Computer Systems Science & Engineering
DOI:10.32604/csse.2021.015451
Article

Saddle Point Optimality Criteria of Interval Valued Non-Linear Programming Problem

Md Sadikur Rahman[1], Emad E. Mahmoud[2], Ali Akbar Shaikh[1,*], Abdel-Haleem Abdel-Aty[3,4] and Asoke Kumar Bhunia[1]

[1]Department of Mathematics, The University of Burdwan, Burdwan, 713104, India
[2]Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif, 21944, Saudi Arabia
[3]Department of Physics, College of Sciences, University of Bisha, P.O. Box 344, Bisha, 61922, Saudi Arabia
[4]Physics Department, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt
*Corresponding Author: Ali Akbar Shaikh. Email: aliashaikh@math.buruniv.ac.in
Received: 22 November 2020; Accepted: 17 February 2021

Abstract: The present paper aims to develop the Kuhn-Tucker and Fritz John criteria for saddle point optimality of interval-valued nonlinear programming problem. To achieve the study objective, we have proposed the definition of minimizer and maximizer of an interval-valued non-linear programming problem. Also, we have introduced the interval-valued Fritz-John and Kuhn Tucker saddle point problems. After that, we have established both the necessary and sufficient optimality conditions of an interval-valued non-linear minimization problem. Next, we have shown that both the saddle point conditions (Fritz-John and Kuhn-Tucker) are sufficient without any convexity requirements. Then with the convexity requirements, we have established that these saddle point optimality criteria are the necessary conditions for optimality of an interval-valued non-linear programming with real-valued constraints. Here, all the results are derived with the help of interval order relations. Finally, we illustrate all the results with the help of a numerical example.

Keywords: Convexity of interval valued function; extended Fritz-John theorem; Interval order relation; Karlin's constraint; saddle point optimality

1 Introduction

The optimality conditions of a constrained nonlinear programming problem with differentiability (especially, Karush-Kuhn-Tucker conditions) and without differentiability (Kuhn-Tucker and Fritz John optimality criteria) play important roles in the area of nonlinear programming. A few decades ago, these familiar results of optimization had been developed in the crisp environment. However, because of the fluctuation and the randomness of the parameters of a real-life decision-making problem, it has become a difficult task for the decision-makers to develop the optimality conditions of such decision-making problems, including optimization problems in which most of the parameters are imprecise. Thus, the study of optimality with or without differentiability of an imprecise optimization problem is an important research topic.

Depending upon the nature of different parameters of a real-life optimization problem, the following types are categorized

- Crisp optimization problem
- Fuzzy optimization problem
- Stochastic optimization problem
- Interval optimization problem

In a crisp optimization problem, the objective function and all the constraints are deterministic. The generalized form of a crisp optimization problem is

Find $\bar{x}(\in X)$ if it exists such that

$$f(\bar{x}) = \min_{x \in X} f(x),$$

where $X = \{x : x \in T,\ g_i(x) \leq 0,\ i = 1, 2, \ldots, m\}$

and $f, g_i : T(\subseteq \mathbb{R}^n) \to \mathbb{R}$

The equivalent saddle point problem of the above-mention minimization problem is Find $x^* \in T,\ s^* = \left(s_k^* : k = 1, 2, \ldots, l\right) \in \mathbb{R}^l,\ s_k^* \geq 0$,

if exist such that $\psi(x^*, s) \leq \psi(x^*, s^*) \leq \psi(x, s^*),\ s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l, s_k \geq 0,\ \forall x \in T$

where $\psi(x, s) = f(x) + \sum_{k=1}^{l} s_k g_k(x)$ and $f, g_k : T(\subseteq \mathbb{R}^n) \to \mathbb{R}$.

Here, the point (x^*, s^*) satisfying the above inequality is called the saddle point of $\psi(x, s)$. Using this saddle point criterion or differentiability assumption, several researchers established optimality criteria of a nonlinear optimization problem. In this area, Karush [1] derived optimality conditions of constrained nonlinear programming. A few years later, the same conditions were developed independently by Kuhn and Tucker [2]. From that time onwards, these conditions were as familiar as KKT conditions. However, a few years ago, using inequality constraint qualifications and saddle point criteria, John [3] developed the same in a different approach before Kuhn and Tucker.

In a fuzzy optimization problem, the objective function $f(x)$ and all the constraints $g_i(x)$ are considered either as fuzzy sets or fuzzy numbers and the inequality of the condition of saddle points is not an ordinary sign—it depends upon the ordering of fuzzy numbers. In this area, Bellman and Zadeh [4] first introduced the concept of fuzzy in the decision-making problem. Then, Delgado et al. [5] proposed the advancement of fuzzy optimization. On the other hand, Wu [6] introduced the saddle point optimality criteria of the fuzzy optimization problem. After that, Gong and Li [7] derived the same in the fuzzy optimization problem. Recently, Li et al. [8] and Bao and Bai [9] made their significant contributions to fuzzy nonlinear programming. In a stochastic optimization problem, the objective function $f(x)$and all the constraints $g_i(x)$ are taken as random variables with proper probability density functions and the inequality sign in the definition of the saddle point is dependent on the nature of random variables. Here, a number of researchers, including Nemirovski et al. [10] Chen et al. [11,12], Bedi et al. [13], Nemirovski and Rubinstein [14], and others contributed their works in non-linear stochastic programming.

Alternatively, if the parameters involved in a nonlinear programming problem are in interval form, then the objective function or constraints or both of the corresponding nonlinear programming problems are in interval form. Thus, a nonlinear programming problem in an interval environment is of the form:

Find $\bar{x} \in X$, if exists, such that

$$f(\bar{x}) = \left[\underline{f}(\bar{x}), \bar{f}(\bar{x})\right] = \langle f_c(\bar{x}), f_r(\bar{x})\rangle = \min_{x \in X}\left[\underline{f}(x),\ \bar{f}(x)\right] = \min_{x \in X}\langle f_c(x), f_r(x)\rangle,$$

where $X = \{x : x \in T,\ g_k(x) = \langle g_{kc}(x),\ g_{kr}(x)\rangle \leq^{\min} \langle 0, 0\rangle,\ k = 1, 2, \ldots, l\}$,
f, g_k are interval − valued function defined on $T(\subseteq \mathbb{R}^n)$
and f_c, g_{kc} and f_r, g_{kr} are centre and radius of f *and* g_i, respectively.

