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ARTICLE

Some Formulas Involving Hypergeometric Functions in Four Variables

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ABSTRACT

Several (generalized) hypergeometric functions and a variety of their extensions have been presented and investigated in the literature by many authors. In the present paper, we investigate four new hypergeometric functions in four variables and then establish several recursion formulas for these new functions. Also, some interesting particular cases and consequences of our results are discussed.

KEYWORDS

Recursion formula; quadruple hypergeometric functions; pascal; identity

AMS Subject Classification: 15A15; 33C65

1 Introduction

In recent years, many researchers introduced and studied several extensions and generalizations of various special functions due to its applications in diverse areas of mathematical, physical, engineering, etc. Agarwal et al. [1,2] established some properties for generalized Gauss hypergeometric functions, which were introduced by Özergin et al. Later, Agarwal et al. [3] and Çetinkaya et al. [4] introduced and investigated further extensions of Appell's hypergeometric functions of two variables and Lauricella's hypergeometric functions of three variables by using the generalized Beta type function. Purohit et al. [5] investigated Chebyshev type inequalities involving fractional integral operator containing a multi-index Mittag-Leffler function in the kernel. Suthar et al. [6] introduced certain generalized forms of the fractional kinetic equation pertaining to the (p, q)-Mathieu-type power series using the Laplace transforms technique. Chandola et al. [7] defined a new extension of beta function using the Appell series and the Lauricella function. The interested reader may be referred to several recent papers on the subject (see, e.g., [8–11] and the references cited therein).

Hypergeometric functions in several variables have many applications in applied problems (see, e.g., [12–16]). Also, multidimensional hypergeometric functions are used to solve boundary value problems (Dirichlet problem, Neumann problem, Holmgren problem, etc) for multidimensional degenerate differential equations (see [17–19]). In [20], Exton defined twenty one complete hypergeometric functions in four variables denoted by the symbols $K_1, K_2, \ldots, K_{21}$. In [21], Sharma et al. introduced eighty three complete quadruple hypergeometric functions, namely $F_1^{(4)}, F_2^{(4)}, \ldots, F_{83}^{(4)}$. Very recently, Younis et al. [22] introduced and studied further quadruple hypergeometric functions denoted by $X_{85}^{(4)}, X_{86}^{(4)}, \ldots, X_{90}^{(4)}$. Each quadruple hypergeometric function in [20–22] is of the form:

$$X^{(4)}(.) = \sum_{m,n,p,q=0}^{\infty} \Delta(m,n,p,q) \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$

where $\Delta(m, n, p, q)$ is a certain sequence of complex parameters and there are twelve parameters in each series $X^{(4)}(.)$ (eight $a's$ and four $c's$). The 1st, 2nd, 3rd and 4th parameters in $X^{(4)}(.)$ are connected with the integers m, n, p and q, respectively. Each repeated parameter in the series $X^{(4)}(.)$ points out a term with double parameters in $\delta(m, n, p, q)$. For example, $X^{(4)}(\sigma_1,\sigma_1,\sigma_2,\sigma_2,\sigma_3,\sigma_3,\sigma_4,\sigma_5)$ mean that $(\sigma_1)_{m+n}(\sigma_2)_{p+q}$ $(\sigma_3)_{m+n}(\sigma_4)_p(\sigma_5)_q$ includes the term. Similarly, $X^{(4)}(\sigma_1,\sigma_1,\sigma_1,\sigma_2,\sigma_1,\sigma_1,\sigma_2,\sigma_3)$ points out the term $(\sigma_1)_{2m+2n+p}$ $(\sigma_2)_{p+q}$ $(\sigma_3)_q$ and $X^{(4)}(\sigma_1, \sigma_1, \sigma_2, \sigma_4, \sigma_1, \sigma_2, \sigma_3, \sigma_5)$ shows the existence of the term $(\sigma_1)_{2m+n}(\sigma_2)_{n+p}(\sigma_3)_p(\sigma_4)_q(\sigma_5)_q$. Thus, it is possible to form various combinations of indices. There seems to be no way of establishing independently the number of distinct Gaussian hypergeometric series for any given integer $n \geq 2$ without stating explicitly all such series. Thus, in every situation with $n = 4$, one ought to begin by actually constructing the set just as in the case $n = 3$ (see [23]). Motivated by the works [20–22], we decide to define further hypergeometric functions in four variables as follows:

$$\begin{aligned} &X_{91}^{(4)}(\sigma_1,\sigma_1,\sigma_2,\sigma_4,\sigma_1,\sigma_2,\sigma_3,\sigma_5;\rho_1,\rho_1,\rho_2,\rho_1;x,y,z,u) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(\sigma_1)_{2m+n}(\sigma_2)_{n+p}(\sigma_3)_p(\sigma_4)_q(\sigma_5)_q}{(\rho_1)_{m+n+p}(\rho_2)_q} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \\ &\quad \left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right); \end{aligned} \tag{1}$$

$$\begin{aligned} &X_{92}^{(4)}(\sigma_1,\sigma_1,\sigma_2,\sigma_4,\sigma_1,\sigma_2,\sigma_3,\sigma_5;\rho_1,\rho_2,\rho_1,\rho_1;x,y,z,u) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(\sigma_1)_{2m+n}(\sigma_2)_{n+p}(\sigma_3)_p(\sigma_4)_q(\sigma_5)_q}{(\rho_1)_{m+p+q}(\rho_2)_n} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \\ &\quad \left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right); \end{aligned} \tag{2}$$

