

## **A comparison of various basis functions to linear stability of circular jet using MLPG method**

**Xie Ming-Liang<sup>1</sup>**

### **Summary**

Various basis function based on Fourier-Chebyshev Petrov-Galerkin spectral method is described for computation of temporal linear stability in a circular jet. Basis functions presented here are exponentially mapped Chebyshev functions. There is a linear dependence between the components of the vector field according to the perturbation continuum equation. Therefore, there are only two degrees of freedom. According to the principle of permutation and combination, the basis function has three basic forms, i.e., the radial, azimuthal or axial component is free. The results show that three eigenvalues for various cases are consistent, but the basis function in case I is preferable for numerical computation.

**Keywords:** hydrodynamic stability; circular jet; coordinate transformation; spectral-Galerkin method

### **Introduction**

Jets are important in many practical applications, e.g., related to combustion, propulsion, mixing and aeroacoustic. The round jet results when fluid is emitted, with a given initial momentum, out of a circular orifice into a large space. At sufficiently high Reynolds numbers this jet will be turbulent. The stability properties of the flow play a fundamental role in the transition to turbulence and the formation of coherent vortex structures in a turbulent fluid.

Frequently, the choice of independent variables is motivated by the symmetry of circular jet, then cylindrical coordinates are likely most appropriate. However, the choice of a particular set of independent variable might inadvertently introduce mathematically allowable but physically unrealistic terms, e.g., singularities. For circular jet, the unbounded domain is another problem to be overcome.

The treatment of the geometrical singularity in cylindrical has been a difficulty in the development of accurate schemes for many years. The use of a spectral representation is often to be preferred for the accurate solution of problems with simple geometry [1-5]. It shows that the Meshless Local Petrov-Galerkin(MLPG) method is very promising to solve the Navier-Stokes equations and fluid mechanics problems [6-7]. To construct basis function for unbounded domains, it is necessary to assume the asymptotic behavior of the approximated functions for large radius

---

<sup>1</sup>The State Key Laboratory of Coal Combustion, Huazhong University of Science and Technology, Wuhan, 430074, China

$r$ . It can be made more efficient if additional mappings are used, so that standard spectral basis functions such as Chebyshev polynomial can be used [6-7].

There are many different ways to obtain divergence-free fields in polar coordinates. The solenoidal condition introduces a linear dependence between the radial, azimuthal and axial components of the fields. Therefore, there are only two degrees of freedom. We will distinguish three different situations, radial dependent, azimuthal depend or axial dependent, based on the principle of combination. And a comparison between various situations is present in this work.

### The mathematical formulation

We investigate the utility of mappings to solve the linear stability problems of round jet numerically in infinite regions. To expand this class of functions, we consider the exponential mapping:

$$x = \frac{1 - e^{-r/L}}{1 + e^{-r/L}}; r \in (-\infty, \infty) \quad (1)$$

where  $L$  is the map parameter. We adopted the map parameter values with  $L=3$  for calculation; this value represents the best compromise between the competing demands of the accuracy and the cost of computation (Xie *et al.*, 2008a, 2009). Then the linearized Navier-Stokes equations in cylindrical polar coordinates are:

$$-ikc\mathbf{u} = -Dp + \ell[\mathbf{u}] \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

In which  $\ell$  stands for the linear operator of linear stability equations

$$\ell[\cdot] = \frac{1}{Re} \Delta[\cdot] - \mathbf{u}_B \cdot \nabla[\cdot] - [\cdot] \cdot \nabla \mathbf{u}_B \quad (4)$$

where  $\mathbf{u}_B$  are the basic flow velocity vector  $(0, 0, U_z)$ . And  $\mathbf{u} = (u_r(x), u_\theta(x), u_z(x))$  and  $p(x)$  are the amplitudes of the corresponding disturbances;  $n$  is the azimuthal mode of the disturbance;  $k$  is the axial wavenumber of disturbance;  $c$  is the wave amplification factor. These equations are non-dimensionalised with respect to length scale  $L^*$ , velocity scale  $U^*$ , and Reynolds number is  $Re = L^* U^* / \nu$ . The length scale and velocity scale is usually based on the jet core velocity and momentum thickness.