And the equivalent saddle point problem is

Find $\bar{x} \in T,\ s^* = (s_k^* : k = 1, 2, \ldots, l) \in \mathbb{R}^l,\ s_k^* \geq 0$,

if exist, such that

$$\psi(x^*, s) \leq^{\min} \psi(x^*, s^*) \leq^{\min} \psi(x, s^*),\ s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l, s_k \geq 0,\ \forall x \in T$$

where $\psi(x, s) = \langle \psi_c(x, s), \psi_r(x, s)\rangle = \left\langle f_c(x) + \sum_{k=1}^{l} s_k g_{kc}(x), f_r(x) + \sum_{k=1}^{l} s_k g_{kr}(x) \right\rangle$.

The inequality $\leq^{\min}$ involved in the above-mentioned problem is not the usual inequality sign. This inequality is dependent on an interval order relation. In this area, Wu [15] derived the KKT conditions of interval-valued non-linear programming problems. In his work, he introduced two different optimization techniques with the help of Ishibuchi and Tanaka [16] partial interval order relations. Recently, Rahman et al. [17] established the optimality conditions of nonlinear interval-valued programming using Bhunia and Samanta's [18] interval ranking. However, no one has derived the Saddle point optimality criteria for an interval-valued non-linear programming problem till now.

2 Research Gap and Contribution

In the existing literature, several researchers contributed their works on interval analysis (especially, interval ordering). Among them, Bhunia and Samanta [18] proposed a complete interval order relation. There are lots of applications of Bhunia and Samanta [18] order relation in the area of inventory management. Among those, the works of Shaikh and Bhunia [19], Shaikh et al. [20], Rahman et al. [21,22], … etc. are worth-mentioning. The above-mentioned works are the application of interval analyses in inventory control. To the best of our knowledge, no one can apply the interval technique in the other part of the optimization and operations research. The major of parameters of the real-life problems, especially optimization problems are imprecise due to uncertainty. Currently, the development of optimization theory in imprecise environments (Fuzzy, Stochastic, and Interval) has become a popular research topic. Hence, this topic has opened a new horizon in the world of mathematics. In this work, for the first time, the saddle point optimality criteria (like Extended Kuhn Tucker and Fritz-John) of interval-valued non-linear programming problems have been established.

This work is enhanced by introducing the concepts of interval order relations in derivative-free optimization. With the help of Bhunia and Samanta's [18] interval ranking, the definitions of the minimizer, maximizer, and some beautiful concepts of interval non-linear programming have been proposed. With these concepts, the Interval Fritz-John Saddle point problem and Interval Kuhn-Tucker Saddle point problem are defined. After that, the necessary and sufficient optimality criteria of those problems are derived. Finally, using these saddle optimality criteria, the optimality conditions of a non-linear programming problem have been established. These are the contributions of this work.

3 Some Basic Definitions and Results

In this section, we have mentioned Bhunia and Samanta's [18] interval order relations. Then, using these definitions of order relations, we have brought into the definitions of convexity, minimizer of an interval-valued function, and some simple results.

3.1 Interval Order Relations

The definitions of Bhunia and Samanta's [18] ordering, $\geq^{\max}$ and $\leq^{\min}$ between two intervals in $I(\mathbb{R})$ for both maximization and minimization problem are given below.

where, $I(\mathbb{R}) = \{[\bar{a}, \underline{a}] : \underline{a}, \bar{a} \in \mathbb{R}\, and\, \underline{a} \leq \bar{a}\}$

Definition 1. Let $C = [\underline{c}, \bar{c}] = \langle c_c, c_r \rangle$, B = $D = \left[\underline{d}, \bar{d}\right] = \langle d_c, d_r \rangle \in I(\mathbb{R})$.

Then, $C \geq^{\max} D \Leftrightarrow \begin{cases} c_c \geq d_c, \text{ if } c_c \neq d_c \\ c_r \leq d_r, \text{ if } c_c = d_c \end{cases}$ and $C >^{\max} D \Leftrightarrow C \geq^{\max} D \;\&\; C \neq D$

Definition 2.

$C \leq^{\min} D \Leftrightarrow \begin{cases} c_c \leq d_c, \text{ if } c_c \neq d_c \\ c_r \leq d_r, \text{ if } c_c = d_c \end{cases}$ and C $<^{\min}$ D $\Leftrightarrow C \leq^{\min} D \;\&\; C \neq D$

3.2 Minimizer and Convexity of an Interval-valued Function

Let $T \subseteq \mathbb{R}^n$ and $G : T \to I(\mathbb{R})$ be an interval valued function defined by $G(x) = \left[\underline{g}(x), \bar{g}(x)\right] = \langle g_c(x), g_r(x) \rangle$,

where $g_c(x) = \dfrac{\bar{g}(x) + \underline{g}(x)}{2}$, $g_r(x) = \dfrac{\bar{g}(x) - \underline{g}(x)}{2}$,

Definition 3. A point $x^* \in T$ is the local minimizer of the interval valued function $G(x)$ if $\exists\; a\; \delta > 0$ such that $\left[\underline{g}(x^*), \bar{g}(x^*)\right] \leq^{\min} \left[\underline{g}(x), \bar{g}(x)\right]$, $\forall x \in B(x^*, \delta) \cap T$,

where $B(x^*, \delta)$ is an open ball whose center is at x^* and radius δ.

Definition 4. A point $x^* \in T$ is a global minimizer of $G(x)$ if

$\left[\underline{g}(x^*), \bar{g}(x^*)\right] \leq^{\min} \left[\underline{g}(x), \bar{g}(x)\right]$, $\forall x \in T$.

Proposition 1. The point$x^* \in T$ is a local minimizer of $G(x)$ iff

$$\begin{cases} x^* \text{ is local minimizer of } g_c(x), \text{when } g_c(x) \neq \text{constant} \\ x^* \text{ is local minimizer of } g_r(x), \text{when } g_c(x) = \text{constant} \end{cases}$$

Proof. The proof is immediately followed from the definition of interval ordering.

Definition 5. The interval-valued function G is said to be c-r convex over a convex subset T if $G\,(\lambda x_1 + (1 - \lambda) x_2) \leq^{\min} \lambda\, G(x_1) + (1 - \lambda)\, G(x_2)$ for each $\lambda \in (0, 1)$ and $\forall x_1, x_2 \in T$.

Proposition 2. Let $T \subseteq \mathbb{R}^n$ be convex set and G be an interval valued function of the form $G(x) = \langle g_c(x), g_r(x) \rangle$. If g_c and g_r are convex, then $G(x)$ is c-r convex.

Proof. The proof follows from the definition of c-r convex and the $\leq^{\min}$ order relation.