$$\begin{aligned} &X_{93}^{(4)}(\sigma_1,\sigma_1,\sigma_2,\sigma_4,\sigma_1,\sigma_2,\sigma_3,\sigma_5;\rho_2,\rho_1,\rho_1,\rho_1;x,y,z,u) \\ &\quad = \sum_{m,n,p,q=0}^{\infty} \frac{(\sigma_1)_{2m+n}(\sigma_2)_{n+p}(\sigma_3)_p(\sigma_4)_q(\sigma_5)_q}{(\rho_1)_{n+p+q}(\rho_2)_m} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!}, \\ &\quad \left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right); \end{aligned} \tag{3}$$

$$X_{94}^{(4)}(\sigma_1, \sigma_1, \sigma_2, \sigma_4, \sigma_1, \sigma_2, \sigma_3, \sigma_5; c, c, c, c; x, y, z, u)$$
$$= \sum_{m,n,p,q=0}^{\infty} \frac{(\sigma_1)_{2m+n}(\sigma_2)_{n+p}(\sigma_3)_p(\sigma_4)_q(\sigma_5)_q}{(c)_{m+n+p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{u^q}{q!},$$
$$\left(|x| < \frac{1}{4}, |y| < 1, |z| < 1, |u| < 1\right), \tag{4}$$

where

$$(a)_m := \frac{\Gamma(a+m)}{\Gamma(a)}, (a+m \in \mathbb{C}\backslash\mathbb{Z}_0^-)$$
$$= \begin{cases} 1 & (m=0) \\ a(a+1)\ldots(a+m-1) & (m=n\in\mathbb{N}). \end{cases}$$

Here, $\mathbb{C}, \mathbb{Z}_0^-$ and $\mathbb{N}$ denote the sets of complex numbers, non-positive integers, and positive integers, respectively.

Recently, many authors have obtained several recursion formulas involving hypergeometric functions in several variables. In Opps et al. [24], introduced the recursion formulas for the Appell's function F_2 and gave its applications to radiation field problems. Wang [25] presented the recursion formulas for Appell functions F_1, F_2, F_3 and F_4. Sahai et al. [26,27] established the recursion formulas for Lauricella's triple functions, Srivastava hypergeometric functions in three variables, k-variable Lauricella functions and the Srivastava-Daoust and related multivariable hypergeometric functions. Shehata et al. [28] discussed and derived new recursion relations for the Horn's hypergeometric functions. In this present paper, we aim to establish several recursion formulas for the new hypergeometric functions in four variables defined by (1.1)–(1.4).

The following abbreviated notations are used in this paper. We, for example, write $X_{91}^{(4)}$ for the series $X_{91}^{(4)}(\sigma_1, \sigma_1, \sigma_2, \sigma_4, \sigma_1, \sigma_2, \sigma_3, \sigma_5; \rho_1, \rho_1, \rho_2, \rho_1; x, y, z, u)$ and $X_{91}^{(4)}(\sigma_1 + n)$ for $X_{91}^{(4)}(\sigma_1 + n, \sigma_1 + n, \sigma_2, \sigma_4, \sigma_1, \sigma_2, \sigma_3, \sigma_5; \rho_1, \rho_1, \rho_2, \rho_1; x, y, z, u)$. The notation $X_{91}^{(4)}(\sigma_1 + n, \sigma_2 + n_1)$ stands for $X_{91}^{(4)}(\sigma_1 + n, \sigma_1 + n, \sigma_2 + n_1, \sigma_4, \sigma_1 + n, \sigma_2 + n_1, \sigma_3, \sigma_5; \rho_1, \rho_1, \rho_2, \rho_1; x, y, z, u)$ and $X_{91}^{(4)}(\sigma_1 + n, \sigma_2 + n_1, \rho_1 + n_2)$ stands for $X_{91}^{(4)}(a_1 + n, \sigma_1 + n, \sigma_2 + n_1, \sigma_4, \sigma_1 + n, \sigma_2 + n_1, \sigma_3, \sigma_5; \rho_1 + n_2, \rho_1 + n_2, \rho_2, \rho_1; x, y, z, u)$, etc.

2 Main Results

Here, we establish several recursion formulas for our hypergeometric functions in four variables.