And the boundary conditions for first azimuthal ( $n = 1$ ) mode become:

$$\begin{aligned} u_r(0) + u_\theta(0) &= Du_z(0) = Dp(0) = 0; \\ u_r(1) &= u_\theta(1) = u_z(1) = p(1) = 0 \end{aligned} \quad (5)$$

where  $D=(1-x^2)(d/dx)/2L$ .

### Solenoidal Petrov-Galerkin discretisation

In order to have spectral accuracy in the numerical approximation of the eigenvalue problem, the solenoidal basis for the approximation of the perturbation vector field takes the form Priymak & Miyazaki [2]:

$$\begin{aligned} \mathbf{u} &= e^{i(kz+n\theta-kct)} \sum_{m=0}^M a_m^{(1)} \mathbf{w}_m^{(1)}(x) \mathbf{u}_m^{(1)}(x) \\ &+ e^{i(kz+n\theta-kct)} \sum_{m=0}^M a_m^{(2)} \mathbf{w}_m^{(2)}(x) \mathbf{u}_m^{(2)}(x) \end{aligned} \quad (6)$$

where  $\mathbf{u}_m$  belongs to the physical or trial space and  $\mathbf{w}_m$  is a solenoidal vector field belongs to the test or projection space. There is a linear dependence between the components of the vector field according to the perturbation continuum equation. Therefore, there are only two degrees of freedom. According to the principle of permutation and combination,  $\mathbf{u}_m$  and  $\mathbf{w}_m$  have three basic forms:

Case I: rendering the azimuthal free, then the physical or trial basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} \begin{bmatrix} -inr^{n-1}g_m(x) \\ D[r^n g_m(x)] \\ 0 \end{bmatrix} \end{pmatrix}, \mathbf{u}_m^{(2)} = \begin{pmatrix} \begin{bmatrix} 0 \\ -ikr^n h_m(x) \\ inr^{n-1}h_m(x) \end{bmatrix} \end{pmatrix} \quad (7)$$

Case II: rendering the axial free, the physical basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} \begin{bmatrix} -ikr^{n-1}g_m(x) \\ 0 \\ D[r^n g_m(x)]/r \end{bmatrix} \end{pmatrix}, \mathbf{u}_m^{(2)} = \begin{pmatrix} \begin{bmatrix} 0 \\ -ikr^n h_m(x) \\ inr^{n-1}h_m(x) \end{bmatrix} \end{pmatrix} \quad (8)$$

Case III: rendering the radial free, the physical basis is:

$$\mathbf{u}_m^{(1)} = \begin{pmatrix} \begin{bmatrix} -inr^{n-1}g_m(x) \\ D[r^n g_m(x)] \\ 0 \end{bmatrix} \end{pmatrix}, \mathbf{u}_m^{(2)} = \begin{pmatrix} \begin{bmatrix} -ikr^{n-1}h_m(x) \\ 0 \\ D[r^n h_m(x)]/r \end{bmatrix} \end{pmatrix} \quad (9)$$

In which  $r/L=\ln(1+x)/(1-x)$ . The projection fields,  $\mathbf{w}_m$ , are going to have the same structure as the trial fields but the functions will be modified by the Chebyshev weight  $(1-x^2)^{-1/2}$ .

The Petrov-Galerkin projection scheme is carried out by substituting the spectral series in equations and projecting over the dual space. This procedure leads to a discretized generalized eigenvalues problem, and the coefficient  $a_m^{(1,2)}$  govern the temporal behavior.

$$\mathbf{A}\mathbf{X} = -ikc\mathbf{B}\mathbf{X} \quad (10)$$

where the matrixes  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{X}$  represent as follows:

$$\mathbf{A} = \begin{bmatrix} \left[ \begin{pmatrix} \mathbf{w}_m^{(1)} \cdot \ell[\mathbf{u}_m^{(1)}] \\ \mathbf{w}_m^{(2)} \cdot \ell[\mathbf{u}_m^{(1)}] \end{pmatrix} \right] & \left[ \begin{pmatrix} \mathbf{w}_m^{(1)} \cdot \ell[\mathbf{u}_m^{(2)}] \\ \mathbf{w}_m^{(2)} \cdot \ell[\mathbf{u}_m^{(2)}] \end{pmatrix} \right] \end{bmatrix};$$