Lemma 1. Let $A = [\underline{a}, \bar{a}] = \langle a_c, a_r \rangle$, $B = \left[\underline{b}, \bar{b}\right] = \langle b_c, b_r \rangle$ and $C = [\underline{c}, \bar{c}] = \langle c_c, c_r \rangle \in I(\mathbb{R})$.

Then, $A \leq^{\min} B \leq^{\min} C$ iff $\begin{cases} a_c \leq b_c \leq c_c \text{ if } a_c \neq b_c \neq c_c \\ a_c \leq b_c \text{ and } b_r \leq c_r \text{ if } a_c \neq b_c = c_c \\ a_r \leq b_r \text{ and } b_c \leq c_c \text{ if } a_c = b_c \neq c_c \\ a_r \leq b_r \leq c_r \text{ if } a_c = b_c = c_c \end{cases}$

Proof. The proof of this Lemma follows from the definitions of interval order relations.

4 The Interval-Valued Minimization and Saddle Point Problems

Here, we have introduced Interval-valued Minimization Problem (IMP), local interval-valued minimization problem, and interval-valued saddle points (Fritz-John and Kuhn-Tucker) problems respectively. Then, we have established the relation between their solutions.

Let $T \subseteq \mathbb{R}^n$ and $f,\ g_i : T \to I(\mathbb{R})$ is the interval-valued functions of the form:

$$F(x) = \left[\underline{f}(x),\ \bar{f}(x)\right] = \langle f_c(x),\ f_r(x)\rangle$$

$$G_i(x) = \left[\underline{g}_k(x),\ \bar{g}_k(x)\right] = \langle g_{kc}(x),\ g_{kr}\rangle,\ k = 1, 2, \ldots, l.$$

4.1 The Interval-Valued Minimization Problem (IMP)

(IMP)

Find $\bar{x} \in X$, if exists, such that

$$F(x^*) = \langle f_c(x^*), f_r(x^*)\rangle = \min_{x\in X} F(x) = \min_{x\in X}\ \langle f_c(x),\ f_r(x)\rangle,$$

where $X = \left\{x : x \in T,\ g_k(x) = \langle g_{kc}(x), g_{kr}(x)\rangle \leq^{\min} \langle 0, 0\rangle,\ k = 1, 2, \ldots, l\right\}$

The set X is called the feasible region, x^* is the solution and $F(x^*)$ is the minimum of the problem **IMP**.

4.2 The Local Interval-Valued Minimization Problem (LIMP)

(LIMP)

Find x^* in$X$, such that there exists some open ball $B(x^*, \delta)$ centre at x^* with radious $\delta > 0$ $x \in B(x^*,\ \delta) \cap X \Rightarrow F(x^*) \leq^{\min} F(x)$

4.3 The Interval-Valued Fritz John Saddle-Point Problem (IFJSP)

(IFJSP)

Find $x^* \in T,\ r_o^* \in \mathbb{R},\ r^* = \left(r_k^* : k = 1, 2, \ldots, l\right) \in \mathbb{R}^l,\ r_o^*, r_k^* \geq 0,$

If exist, such that

$$\pi\left(x^*, r_o^*, r\right) \leq^{\min} \pi\left(x^*, r_o^*, r^*\right) \leq^{\min} \pi\left(x, r_o^*, r^*\right),\ r = (r_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l, r_k \geq 0,\ \forall x \in T$$

where

$$\pi(x, r_o, r) = \ \langle \pi_c(x, r_o, r), \pi_r(x, r_o, r)\rangle = r_o F(x) + \sum_{k=1}^{l} r_k G_k(x)$$

$$= \left\langle r_o f_c(x) \sum_{k=1}^{l} r_k\ g_{kc}\ (x),\ r_o f_r\ (x) + \sum_{k=1}^{m} r_k\ g_{kr}\ (x) \right\rangle$$

4.4 The Interval-Valued Kuhn-Tucker Saddle-Point Problem (IKTSP)

(IKTSP)

Find $x^* \in T,\ s^* = \left(s_k^* : k = 1, 2, \ldots, l\right) \in \mathbb{R}^l,\ s_k^* \geq 0,$

if exist, such that

$$\psi(x^*, s) \leq^{\min} \psi(x^*, s^*) \leq^{\min} \psi(x, s^*),\ s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l, s_k \geq 0,\ \forall x \in T$$

where $\psi(x, s) = \langle \psi_c(x, s), \psi_r(x, s)\rangle = F(x) + \sum_{k=1}^{l} s_k G_k(x) = \left\langle f_c(x) + \sum_{k=1}^{l} s_k g_{kc}(x),\ f_r(x) + \sum_{k=1}^{l} s_k g_{kr}(x) \right\rangle$

Theorem 1.

If $\left(x^*, r_o^*, r^*\right)$ is the solution of **IFJSP** and $r_o^* > 0$, then $\left(x^*, r^*/r_o^*\right)$ is a solution of **IKTSP**. Conversely, if (x^*, s^*) is the solution of **IKTSP,** then $(x^*, 1, s^*)$ is the solution of **IFJSP.**

Proof.

First, let $\left(x^*, r_o^*, r^*\right)$ be a solution of **IFJSP,** then

$$\pi\left(x^*, r_o^*, r\right) \leq^{\min} \pi\left(x^*, r_o^*, r^*\right) \leq^{\min} \pi\left(x, r_o^*, r^*\right),\ r = (r_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l, r_k \geq 0,\ \forall x \in T$$

Now, by Lemma 1, four cases may arise:

$$\text{Case} - 1 : \pi_c\left(x^*, r_o^*, r\right) \neq \pi_c\left(x^*, r_o^*, r^*\right) \neq \pi_c\left(x, r_o^*, r^*\right)$$
$$\text{Case} - 2 : \pi_c\left(x^*, r_o^*, r\right) \neq \pi_c\left(x^*, r_o^*, r^*\right) = \pi_c\left(x, r_o^*, r^*\right)$$
$$\text{Case} - 3 : \pi_c\left(x^*, r_o^*, r\right) = \pi_c\left(x^*, r_o^*, r^*\right) \neq \pi_c\left(x, r_o^*, r^*\right)$$
$$\text{Case} - 4 : \pi_c\left(x^*, r_o^*, r\right) = \pi_c\left(x^*, r_o^*, r^*\right) = \pi_c\left(x, r_o^*, r^*\right)$$