Theorem 2.1 The following recursion formulas hold true for the numerator parameter σ_1, σ_2, σ_3, σ_4, σ_5 of the $X_{91}^{(4)}$:

$$X_{91}^{(4)}(\sigma_1 + n) = X_{91}^{(4)} + \frac{2x}{\rho_1} \sum_{n_1=1}^{n} (\sigma_1 + n_1) X_{91}^{(4)}(\sigma_1 + 1 + n_1, \rho_1 + 1)$$
$$+ \tfrac{y\sigma_2}{\rho_1} \sum_{n_1=1}^{n} X_{91}^{(4)}(\sigma_1 + n_1, \sigma_2 + 1, \rho_1 + 1), \tag{5}$$

$$X_{91}^{(4)}(\sigma_1 - n) = X_{91}^{(4)} - \frac{2x}{\rho_1}\sum_{n_1=1}^{n}(\sigma_1+1-n_1)X_{91}^{(4)}(\sigma_1+2-n_1, \rho_1+1)$$
$$-\frac{y\sigma_2}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_1+1-n_1, \sigma_2+1, \rho_1+1), \tag{6}$$

$$X_{91}^{(4)}(\sigma_2 + n) = X_{91}^{(4)} + \frac{y\sigma_1}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_1+1, \sigma_2+n_1, \rho_1+1)$$
$$+\frac{z\sigma_3}{\rho_2}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_2+n_1, \sigma_3+1, \rho_2+1), \tag{7}$$

$$X_{91}^{(4)}(\sigma_2 - n) = X_{91}^{(4)} - \frac{y\sigma_1}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_1+1, \sigma_2+1-n_1, \rho_1+1)$$
$$-\frac{z\sigma_3}{\rho_2}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_2+1-n_1, \sigma_3+1, \rho_2+1), \tag{8}$$

$$X_{91}^{(4)}(\sigma_3 + n) = X_{91}^{(4)} + \frac{z\sigma_2}{\rho_2}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_2+1, \sigma_3+n_1, \rho_2+1), \tag{9}$$

$$X_{91}^{(4)}(\sigma_3 - n) = X_{91}^{(4)} - \frac{z\sigma_2}{\rho_2}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_2+1, \sigma_3+1-n_1, \rho_2+1), \tag{10}$$

$$X_{91}^{(4)}(\sigma_4 + n) = X_{91}^{(4)} + \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_5+1, \sigma_4+n_1, \rho_1+1), \tag{11}$$

$$X_{91}^{(4)}(\sigma_4 - n) = X_{91}^{(4)} - \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_5+1, \sigma_4+1-n_1, \rho_1+1), \tag{12}$$

$$X_{91}^{(4)}(\sigma_5 + n) = X_{91}^{(4)} + \frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_4+1, \sigma_5+n_1, \rho_1+1), \tag{13}$$

$$X_{91}^{(4)}(\sigma_5 - n) = X_{91}^{(4)} - \frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{91}^{(4)}(\sigma_4+1, \sigma_5+1-n_1, \rho_1+1). \tag{14}$$

Proof. From the definition of the hypergeometric function $X_{91}^{(4)}$ and the relation

$$(\sigma_1+1)_{2m+n} = (\sigma_1)_{2m+n}(1+\frac{2m}{\sigma_1}+\frac{n}{\sigma_1}) \tag{15}$$

we obtain the following contiguous relation:

$$X_{91}^{(4)}(\sigma_1+1) = X_{91}^{(4)} + \frac{2x}{\rho_1}(\sigma_1+1)\,X_{91}^{(4)}(\sigma_1+2,\,\rho_1+1) + \frac{y\sigma_2}{\rho_1}X_{91}^{(4)}(\sigma_1+1,\sigma_2+1,\rho_1+1). \tag{16}$$

To find a contiguous relation for $X_{91}^{(4)}(\sigma_1+2)$, we replace σ_1 by σ_1+1 in (16) and simplify. This leads to:

$$\begin{aligned}X_{91}^{(4)}(\sigma_1+2) = X_{91}^{(4)} &+ \tfrac{2x}{\rho_1}\sum_{n_1=1}^{2}(\sigma_1+n_1)\,X_{91}^{(4)}(\sigma_1+n_1+1,\,\rho_1+1)\\ &+\tfrac{y\sigma_2}{\rho_1}\sum_{n_1=1}^{2}X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+1,\rho_1+1).\end{aligned} \tag{17}$$

Iterating this process n-times, we obtain (5). For the proof of (6), replace the parameter σ_1 by $\sigma_1 - 1$ in (15). This implies that

$$X_{91}^{(4)}(\sigma_1-1) = X_1 - \frac{2x}{\rho_1}\sigma_1\,X_{91}^{(4)}(\sigma_1+1,\rho_1+1) - \frac{y\sigma_2}{\rho_1}X_{91}^{(4)}(\sigma_2+1,\,\rho_1+1). \tag{18}$$

Iteratively, we get (6).

The recursion formulas from (7)–(14) can be proved in a similar manner.

Theorem 2.2 The following recursion formulas hold true for the numerator parameter σ_2, σ_3, σ_4, σ_5 of the $X_{91}^{(4)}$:

$$\begin{aligned}&X_{91}^{(4)}(\sigma_2+n)\\ &= \sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\,\rho_2+n_2),\end{aligned} \tag{19}$$

$$\begin{aligned}&X_{91}^{(4)}(\sigma_2-n)\\ &= \sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}(-y)^{n_1}(-z)^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}X_{91}^{(4)}(\sigma_1+n_1,\,\sigma_3+n_2,\,\rho_1+n_1,\,\rho_2+n_2),\end{aligned} \tag{20}$$

$$X_{91}^{(4)}(\sigma_3+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}z^{n_1}}{(\rho_2)_{n_2}}X_{91}^{(4)}(\sigma_2+n_1,\sigma_3+n_1,\,\rho_2+n_1), \tag{21}$$

