

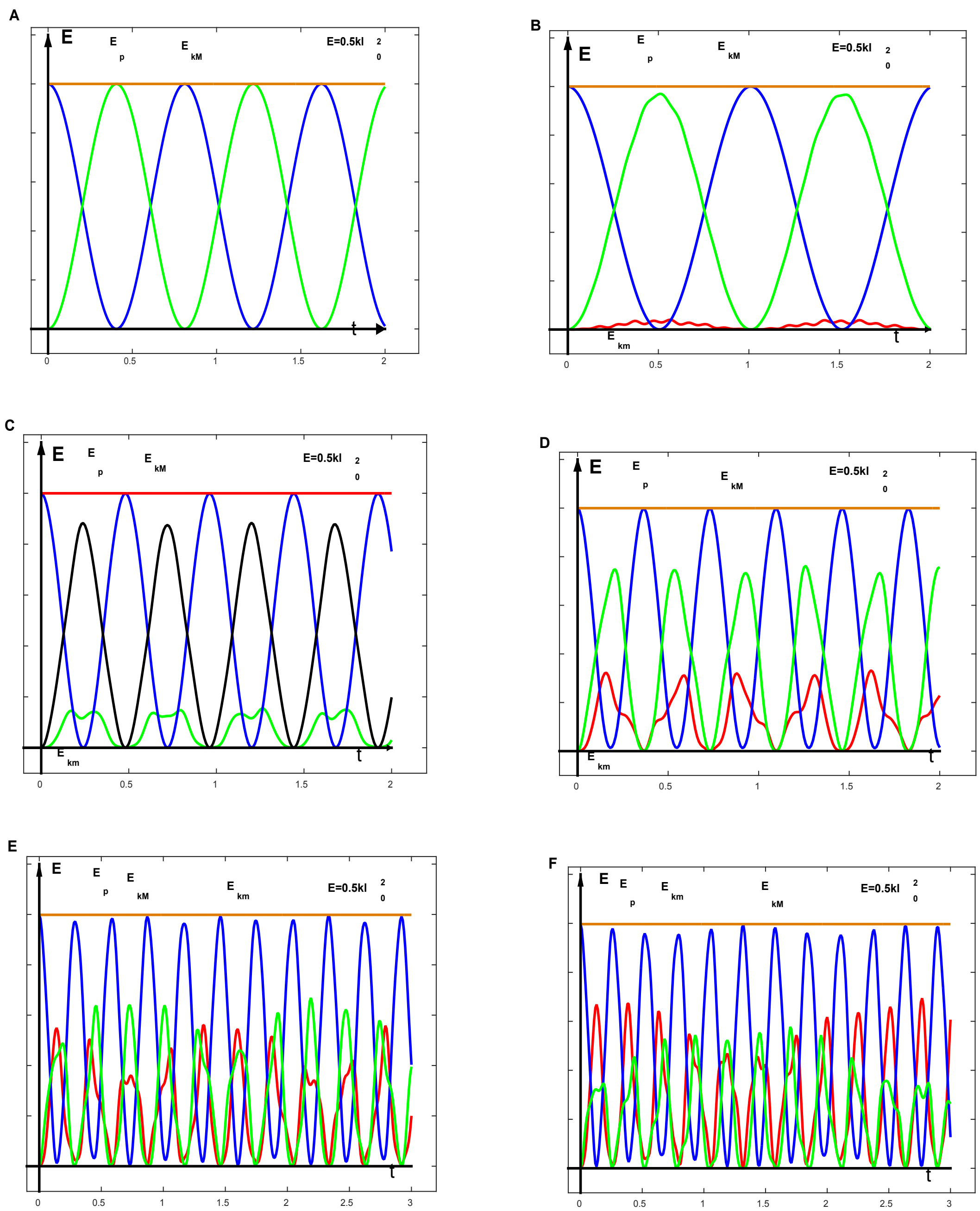

Figure 5: The total energy of a spring E with different ratios m_s/M. A. $m_s/M = 0.1$, B. $m_s/M = 0.5$, C. $m_s/M = 1.0$, D. m_s/M 3.0

4 Conclusions

In conclusion, this study theoretically investigates the kinetic energy and potential energy of a spring, and kinetic energy mass *M* with different ratios m_s/M. The kinetic energy and potential energy of a spring, and the kinetic energy mass *M* are numerically simulated. The kinetic energy and potential energy can transform each other, whereas the total energy of the system is conserved. The change behaviors of the spring's kinetic energy are the same as that of the oscillator energy with small ratio m_s/M, while kinetic energy of the oscillator gets its maximum before the spring kinetic energy for big ratio m_s/M. Moreover, the maximum of the spring kinetic energy is greater than that of the oscillator when the ratio m_s/M is big enough.

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