Case-1 If $\pi_c\left(x^*, r_o^*, r\right) \neq \pi_c\left(x^*, r_o^*, r^*\right) \neq \pi_c\left(x, r_o^*, r^*\right)$,

then, $\pi_c\left(x^*, r_o^*, r\right) < \pi_c\left(x^*, r_o^*, r^*\right) < \pi_c\left(x, r_o^*, r^*\right)$

i.e.,

$$r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k g_{kc}(x^*) < r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_{kc}(x^*) < r_o^* f_c(x) + \sum_{k=1}^{l} r_k^* g_{kc}(x)$$

i.e.,

$$f_c(x^*) + \sum_{k=1}^{l} (r_k/r_o^*) g_{kc}(x^*) < f_c(x^*) + \sum_{k=1}^{l} (r_k/r_o^*) g_{kc}(x^*) < f_c(x) + \sum_{k=1}^{l} (r_k/r_o^*) g_{kc}(x)$$

i.e.,

$$\psi_c\left(x^*, r/r_o^*\right) < \psi_c\left(x^*, r^*/r_o^*\right) < \psi_c\left(x, r^*/r_o^*\right)\ \ i.e.,\ \psi\left(x^*, r/r_o^*\right) \leq^{\min} \psi\left(x^*, r^*/r_o^*\right) \leq^{\min}\ \psi_c\left(x, r^*/r_o^*\right)$$

Case-2 If $\pi_c\left(x^*, r_o^*, r\right) \neq \pi_c\left(x^*, r_o^*, r^*\right) = \pi_c\left(x, r_o^*, r^*\right)$,

then,

$$\pi_c\left(x^*, r_o^*, r\right) < \pi_c\left(x^*, r_o^*, r^*\right)\ \text{and}\ \ \pi_r\left(x^*, r_o^*, r^*\right) \leq \pi_r\left(x, r_o^*, r^*\right)$$

$$r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k g_{kc}(x^*) < r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_{kc}(x^*)\ \text{and}\ r_o^* f_r(x^*) + \sum_{k=1}^{l} r_k g_{kr}(x^*) \leq r_o^* f_r(x) + \sum_{k=1}^{l} r_k^* g_{kr}(x)$$

i.e.,

$$f_c(x^*) + \sum_{k=1}^{l} \left(r_k/r_o^*\right) g_{kc}(x^*) < f_c(x^*) + \sum_{k=1}^{l} \left(r_k/r_o^*\right) g_{kc}(x^*)$$

and

$$f_r(x^*) + \sum_{k=1}^{l} \left(r_k/r_o^*\right) g_{kr}(x^*) \leq f_r(x^*) + \sum_{k=1}^{l} \left(r_k/r_o^*\right) g_{kr}(x^*) \left[\text{since } r_o^* > 0\right]$$

i.e., $\psi_c\left(x^*, r_i/r_o^*\right) < \psi_c\left(x^*, r_i^*/r_o^*\right)$ and $\psi_r\left(x^*, r_i^*/r_o^*\right) \leq \psi_r\left(x, r_i^*/r_o^*\right)$

i.e., $\psi\left(x^*, r_i/r_o^*\right) \leq^{\min} \psi\left(x^*, r_i^*/r_o^*\right) \leq^{\min} \psi_r\left(x, r_i^*/r_o^*\right)$.

Case-3 If $\pi_c\left(x^*, r_o^*, r\right) = \pi_c\left(x^*, r_o^*, r^*\right) \neq \pi_c\left(x, r_o^*, r^*\right)$

then, similarly as **Case-2,** we have obtained $\psi\left(x^*, r_i/r_o^*\right) \leq^{\min} \psi\left(x^*, r_i^*/r_o^*\right) \leq^{\min} \psi_r\left(x, r_i^*/r_o^*\right)$.

Case-4 If $\pi_c(x^*, r_o^*, r) = \pi_c(x^*, r_o^*, r^*) = \pi_c(x, r_o^*, r^*)$, then similarly as **Case-1,** we get $\psi_r(x^*, r_i/r_o^*) \leq \psi_r(x^*, r_i^*/r_o^*) \leq \psi_r(x, r_i^*/r_o^*)$

i.e., $\psi(x^*, r_i/r_o^*) \leq^{\min} \psi(x^*, r_i^*/r_o^*) \leq^{\min} \psi_r(x, r_i^*/r_o^*)$.

Hence, combining all the cases first part of the theorem is proved.

Conversely, let (x^*, s^*) be a solution of **IKTSP.**

Then,

$$\psi(x^*, s) \leq^{\min} \psi(x^*, s^*) \leq^{\min} \psi(x, s^*),\ s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l,\ s_k \geq 0,\ \forall x \in T.$$

where $= 1.F(x) + \sum_{k=1}^{l} s_k G_k(x) = \left\langle 1.f_c(x) + \sum_{k=1}^{l} s_k g_{kc}(x), 1.f_r(x) + \sum_{k=1}^{l} s_k g_{kr}(x) \right\rangle = \pi(x, 1, s)$

Hence, $\pi(x^*, 1, s) \leq^{\min} \pi(x^*, 1, s^*) \leq^{\min} \pi(x, 1, s^*),\ s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l,\ s_k \geq 0,\ \forall x \in T$.

This completes the proof.

5 Optimality Conditions of IMP

5.1 Sufficient Optimality of IMP

The sufficient optimality criterion has been derived without convexity assumption of the interval minimization problem (IMP).

Theorem 2. If (x^*, s^*) is the solution of **IKTSP,** then x^* is a solution of **IMP.** If (x^*, s_o^*, s^*) is a solution of **IFJSP** and $s_o^* > 0$, then x^* is a solution of **IMP.**

Proof.

First Part.

Let (x^*, s^*) be a solution of **IKTSP**.

Then, $\forall s = (s_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l,\ s_k \geq 0,\ \forall x \in T,\ \ \psi(x^*, s) \leq^{\min} \psi(x^*, s^*) \leq^{\min} \psi(x, s^*)$. where $\psi(x, s) = \langle \psi_c(x, s), \psi_r(x, s) \rangle = F(x) + \sum_{k=1}^{l} s_k G_k(x) = \left\langle f_c(x) + \sum_{k=1}^{l} s_k g_{kc}(x),\ f_r(x) + \sum_{k=1}^{l} s_k g_{kr}(x) \right\rangle$.

Then, by Lemma 1., four cases may arise.

Case-1. If $\psi_c(x^*, s) \neq \psi_c(x^*, s^*) \neq \psi_c(x, s^*)$,

then $\psi_c(x^*, s) < \psi_c(x^*, s^*) < \psi_c(x, s^*)$, $\forall x \in T,\ s \in \mathbb{R}^l$, where, $\psi_c(x, s) = f_c(x) + \sum_{k=1}^{l} s_k g_{kc}(x)$. By the Sufficient Optimality Criteria for real-valued objective function, we can say that $f_c(x^*) < f_c(x)$, i.e., $F(x^*) \leq^{\min} F(x)$.