$$X_{91}^{(4)}(\sigma_3-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}(-z)^{n_1}}{(\rho_2)_{n_1}}X_{91}^{(4)}(\sigma_2+n_1,\,\rho_2+n_1), \tag{22}$$

$$X_{91}^{(4)}(\sigma_4+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{91}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\,\rho_1+n_1), \tag{23}$$

$$X_{91}^{(4)}(\sigma_4-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{91}^{(4)}(\sigma_5+n_1,\,\rho_1+n_1), \tag{24}$$

$$X_{91}^{(4)}(\sigma_5+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{91}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\,\rho_1+n_1), \tag{25}$$

$$X_{91}^{(4)}(\sigma_5-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{91}^{(4)}(\sigma_4+n_1,\,\rho_1+n_1), \tag{26}$$

where $\binom{n}{n_1,n_2}=\frac{n!}{n_1!n_2!(n-n_1-n_2)!}$ and $N_2=n_1+n_2$.

Proof. The proof of (19) is based upon the principle of a mathematical induction on $n\in\mathbb{N}$. For $n=1$, the result (19) is true obviously following (7). Suppose (19) is true for $n=m$, that is,

$$\begin{aligned}&X_{91}^{(4)}(\sigma_2+m)\\&\quad=\sum_{N_2\leq m}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\rho_2+n_2),\end{aligned} \tag{27}$$

Replacing σ_2 with σ_2+1 in (27) and using the contiguous relation (7) for $n=1$, we get

$$\begin{aligned}&X_{91}^{(4)}(\sigma_2+m+1)\\&=\sum_{N_2\leq m}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}\\&\quad\times\Big\{X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\rho_2+n_2)\\&\quad+\frac{(\sigma_1+n_1)y}{(\rho_1+n_1)}X_{91}^{(4)}(\sigma_1+n_1+1,\sigma_2+N_2+1,\,\sigma_3+n_2,\,\rho_1+n_1+1,\rho_2+n_1)\\&\quad+\frac{(\sigma_3+n_2)z}{(\rho_2+n_2)}X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2+1,\,\sigma_3+n_2+1,\,\rho_1+n_1,\rho_2+n_2+1)\Big\}.\end{aligned} \tag{28}$$

By a simplification, (28) takes the form

$$\begin{aligned}&X_{91}^{(4)}(a_2+m+1)\\&=\sum_{N_2\leq m}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}\\&\quad\times X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\rho_2+n_2)\\&\quad+\sum_{N_2\leq m+1}\binom{n}{n_1-1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}\\&\times X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\rho_2+n_2)\\&\quad+\sum_{N_2\leq m+1}\binom{n}{n_1,n_2-1}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}\\&\quad\times X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\,\sigma_3+n_2,\,\rho_1+n_1,\rho_2+n_2).\end{aligned} \tag{29}$$

Using the Pascal's identity in (29), we have

$$X_{91}^{(4)}(\sigma_2+m+1)$$
$$=\sum_{N_2\leq m+1}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_1}(\rho_2)_{n_2}}$$
$$\times X_{91}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\sigma_3+n_2,\rho_1+n_1,\rho_2+n_2).$$

This establishes (19) for $n=m+1$. Hence, by induction, the result given in (19) is true for all values of n. The recursion formulas (20)–(26) can be proved in a similar manner.

Theorem 2.3 The following recursion formulas hold true for the denominator parameter ρ_1, ρ_2 of the $X_{91}^{(4)}$:

$$\begin{aligned}X_{91}^{(4)}(\rho_1-n)=X_{91}^{(4)}&+(\sigma_1)_2x\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{91}^{(4)}(\sigma_1+2,\rho_1+2-n_1)\\&+\sigma_1\sigma_2y\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{91}^{(4)}(\sigma_1+1,\sigma_2+1,\rho_1+2-n_1)\\&+\sigma_4\sigma_5u\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{91}^{(4)}(\sigma_4+1,\sigma_5+1,\rho_1+2-n_1),\end{aligned}\tag{30}$$

$$\begin{aligned}&X_{91}^{(4)}(\rho_2-n)\\&=X_{91}^{(4)}+\sigma_4\sigma_5u\sum_{n_1=1}^{n}\frac{1}{(\rho_2-n_1)(\rho_2+1-n_1)}X_{91}^{(4)}(\sigma_4+1,\sigma_5+1,\rho_2+2-n_1).\end{aligned}\tag{31}$$

Proof. Applying the definition of the hypergeometric function $X_{91}^{(4)}$ and the relation

$$\frac{1}{(\rho_1-1)_{m+n+q}}=\frac{1}{(\rho_1)_{m+n+q}}\left(1+\frac{m}{\rho_1-1}+\frac{n}{\rho_1-1}+\frac{q}{\rho_1-1}\right),\tag{32}$$

we have:

$$\begin{aligned}&X_{91}^{(4)}(\rho_1-1)=X_{91}^{(4)}+\frac{(\sigma_1)_2x}{\rho_1(\rho_1-1)}X_{91}^{(4)}(\sigma_1+2,\rho_1+1)+\frac{\sigma_1\sigma_2y}{\rho_1(\rho_1-1)}\\&X_{91}^{(4)}(\sigma_1+1,\sigma_2+1,\rho_1+2-n_1)+\frac{\sigma_4\sigma_5u}{\rho_1(\rho_1-1)}X_{91}^{(4)}(\sigma_4+1,\sigma_5+1,\rho_1+2-n_1).\end{aligned}\tag{33}$$

Using this contiguous relation to the $X_{91}^{(4)}$ with the parameter ρ_1-n for n-times, we get the result (30). The recursion formula (31) can be proved in a similar manner.