$$\mathbf{B} = \begin{bmatrix} \left[ \begin{pmatrix} \mathbf{w}_m^{(1)} \cdot \mathbf{u}_m^{(1)} \\ \mathbf{w}_m^{(2)} \cdot \mathbf{u}_m^{(1)} \end{pmatrix} \right] & \left[ \begin{pmatrix} \mathbf{w}_m^{(1)} \cdot \mathbf{u}_m^{(2)} \\ \mathbf{w}_m^{(2)} \cdot \mathbf{u}_m^{(2)} \end{pmatrix} \right] \end{bmatrix};$$

$$\mathbf{X} = \begin{bmatrix} \begin{bmatrix} a_m^{(1)} \\ a_m^{(2)} \end{bmatrix} \end{bmatrix}$$

The pressure term should be formally included in the operator  $\ell$ , but it is cancelled when projecting it over  $\mathbf{w}$ , that is  $(\mathbf{w}, \mathbf{p})=0$ .

## Results and discussion

The generalized eigenvalues problem in Eq.(9) can be computed exactly by Gauss-Chebyshev-Lobatto quadrature formulas. In the present study, the temporal instability of round jet is considered. Hence,  $k$  and  $n$  is real quantity while  $c = c_r + ic_i$  is generally complex. The disturbances will grow with time if  $c_i > 0$  and will decay if  $c_i < 0$ . The neutral disturbances are then characterized by  $c_i = 0$ . Table 1 shows the comparison of eigenvalues for various cases under critical conditions ( $Re = 37.64$ ,  $k = 0.469$ ), and theses eigenvalues are consistent generally. From the point of computation cost, the basis function in case II has an additional division operation and the basis function in case III need more differential operators than that in case I. Therefore the basis functions in case I is more preferable in numerical simulation [1-2, 6-7].

Table 1: Comparison of real eigenvalues

	$Re$	$k$	$c_r$	$c_i$
Case I	37.64	0.469	0.118582	-1.98e-6
Case II	37.64	0.469	0.118575	-1.21e-5
Case III	37.64	0.469	0.118655	-4.93e-4

### **Acknowledgement**

This work was supported by the National Natural Science Foundation of China with Grant No. 50806023 and the National Natural Science Foundation of China with Grant No. 50721005 and the Programmer of Introducing Talents of Discipline to Universities (“111” project No.B06019), China

### **References**

1. A. Meseguer, L.N. Trefethen, Linearized pipe flow to Reynolds number  $10^7$ . **J. Comput. Phys.**, 186(2003): 178–197
2. V.G. Priymak, T. Miyazakiy, Accurate Navier- Stokes investigation of transitional and turbulent flows in a circular pipe. **J. Comput. Phys.**, 142 (1998): 370–411
3. M.L. Xie, T.L. Chan, Y.D. Zhang, H.C. Zhou, Numerical analysis of nonlinear stability of two phase flow in the Blasius boundary layer. **Int. J. Nonlin. Sci. Num.**, 9(2008): 423-434
4. M.L. Xie, J.Z. Lin, H.C. Zhou, The effect of nonlinear interaction between gas and particle velocities on the hydrodynamic stability in the Blasius boundary layer. **Int. J. Non-Lin. Mech.**, 44(2009): 106-114.
5. M.L. Xie, J.Z. Lin, H.C. Zhou, Temporal stability of a particle-laden Blasius boundary layer. **Mod. Phys. Lett. B**, 23 (2009): 203-216
6. M.L. Xie, J.Z. Lin, An efficient numerical solution for linear stability of circular jet: a combination of Petrov Galerkin spectral method and exponential coordinate transformation based on Fornberg’s treatment. **Int. J. Numer. Meth. Fl.**, DOI: 10.1002/fld.1975 (2008)
7. M.L. Xie, H.C. Zhou, T.L. Chan, An improved Petrov-Galerkin spectral collocation solution for linear stability of circular jet-figure. **CMES: Comp. Model. Eng.**, 46(2009): 271-289