Case-2. If $\psi_c(x^*, s) \neq \psi_c(x^*, s^*) = \psi_c(x, s^*)$,

then $\psi_c(x^*,\ s) < \psi_c(x^*,\ s^*)$ and $\psi_r(x^*,\ s^*) \leq \psi_r(x,\ s^*)$, $\forall x \in T,\ s \in \mathbb{R}^l$.

Now, from first inequality, we have

$$f_c(x^*) + \sum_{k=1}^{l} s_k g_{kc}(x^*) < f_c(x^*) + \sum_{k=1}^{l} s_k^* g_{kc}(x^*) \tag{1}$$

$$\Rightarrow \sum_{k=1}^{l} (s_k - s_k^*) g_{kc}(x^*) < 0\ \ \forall s_k \geq 0,\ k = 1, 2, \ldots, l$$

Now, for any j, $1 \leq j \leq l$, let $s_k = s_k^*$, $i = 1, 2, \ldots, j-1, j+1, \ldots, m$, $s_j = s^*{}_j + 1$

Which gives $g_{jc}(x^*) < 0$. Repeating this $\forall k$, we get $g_{kc}(x^*) < 0$.

$$\text{Now, since, } s_k^* \geq 0, \text{ and } g_{kc}(x^*) < 0, \ \sum_{k=1}^{l} s_k^* g_{kc}(x^*) \leq 0 \tag{2}$$

But, again from (1) by setting $s_k = 0$,we obtain

$$\begin{aligned} & f_c(x^*) < f_c(x^*) + \sum_{k=1}^{l} s_k^* g_{kc}(x^*) \\ & or, \ \sum_{k=1}^{l} s_k^* g_{kc}(x^*) > 0 \end{aligned} \tag{3}$$

Hence, from (2) and (3), we have $\sum_{k=1}^{l} s_k^* g_{kc}(x^*) = 0$

Now, from $\psi_c(x^*, s^*) = \psi_c(x, s^*)$, we get

$$f_c(x^*) + \sum_{k=1}^{l} s_k^* g_{kc}(x^*) = f_c(x) + \sum_{k=1}^{l} s_k^* g_{kc}(x)$$

$$i.e., \ f_c(x^*) = f_c(x) + \sum_{k=1}^{l} s_k^* g_{kc}(x) \quad \left[\because \sum_{k=1}^{l} s_k^* g_{kc}(x) = 0\right], \ \forall x \in T$$

which is possible only if $s_k^* = 0, \forall k = 1, 2, \ldots, l$ and $f_c(x)$ is constant function.

So, in this case $f_c(x^*) = f_c(x)$.

Thus, from $\psi_r(x^*, u^*) \leq \psi_r(x, u^*)$, we get

$$f_r(x^*) + \sum_{k=1}^{l} s_k^* g_{kr}(x^*) \leq f_r(x) + \sum_{k=1}^{l} s_k^* g_{kr}(x)$$

$$i.e., \ f_r(x^*) \leq f_r(x), \ [\because s_k^* = 0]$$

Hence, $F(x^*) \leq^{\min} F(x)$.

Case-3 If $\psi_c(\bar{x}, u) = \psi_c(\bar{x}, \bar{u}) \neq \psi_c(x, \bar{u})$

then,

$\psi_r(x^*, s) \leq \psi_r(x^*, s^*)$ and $\psi_c(x^*, s^*) < \psi_c(x, s^*)$, $\forall x \in T, \ s \in \mathbb{R}^l$.

From $\psi_c(x^*, s) = \psi_c(x^*, s^*)$, we get $s_k = 0, \ \forall k = 1, 2, \ldots, l$

and from $\psi_c(x^*, s^*) lt \psi_c(x, s^*)$, we get

$$f_c(x^*) + \sum_{k=1}^{l} s_k^* g_{kc}(x^*) lt f_c(x^*) + \sum_{k=1}^{l} s_k g_{kc}(x), \ \forall s_k$$

$$\Rightarrow f_c(x^*) + \sum_{k=1}^{l} s_k^* g_{kc}(x^*) lt f_c(x) + \sum_{k=1}^{l} s_k^* g_{kc}(x), \text{ for } s_k = s_k^*$$

$$\Rightarrow f_c(x^*) lt f_c(x) \ [\because s_k^* = 0]$$

$$i.e., \ F(x^*) \leq^{\min} F(x)$$

Case-4. If $\psi_c(x^*, s) = \psi_c(x^*, s^*) = \psi_c(x, s^*)$,

Then, $\psi_r(x^*, s) \leq \psi_r(x^*, s^*) \leq \psi_r(x, s^*)$, $\forall x \in T$, $s = (s_k : k = 1, 2, \ldots, l)$, $s_k \geq 0$

Similar to case-1, we can say that $F(x^*) \leq^{\min} F(x)$.

Combining all the cases, the proof of the first part completes.

Second Part. The proof of this part follows from Theorem 1. and First part of this theorem.

5.2 Extended Fritz-John Saddle-Point Optimality Theorem

Here, we have derived the conditions for which the solution of **IMP** will be necessarily the solution of **IFJSP**. For this purpose, we have stated and proved Extended Fritz-John saddle point necessary optimality theorem. Before stating the theorem, we have stated the following Lemma (Mangasarian [23]):

Lemma 2. Let $T(\neq \phi) \subseteq \mathbb{R}^n$. Also, let f_1, f_2 and f_3 be $m_1,\ m_2,\ m_3$ dimensional convex vector-valued function on T and $g_k\ (k = 1, 2, \ldots, l)$ be convex functions on T.