Theorem 2.4 The following recursion formulas hold true for the numerator parameter σ_1, σ_2, σ_3, σ_4, σ_5 of the $X_{92}^{(4)}$:

$$\begin{aligned}X_{92}^{(4)}(\sigma_1+n)=X_{92}^{(4)}&+\frac{2x}{\rho_1}\sum_{n_1=1}^{n}(\sigma_1+n_1)X_{92}^{(4)}(\sigma_1+1+n_1,\rho_1+1)\\&+\frac{y\sigma_2}{\rho_2}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_1+n_1,\sigma_2+1,\rho_2+1),\end{aligned}\tag{34}$$

$$X_{92}^{(4)}(\sigma_1 - n) = X_{92}^{(4)} - \frac{2x}{\rho_1}\sum_{n_1=1}^{n}(\sigma_1 + 1 - n_1)X_{92}^{(4)}(\sigma_1 + 2 - n_1, \rho_1 + 1)$$
$$-\frac{y\sigma_2}{\rho_2}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_1 + 1 - n_1, \sigma_2 + 1, \rho_2 + 1), \tag{35}$$

$$X_{92}^{(4)}(\sigma_2 + n) = X_{92}^{(4)} + \tfrac{y\sigma_1}{\rho_2}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_1 + 1, \sigma_2 + n_1, \rho_2 + 1)$$
$$+\frac{z\sigma_3}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_2 + n_1, \sigma_3 + 1, \rho_1 + 1), \tag{36}$$

$$X_{92}^{(4)}(\sigma_2 - n) = X_{92}^{(4)} - \frac{y\sigma_1}{\rho_2}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_1 + 1, \sigma_2 + 1 - n_1, \rho_2 + 1)$$
$$-\frac{z\sigma_3}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_2 + 1 - n_1, \sigma_3 + 1, \rho_1 + 1), \tag{37}$$

$$X_{92}^{(4)}(\sigma_3 + n) = X_{92}^{(4)} + \frac{z\sigma_2}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_2 + 1, \sigma_3 + n_1, \rho_1 + 1), \tag{38}$$

$$X_{92}^{(4)}(\sigma_3 - n) = X_{92}^{(4)} - \frac{z\sigma_2}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_2 + 1, \sigma_3 + 1 - n_1, \rho_1 + 1), \tag{39}$$

$$X_{92}^{(4)}(\sigma_4 + n) = X_{92}^{(4)} + \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_5 + 1, \sigma_4 + n_1, \rho_1 + 1), \tag{40}$$

$$X_{92}^{(4)}(\sigma_4 - n) = X_{92}^{(4)} - \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_5 + 1, \sigma_4 + 1 - n_1, \rho_1 + 1) \tag{41}$$

$$X_{92}^{(4)}(\sigma_5 + n) = X_{92}^{(4)} + \frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_4 + 1, \sigma_5 + n_1, \rho_1 + 1), \tag{42}$$

$$X_{92}^{(4)}(\sigma_5 - n) = X_{92}^{(4)} - \frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{92}^{(4)}(\sigma_4 + 1, \sigma_5 + 1 - n_1, \rho_1 + 1). \tag{43}$$

Theorem 2.5 The following recursion formulas hold true for the numerator parameter σ_2, σ_3, σ_4, σ_5 of the $X_{92}^{(4)}$:

$$X_{92}^{(4)}(\sigma_2+n) \\ = \sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{n_2}(\rho_2)_{n_1}}X_{92}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\sigma_3+n_2,\rho_1+n_2,\rho_2+n_1), \tag{44}$$

$$X_{92}^{(4)}(\sigma_2-n) \\ = \sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}(-y)^{n_1}(-z)^{n_2}}{(\rho_1)_{n_2}(\rho_2)_{n_1}}X_{92}^{(4)}(\sigma_1+n_1,\sigma_3+n_2,\rho_1+n_2,\rho_2+n_1), \tag{45}$$

$$X_{92}^{(4)}(\sigma_3+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}z^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_2+n_1,\sigma_3+n_1,\rho_1+n_1), \tag{46}$$

$$X_{92}^{(4)}(\sigma_3-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}(-z)^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_2+n_1,\rho_1+n_1), \tag{47}$$

$$X_{92}^{(4)}(\sigma_4+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\rho_1+n_1), \tag{48}$$

$$X_{92}^{(4)}(\sigma_4-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_5+n_1,\rho_1+n_1), \tag{49}$$

$$X_{92}^{(4)}(\sigma_5+n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\rho_1+n_1), \tag{50}$$

$$X_{92}^{(4)}(\sigma_5-n) = \sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{92}^{(4)}(\sigma_4+n_1,\rho_1+n_1), \tag{51}$$

where $N_2=n_1+n_2$.

Theorem 2.6 The following recursion formulas hold true for the denominator parameter ρ_1, ρ_2 of the $X_{92}^{(4)}$:

$$\begin{aligned} X_{92}^{(4)}(\rho_1-n) = X_{92}^{(4)} &+ (\sigma_1)_2 x\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{92}^{(4)}(\sigma_1+2,\rho_1+2-n_1) \\ &+\sigma_2\sigma_3 z\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{92}^{(4)}(\sigma_2+1,\sigma_3+1,\rho_1+2-n_1) \\ &+\sigma_4\sigma_5 u\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{92}^{(4)}(\sigma_4+1,\sigma_5+1,\rho_1+2-n_1), \end{aligned} \tag{52}$$

$$X_{92}^{(4)}(\rho_2 - n) = X_{92}^{(4)} + \sigma_1\sigma_2 y \sum_{n_1=1}^{n} \frac{1}{(\rho_2 - n_1)(\rho_2 + 1 - n_1)} X_{92}^{(4)}(\sigma_1 + 1, \sigma_2 + 1, \rho_2 + 2 - n_1). \tag{53}$$

Theorem 2.7 The following recursion formulas hold true for the numerator parameter σ_1, σ_2, σ_3, σ_4, σ_5 of the $X_{93}^{(4)}$:

$$\begin{aligned} X_{93}^{(4)}(\sigma_1 + n) = X_{93}^{(4)} &+ \frac{2x}{\rho_1}\sum_{n_1=1}^{n}(\sigma_1 + n_1)X_{93}^{(4)}(\sigma_1 + 1 + n_1, \rho_1 + 1) \\ &+ \frac{y\sigma_2}{\rho_2}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_1 + n_1, \sigma_2 + 1, \rho_2 + 1), \end{aligned} \tag{54}$$

$$\begin{aligned} X_{93}^{(4)}(\sigma_1 - n) = X_{93}^{(4)} &- \frac{2x}{\rho_1}\sum_{n_1=1}^{n}(\sigma_1 + 1 - n_1)X_{93}^{(4)}(\sigma_1 + 2 - n_1, \rho_1 + 1) \\ &- \frac{y\sigma_2}{\rho_2}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_1 + 1 - n_1, \sigma_2 + 1, \rho_2 + 1), \end{aligned} \tag{55}$$

$$\begin{aligned} X_{93}^{(4)}(\sigma_2 + n) = X_{93}^{(4)} &+ \frac{y\sigma_1}{\rho_2}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_1 + 1, \sigma_2 + n_1, \rho_2 + 1) \\ &+ \frac{z\sigma_3}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_2 + n_1, \sigma_3 + 1, \rho_1 + 1), \end{aligned} \tag{56}$$

$$\begin{aligned} X_{93}^{(4)}(\sigma_2 - n) = X_{93}^{(4)} &- \frac{y\sigma_1}{\rho_2}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_1 + 1, \sigma_2 + 1 - n_1, \rho_2 + 1) \\ &- \frac{z\sigma_3}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_2 + 1 - n_1, \sigma_3 + 1, \rho_1 + 1), \end{aligned} \tag{57}$$

$$X_{93}^{(4)}(\sigma_3 + n) = X_{93}^{(4)} + \frac{z\sigma_2}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_2 + 1, \sigma_3 + n_1, \rho_1 + 1), \tag{58}$$

$$X_{93}^{(4)}(\sigma_3 - n) = X_{93}^{(4)} - \frac{z\sigma_2}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_2 + 1, \sigma_3 + 1 - n_1, \rho_1 + 1), \tag{59}$$

$$X_{93}^{(4)}(\sigma_4 + n) = X_{93}^{(4)} + \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_5 + 1, \sigma_4 + n_1, \rho_1 + 1), \tag{60}$$

$$X_{93}^{(4)}(\sigma_4 - n) = X_{93}^{(4)} - \frac{u\sigma_5}{\rho_1}\sum_{n_1=1}^{n} X_{93}^{(4)}(\sigma_5 + 1, \sigma_4 + 1 - n_1, \rho_1 + 1), \tag{61}$$

$$X_{93}^{(4)}(\sigma_5+n)=X_{93}^{(4)}+\frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{93}^{(4)}(\sigma_4+1,\sigma_5+n_1,\rho_1+1), \tag{62}$$

$$X_{93}^{(4)}(\sigma_5-n)=X_{93}^{(4)}-\frac{u\sigma_4}{\rho_1}\sum_{n_1=1}^{n}X_{93}^{(4)}(\sigma_4+1,\sigma_5+1-n_1,\rho_1+1). \tag{63}$$