If $\left\langle \begin{matrix} f_1(x) < 0,\ f_2(x) \leq 0,\ f_3(x) \leq 0 \\ g_k(x) \leq 0,\ \ k = 1, 2, \ldots, l \end{matrix} \right\rangle$ has no solution, $x \in T$ then there exist $p_1 \in \mathbb{R}^{m_1}$, $p_2 \in \mathbb{R}^{m_2}$, $p_3 \in \mathbb{R}^{m_3}$ and $q = (q_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l$

such that

$$\sum_{i=1}^{m_1} p_{1i} f_{1i}(x) + \sum_{i=1}^{m_2} p_{2i} f_{2i}(x) + \sum_{i=1}^{m_3} p_{3i} f_{3i}\ (x) + \sum_{k=1}^{l} q_k g_k(x) \geq 0,\ \forall x \in T \text{ and } p_{1i}, p_{2i}, p_{3i} \geq 0$$

where $f_j = \left(f_{ji} : i = 1, 2, \ldots, m_j, j = 1, 2, 3\right)$, $p_j = \left(p_{ji} : i = 1, 2, \ldots, m_j\right)$

Theorem 3. Let $T \subseteq \mathbb{R}^n$ be a non-empty convex set, f be interval-valued *c-r* convex function on T and $g_k\ (k = 1, 2, \ldots, l)$ be real-valued convex functions on T. If x^* is a solution of **IMP,** then $\left(x^*, r_o^*, r^*\right)$ $\left(r_o^* \in \mathbb{R}, r^* = \left(r_k^* : k = 1, 2, \ldots, l\right), r_o^* \geq 0,\ r_k^* \geq 0\right)$ is a solution of **IFJSP** and $\sum_{k=1}^{l} r_k^* g_k(x^*) = 0$, where $F(x) = \langle f_c(x), f_r(x) \rangle$.

Proof. Since x^* is a solution of **MP,** then

$F(x^*) \leq^{\min} F(x),\ \forall x \in T.$

i.e.,

either, $f_c(x^*) < f_c(x)$ if $f_c(x^*) \neq f_c(x)$

or, $f_r(x^*) \leq f_r(x)$ if $f_c(x^*) = f_c(x)$.

Now, two cases may arise:

Case-1.

If $f_c(x^*) \neq f_c(x)$, then

$\left\langle \begin{matrix} f_c(x) - f_c(x^*) < 0 \\ g_k(x) < 0,\ \ k = 1, 2, \ldots, l \end{matrix} \right\rangle$ has no solution $\forall x \in T$.

By Lemma 2, there exist $r_o^* \in \mathbb{R}$, $r = (r_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l$, $r_o^* \geq 0$, $r_k \geq 0$ such that

$$r_o^*[f_c(x) - f_c(x^*)] + \sum_{k=1}^{l} r_k^* g_k(x) \geq 0 \ \forall x \in T. \tag{4}$$

$$\text{Now, putting } x = x^* \text{ in (4), we have} \sum_{k=1}^{l} r_k^* g_k(x^*) \geq 0 \tag{5}$$

$$\text{But, since } r_k \geq 0,\ g_k(x^*) < 0, \text{ we have} \sum_{k=1}^{l} r_k^* g_k(x^*) \leq 0 \tag{6}$$

Hence from (5) and (6), we have $\sum_{k=1}^{l} r_k^* g_k(x^*) = 0$

Again from (4), we have

$$r_o^*[f_c(x) - f_c(x^*)] + \sum_{k=1}^{l} r_o^* g_k(x) \geq 0$$

$$or,\ r_o^* f_c(x^*) \leq r_o^* f_c(x) + \sum_{k=1}^{l} r_o^* g_k(x)$$

$$or,\ r_o^* f_c(x^*) + \sum_{k=1}^{l} r_o^* g_k(x^*) \ \leq\ r_o^* f_c(x) + \sum_{k=1}^{l} r_o^* g_k(x) \tag{7}$$

$$\begin{aligned} &\text{As } g_k(x^*) \leq 0, \text{ then} \sum_{k=1}^{l} r_k g_k(x^*) \leq 0 \ \forall r = (r_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l,\ r_k \geq 0. \\ &\text{Hence, } r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k g_k(x^*) \leq r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_k(x^*) \ \left[\because \sum_{k=1}^{l} r_k^* g_k(x^*) = 0\right] \end{aligned} \tag{8}$$

Therefore, from (7) and (8), we obtain

$$r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k g_k(x^*) \leq r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_k(x^*) \ \leq r_o^* f_c(x) + \sum_{k=1}^{l} r_k^* g_k(x)$$

Case-2. If $f_c(x^*) = f_c(x)$,

then

$f_r(x^*) \leq f_r(x)$ i.e., $f_r(x^*) \leq f_r(x^*) \leq f_r(x)$

i.e., $r_o^* f_r(x^*) + \sum_{k=1}^{l} r_k.0 \leq r_o^* f_r(x^*) + \sum_{k=1}^{l} r_k^*.0 \leq r_o^* f_r(x) + \sum_{k=1}^{l} r_k^*.0$

Combining both cases, we have obtained

$$\begin{aligned} r_o^* \langle f_c(x^*),\ f_r(x^*) \rangle + \sum_{k=1}^{l} r_k \langle g_k(x^*), 0 \rangle &\leq^{\min} r_o^* \langle f_c(x^*), f_r(x^*) \rangle + \sum_{k=1}^{l} r_k^* \langle g_k(x^*), 0 \rangle \\ &\leq^{\min} r_o^* \langle f_c(x), f_r(x) \rangle + \sum_{k=1}^{l} r_k^* \langle g_k(x), 0 \rangle \end{aligned}$$

Hence, the proof is complete.

5.3 Extended Kuhn-Tucker Saddle-Point Optimality Theorem

Here, we have derived the necessary conditions (Extended Kuhn-Tucker saddle point optimality) for which the solution of (IMP) will be necessarily the solution of (IKTSP). Before stating this theorem, we have first stated Karlin's constraint qualification which will be required as a hypothesis of this theorem:

Karlin's Type Constraint Qualification

Let $T \subseteq \mathbb{R}^n$ be non-empty convex set and $g = (g_k : k = 1, 2, \ldots, l)$ l-dimensional convex vector-valued function on T. Then, g is said to satisfy constraint qualification of Karlin $(\text{on}\, T)$ if there exists no $p \in \mathbb{R}^l$, $p = (p_k : k = 1, 2, \ldots, l)$, $p_k \geq 0$ such that $\sum_{k=1}^{l} p_k g_k(x) \geq 0$, $\forall x \in T$.

Theorem 4. Let T be convex set in $\mathbb{R}^n$, $F(x) = \langle f_c(x), f_r(x) \rangle$ be interval-valued *c-r* convex function defined on T and $g = (g_k : k = 1, 2, \ldots, l)$ be vector-valued function which satisfies Karlin's constraints qualification on T. If x^* is the solution of **IMP,** then (x^*, s^*) $\left(s_o^* \in \mathbb{R},\ s = (s_k : k = 1, 2, \ldots, l),\ s_o^* \geq 0,\ s_i \geq 0\right)$ is a solution of **IKTSP**.

Proof. Since $g = (g_k : k = 1, 2, \ldots, l)$ satisfies the constraint qualification of Karlin, there exists no $p \in \mathbb{R}^l$, $p = (p_k : k = 1, 2, \ldots l)$, $p_k \geq 0$ such that $\sum_{k=1}^{l} p_k g_k(x) \geq 0$, $\forall x \in T$.