Theorem 2.8 The following recursion formulas hold true for the numerator parameter σ_2, σ_3, σ_4, σ_5 of the $X_{93}^{(4)}$:

$$\begin{aligned}&X_{93}^{(4)}(\sigma_2+n)\\&=\sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}y^{n_1}z^{n_2}}{(\rho_1)_{N_2}}X_{93}^{(4)}(\sigma_1+n_1,\sigma_2+N_2,\sigma_3+n_2,\rho_1+N_2),\end{aligned} \tag{64}$$

$$\begin{aligned}&X_{93}^{(4)}(\sigma_2-n)\\&=\sum_{N_2\le n}\binom{n}{n_1,n_2}\frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2}(-y)^{n_1}(-z)^{n_2}}{(\rho_1)_{N_2}}X_{93}^{(4)}(\sigma_1+n_1,\sigma_3+n_2,\rho_1+N_2),\end{aligned} \tag{65}$$

$$X_{93}^{(4)}(\sigma_3+n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}z^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_2+n_1,\sigma_3+n_1,\rho_1+n_1), \tag{66}$$

$$X_{93}^{(4)}(\sigma_3-n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_2)_{n_1}(-z)^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_2+n_1,\rho_1+n_1), \tag{67}$$

$$X_{93}^{(4)}(\sigma_4+n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\rho_1+n_1), \tag{68}$$

$$X_{93}^{(4)}(\sigma_4-n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_5)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_5+n_1,\rho_1+n_1), \tag{69}$$

$$X_{93}^{(4)}(\sigma_5+n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}u^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,\rho_1+n_1), \tag{70}$$

$$X_{93}^{(4)}(\sigma_5-n)=\sum_{n_1\le n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}(-u)^{n_1}}{(\rho_1)_{n_1}}X_{93}^{(4)}(\sigma_4+n_1,\rho_1+n_1), \tag{71}$$

where $N_2=n_1+n_2$.

Theorem 2.9 The following recursion formulas hold true for the denominator parameter ρ_1, ρ_2 of the $X_{93}^{(4)}$:

$$\begin{aligned}X_{93}^{(4)}(\rho_1-n) = X_{93}^{(4)} &+ \sigma_1\sigma_2 y\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{93}^{(4)}(\sigma_1+1,\sigma_2+1,\,\rho_1+2-n_1)\\ &+\sigma_2\sigma_3 z\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{93}^{(4)}(\sigma_2+1,\,\sigma_3+1,\,\rho_1+2-n_1)\\ &+\sigma_4\sigma_5 u\sum_{n_1=1}^{n}\frac{1}{(\rho_1-n_1)(\rho_1+1-n_1)}X_{93}^{(4)}(\sigma_4+1,\,\sigma_5+1,\,\rho_1+2-n_1),\end{aligned} \tag{72}$$

$$\begin{aligned}&X_{93}^{(4)}(\rho_2-n)\\ &= X_{93}^{(4)} + (\sigma_1)_2 x\sum_{n_1=1}^{n}\frac{1}{(\rho_2-n_1)(\rho_2+1-n_1)}X_{93}^{(4)}(\sigma_1+2,\,\rho_2+2-n_1).\end{aligned} \tag{73}$$

Theorem 2.10 The following recursion formulas hold true for the numerator parameter σ_1, σ_2, σ_3, σ_4, σ_5 of the $X_{94}^{(4)}$:

$$\begin{aligned}X_{94}^{(4)}(\sigma_1+n) = X_{94}^{(4)} &+ \frac{2x}{c}\sum_{n_1=1}^{n}(\sigma_1+n_1)X_{94}^{(4)}(\sigma_1+1+n_1,c+1)\\ &+\frac{y\sigma_2}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_1+n_1,\sigma_2+1,c+1),\end{aligned} \tag{74}$$

$$\begin{aligned}X_{94}^{(4)}(\sigma_1-n) = X_{94}^{(4)} &- \frac{2x}{c}\sum_{n_1=1}^{n}(\sigma_1+1-n_1)X_{94}^{(4)}(\sigma_1+2-n_1,c+1)\\ &-\frac{y\sigma_2}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_1+1-n_1,\sigma_2+1,c+1),\end{aligned} \tag{75}$$

$$\begin{aligned}X_{94}^{(4)}(\sigma_2+n) = X_{94}^{(4)} &+ \frac{y\sigma_1}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_1+1,\sigma_2+n_1,c+1)\\ &+\frac{z\sigma_3}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_2+n_1,\sigma_3+1,c+1),\end{aligned} \tag{76}$$

$$\begin{aligned}X_{94}^{(4)}(\sigma_2-n) = X_{94}^{(4)} &- \frac{y\sigma_1}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_1+1,\sigma_2+1-n_1,c+1)\\ &-\frac{z\sigma_3}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_2+1-n_1,\sigma_3+1,c+1),\end{aligned} \tag{77}$$

$$X_{94}^{(4)}(\sigma_3+n) = X_{94}^{(4)} + \frac{z\sigma_2}{c}\sum_{n_1=1}^{n}X_{94}^{(4)}(\sigma_2+1,\sigma_3+n_1,c+1), \tag{78}$$

$$X_{94}^{(4)}(\sigma_3 - n) = X_{94}^{(4)} - \frac{z\sigma_2}{c}\sum_{n_1=1}^{n} X_{94}^{(4)}(\sigma_2+1, \sigma_3+1-n_1, c+1), \tag{79}$$

$$X_{94}^{(4)}(\sigma_4 + n) = X_{94}^{(4)} + \frac{u\sigma_5}{c}\sum_{n_1=1}^{n} X_{94}^{(4)}(\sigma_5+1, \sigma_4+n_1, c+1), \tag{80}$$

$$X_{94}^{(4)}(\sigma_4 - n) = X_{94}^{(4)} - \frac{u\sigma_5}{c}\sum_{n_1=1}^{n} X_{94}^{(4)}(\sigma_5+1, \sigma_4+1-n_1, c+1) \tag{81}$$

$$X_{94}^{(4)}(\sigma_5 + n) = X_{94}^{(4)} + \frac{u\sigma_4}{c}\sum_{n_1=1}^{n} X_{94}^{(4)}(\sigma_4+1, \sigma_5+n_1, c+1), \tag{82}$$

$$X_{94}^{(4)}(\sigma_5 - n) = X_{94}^{(4)} - \frac{u\sigma_4}{c}\sum_{n_1=1}^{n} X_{94}^{(4)}(\sigma_4+1, \sigma_5+1-n_1, c+1). \tag{83}$$

Theorem 2.11 The following recursion formulas hold true for the numerator parameter σ_2, σ_3, σ_4, σ_5 of the $X_{94}^{(4)}$:

$$\begin{aligned} &X_{94}^{(4)}(\sigma_2 + n) \\ &= \sum_{N_2 \le n} \binom{n}{n_1, n_2} \frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2} y^{n_1} z^{n_2}}{(c)_{N_2}} X_{94}^{(4)}(\sigma_1+n_1, \sigma_2+N_2, \sigma_3+n_2, c+N_2), \end{aligned} \tag{84}$$

$$\begin{aligned} &X_{94}^{(4)}(\sigma_2 - n) \\ &= \sum_{N_2 \le n} \binom{n}{n_1, n_2} \frac{(\sigma_1)_{n_1}(\sigma_3)_{n_2} (-y)^{n_1} (-z)^{n_2}}{(c)_{N_2}} X_{94}^{(4)}(\sigma_1+n_1, \sigma_3+n_2, c+N_2), \end{aligned} \tag{85}$$

$$\begin{aligned} &X_{94}^{(4)}(\sigma_3 + n) \\ &= \sum_{n_1 \le n} \binom{n}{n_1} \frac{(\sigma_2)_{n_1} z^{n_1}}{(c)_{n_1}} X_{94}^{(4)}(\sigma_2+n_1, \sigma_3+n_1, c+n_1), \end{aligned} \tag{86}$$

$$X_{94}^{(4)}(\sigma_3 - n) = \sum_{n_1 \le n} \binom{n}{n_1} \frac{(\sigma_2)_{n_1} (-z)^{n_1}}{(c)_{n_1}} X_{94}^{(4)}(\sigma_2+n_1, c+n_1), \tag{87}$$

$$X_{94}^{(4)}(\sigma_4 + n) = \sum_{n_1 \le n} \binom{n}{n_1} \frac{(\sigma_5)_{n_1} u^{n_1}}{(c)_{n_1}} X_{94}^{(4)}(\sigma_4+n_1, \sigma_5+n_1, c+n_1), \tag{88}$$

$$X_{94}^{(4)}(\sigma_4 - n) = \sum_{n_1 \le n} \binom{n}{n_1} \frac{(\sigma_5)_{n_1} (-u)^{n_1}}{(c)_{n_1}} X_{94}^{(4)}(\sigma_5+n_1, c+n_1), \tag{89}$$

$$X_{94}^{(4)}(\sigma_5+n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}u^{n_1}}{(c)_{n_1}}X_{94}^{(4)}(\sigma_4+n_1,\sigma_5+n_1,c+n_1), \tag{90}$$

$$X_{94}^{(4)}(\sigma_5-n)=\sum_{n_1\leq n}\binom{n}{n_1}\frac{(\sigma_4)_{n_1}(-u)^{n_1}}{(c)_{n_1}}X_{94}^{(4)}(\sigma_4+n_1,c+n_1), \tag{91}$$

where $N_2=n_1+n_2$.

Theorem 2.12 The following recursion formulas hold true for the denominator parameter c of the $X_{94}^{(4)}$:

$$\begin{aligned}X_{94}^{(4)}(c-n)=X_{94}^{(4)}&+(\sigma_1)_2x\sum_{n_1=1}^{n}\tfrac{1}{(c-n_1)(c+1-n_1)}X_{94}^{(4)}(\sigma_1+2,c+2-n_1)\\&+\sigma_1\sigma_2y\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{94}^{(4)}(\sigma_1+1,\sigma_2+1,c+2-n_1)\\&+\sigma_2\sigma_3z\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{94}^{(4)}(\sigma_2+1,\sigma_3+1,c+2-n_1)\\&+\sigma_4\sigma_5u\sum_{n_1=1}^{n}\frac{1}{(c-n_1)(c+1-n_1)}X_{94}^{(4)}(\sigma_4+1,\sigma_5+1,c+2-n_1).\end{aligned} \tag{92}$$

3 Conclusion

Hypergeometric functions in several variables play an essential role in diverse areas of science and engineering. The advancements in applied mathematics, mathematical physics, and other areas of science have led to increasing interest in the study of hypergeometric functions. Also, special functions and its properties are used to solve various problems in science and engineering. In this paper, we have derived several recursion formulas for new hypergeometric functions in four variables. Also, some interested particular cases and consequences of our results have been discussed. In the future, these recursion formulas for the hypergeometric functions in four variables may find applications in various branches of mathematics, mathematical physics, engineering and related areas of study.

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