Also, since x^* is a solution of **MP,**

$$F(x^*) \leq^{\min} F(x),\ \forall x \in T.$$
$$i.e.,\ f_c(x^*) < f_c(x) \text{ if } f_c(x^*) \neq f_c(x)$$
$$f_r(x^*) \leq f_r(x) \text{ if } f_c(x^*) = f_c(x).$$

Here, two cases may arise:

Case-1.

$$\text{if } f_c(x^*) \neq f_c(x),\ \text{ then}$$
$$\left\langle \begin{array}{l} f_c(x) - f_c(x^*) < 0 \\ g_k(x) < 0,\ k = 1, 2, \ldots, l \end{array} \right\rangle \text{ has no solution } \forall x \in T.$$

Then, there exist $r_o^* \in \mathbb{R}$, $r = (r_k : k = 1, 2, \ldots, l) \in \mathbb{R}^l$, $r_o^* \geq 0$, $r_i \geq 0$ such that

$$r_o^*[f_c(x) - f_c(x^*)] + \sum_{k=1}^{l} r_o^* g_k(x) \geq 0\ \forall x \in T. \tag{9}$$

Similar to Case-1 of Theorem 3, we obtain

$$\begin{array}{l} \sum_{k=1}^{l} r_o^* g_k(x^*) = 0 \ \text{ and} \\ r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k g_k(x^*) \leq r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_i(x^*)\ \leq \bar{r}_o f_c(x) + \sum_{i=1}^{m} \bar{r}_i g_i(x) \end{array} \tag{10}$$

Let $r_o^* = 0$, then $r_i \geq 0$, [from (9)]

Now, from the second inequality of (10) we get

$$r_o^* f_c(x^*) + \sum_{k=1}^{l} r_k^* g_k(x^*) \le r_o^* f_c(x) + \sum_{k=1}^{l} r_k^* g_k(x)$$

$$or,\ 0 \le 0 + \sum_{k=1}^{l} r_k^* g_k(x) \ \left[\because r_o^* = 0 \text{ and } \sum_{k=1}^{l} r_k^* g_k(x^*) = 0\right]$$

$$or,\ \sum_{k=1}^{l} r_k^* g_k(x) \ge 0,\ \forall x \in T.$$

which is a contradiction (According to Karlin's constraint qualification). Hence, $r_o^* > 0$.

Now, from (10), we have

$$f_c(x^*) + \sum_{k=1}^{l} (r_k/r_o^*) g_k(x^*) \le f_c(x^*) + \sum_{k=1}^{l} (r_k^*/r_o^*) g_k(x^*) \le f_c(x) + \sum_{k=1}^{l} (r_k^*/r_o^*) g_k(x)$$

$$i.e., f_c(x^*) + \sum_{k=1}^{l} s_k g_k(x^*) \le f_c(x^*) + \sum_{k=1}^{l} s_k^* g_k(x^*) \le f_c(x) + \sum_{k=1}^{l} s_k^* g_k(x), \text{ where } \mathrm{s_k} = r_k/r_o^*$$

Case-2. If $f_c(x^*) = f_c(x)$,

then

$$f_r(x^*) \le f_r(x) \ \ i.e., f_r(x^*) \le f_r(x^*) \le f_r(x)$$

$$i.e., f_r(x^*) + \sum_{k=1}^{l} s_k.0 \le f_r(x^*) + \sum_{k=1}^{l} s_k^*.0 \le f_r(x) + \sum_{k=1}^{l} s_k^*.0$$

Combining both cases, we have

$$\langle f_c(x^*), f_r(x^*)\rangle + \sum_{k=1}^{l} s_k \langle g_k(x^*), 0\rangle \le^{\min} \langle f_c(x^*), f_r(x^*)\rangle + \sum_{k=1}^{l} s_k^* \langle g_k(x^*), 0\rangle$$

$$\le^{\min} \langle f_c(x), f_r(x)\rangle + \sum_{k=1}^{l} s_k^* \langle g_k(x), 0\rangle$$

Hence, the proof is completed.

6 Numerical Example

To illustrate the saddle point optimality criteria, we have considered the following simple example:

Find $\bar{x} \in X = \{x \in \mathbb{R} : -x + 3 \le 0\}$, such that $f(\bar{x}) = \min_{x \in X} f(x)$,

where $f(x) = \left[-\left(x^2 + 1\right), 3x^2 + 1\right]$

Solution. $f_c(x) = x^2 \ne$ constant. Hence, minimizers of f_c and f are the same.

Clearly, $\bar{x} = 3$ is the minimizer of $f_c(x)$, and so that of $f(x)$.

Therefore, the minimum value of $f(x)$ is $[-10, 28]$

Now, the saddle point optimality criterion for this problem is that:

A necessary and sufficient condition that $\bar{x} = 3$ is that there exists a real number $\bar{u}$ such that

$$\phi(\bar{x}, u) \leq^{\min} \phi(\bar{x}, \bar{u}) \leq^{\min} \phi(x, \bar{u}), \forall x \in \mathbb{R} \text{ and } \forall u \in \mathbb{R},\ u \geq 0 \tag{11}$$

where $\phi(x, u) = \left[-\left(x^2 + 1\right), 3x^2 + 1\right] + u(-x + 3)$

Clearly, for $\bar{x} = 3,\ \bar{u} = 6,\ \phi(\bar{x}, u) = \phi(\bar{x}, \bar{u})$ and $\phi_c(\bar{x}, \bar{u}) \leq \phi_c(x, \bar{u})$,

then, interval inequality (11) holds for $\bar{x} = 3,\ \bar{u} = 6$.

Hence, $\phi(x, u)$ has saddle point at $\bar{x} = 3,\ \bar{u} = 6$.

7 Conclusion

In this paper, the derived saddle point (Fritz-John & Kuhn-Tucker) optimality criteria of interval-valued non-linear programming are called Extended Fritz-John and Extended Kuhn-Tucker saddle point criteria. Furthermore, we have shown that the Extended saddle point criteria are the sufficient conditions, so the point $\bar{x} \in X$ is the minimizer of the **IMP**. After considering all constraints of the **IMP** as real-valued convex functions and Karlin's constraint qualification, we illustrated that Extended Fritz-John and Extended Kuhn-Tucker Type saddle point criteria are also necessary conditions. For these purposes, the paper has introduced the definition of the minimizer, convexity of an interval-valued function, as well as Interval-valued Fritz-John and Interval-valued Kuhn-Tucker saddle point problems. Here, all the results have been established without differentiability assumptions of the objective function and constraints. Thus, these saddle point optimality criteria are called optimality criteria without differentiability. The concepts of this work will help to solve imprecise real-life problems like inventory control, supply chain management, problems of game theory,… etc.

For future work, one may attempt to establish the duality theory of **IMP**, saddle point optimality criteria of an interval optimization problem with several objective functions. One may also attempt to extend the concept of this paper in fuzzy, Type-2 fuzzy, and Type-2 interval environment [24].

Acknowledgement: The authors acknowledge Taif University Researchers Supporting Project Number (TURSP-2020/20), Taif University, Taif, Saudi Arabia. All authors are thankful to the anonymous reviewers for their valuable comments and suggestions that improved this manuscript.

Funding Statement: Taif University Researchers Supporting Project number (TURSP-2020/20), Taif University, Taif, Saudi Arabia.

Conflicts of Interest: The authors declare that they have no conflicts of interest regarding the present study.

References

[1] W. Karush, "Minima of functions of several variables with inequalities as side constraints," Master's thesis. Dept. of Mathematics, Univ. of Chicago, 1939.

[2] H. W. Kuhn and A. W. Tucker, "Nonlinear Programming," *Traces and Emergence of Nonlinear Programming*. Birkhäuser, Basel, pp. 247–258, 2014.

[3] F. John, "Extremum Problem with inequalities as subsidiary condition," in *Studies and Essays, Courant Anniversary*, K. O. Friedrichs, O. E. Neugebauer and J. J. Stoker, eds. New York: Wiley—Inter science, pp. 187–204, 1948.

[4] R. E. Bellman and L. A. Zadeh, "Decision making in a fuzzy environment," *Management Science*, vol. 17B, no. 4, pp. 141–164, 1970.

[5] M. Delgado, F. Herrera, J. L. Verdegay and M. A. Vila, "Post-optimality analysis on the membership functions of a fuzzy linear programming problem," *Fuzzy Sets and Systems*, vol. 53, no. 3, pp. 289–297, 1993.

[6] H. C. Wu, "Saddle point optimality conditions in fuzzy optimization problems," *Fuzzy Optimization and Decision Making*, vol. 2, no. 3, pp. 261–273, 2003.

[7] Z. T. Gong and H. X. Li, "Saddle point optimality conditions in fuzzy optimization problems," *Advances in Intelligent and Soft Computing*, vol. 54, pp. 7–14, 2009.

[8] H. X. Li, Y. J. Bai and Z. T. Gong, "The convexity of n-dimensional fuzzy mappings and the saddle point conditions of the fuzzy optimization problems," *Journal of Computational Analysis & Applications*, vol. 25, no. 8, pp. 1480–1489, 2018.

[9] Y. E. Bao and E. D. Bai, "Optimality conditions for fuzzy programming problems with differentiable fuzzy objective mappings," *Journal of Information Science & Engineering*, vol. 35, no. 6, pp. 1311–1327, 2019.

[10] A. Nemirovski, A. Juditsky, G. Lan and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1574–1609, 2009.

[11] P. Chen, A. Quarteroni and G. Rozza, "Stochastic optimal Robin boundary control problems of advection-dominated elliptic equations," *SIAM Journal on Numerical Analysis*, vol. 51, no. 5, pp. 2700–2722, 2013.

[12] Y. Chen, G. Lan and Y. Ouyang, "Optimal primal-dual methods for a class of saddle point problems," *SIAM Journal on Optimization*, vol. 24, no. 4, pp. 1779–1814, 2014.

[13] A. S. Bedi, A. Koppel and K. Rajawat, "Asynchronous saddle point algorithm for stochastic optimization in heterogeneous networks," *IEEE Transactions on Signal Processing*, vol. 67, no. 7, pp. 1742–1757, 2019.

[14] A. Nemirovski and R. Y. Rubinstein, "An efficient stochastic approximation algorithm for stochastic saddle point problems," In: M. Dror, P. L'Ecuyer and F. Szidarovszky (eds.), *Modeling Uncertainty. Int. Series in Operations Research & Management Science*. Springer, New York, NY: Springer, vol. 46, 2002.

[15] H. C. Wu, "The Karush–Kuhn–Tucker optimality conditions in an optimization problem with interval-valued objective function," *European Journal of Operational Research*, vol. 176, no. 1, pp. 46–59, 2007.

[16] H. Ishibuchi and H. Tanaka, "Multi-objective programming in optimization of the interval objective function," *European Journal of Operational Research*, vol. 48, no. 2, pp. 219–225, 1990.

[17] M. S. Rahman, A. A. Shaikh and A. K. Bhunia, "Necessary and sufficient optimality conditions for non-linear unconstrained and constrained optimization problem with interval valued objective function," *Computers & Industrial Engineering*, vol. 147, no. 1, pp. 106634, 2020.

[18] A. K. Bhunia and S. S. Samanta, "A study of interval metric and its application in multi-objective optimization with interval objectives," *Computers & Industrial Engineering*, vol. 74, no. 1, pp. 169–178, 2014.

[19] A. K. Bhunia and A. A. Shaikh, "Investigation of two-warehouse inventory problems in interval environment under inflation via particle swarm optimization," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 22, no. 2, pp. 160–179, 2016.

[20] A. A. Shaikh, S. C. Das, A. K. Bhunia, G. C. Panda and M. A. A. Khan, "A two-warehouse EOQ model with interval-valued inventory cost and advance payment for deteriorating item under particle swarm optimization," *Soft Computing*, vol. 23, no. 24, pp. 13531–13546, 2019.

[21] M. S. Rahman, A. Duary, A. A. Shaikh and A. K. Bhunia, "An application of parametric approach for interval differential equation in inventory model for deteriorating items with selling-price-dependent demand," *Neural Computing and Applications*, vol. 32, no. 17, pp. 14069–140857, 2020.

[22] M. S. Rahman, A. K. Manna, A. A. Shaikh and A. K. Bhunia, "An application of interval differential equation on a production inventory model with interval-valued demand via center-radius optimization technique and particle swarm optimization," *International Journal of Intelligent Systems*, vol. 35, no. 8, pp. 1280–1326, 2020.

[23] O. L. Mangasarian, *Nonlinear programming*. New York: McGraw-Hill, 1969, ISBN 0-89871-341-2.

[24] M. S. Rahman, A. A. Shaikh and A. K. Bhunia, "On Type-2 interval with interval mathematics and order relations: Its applications in inventory control," *International Journal of Systems Science: Operations & Logistics*, pp. 1–13, 2020. DOI 10.1080/23302674.2020.1754499.