



ARTICLE

Efficient Identification of Heterogeneous Thermal Conductivity Using a Complex-Variable Hyper-Reduced Order Model

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ABSTRACT: Heterogeneous composite material property identification suffers from prohibitive computational overhead and inefficient inversion procedures. To overcome these limitations, this paper presents a complex-variable hyper-reduced order model (CV-HROM) for the efficient identification of multi-peak heterogeneous thermal properties. Built upon the conventional finite element method, the proposed hyper-reduced order model replaces expensive global numerical integration of heterogeneous domain with localized discrete element integration via the discrete empirical interpolation method (DEIM), substantially accelerating forward heat conduction simulations. In the parameter identification process, the Levenberg–Marquardt algorithm is adopted to minimize the loss function, wherein the gradients are evaluated through the complex-variable derivative method (CVDM). Specifically, an infinitesimal complex perturbation is introduced into the hyper-reduced order model, such that the real part of the solution constructs the loss function while the imaginary part directly yields the temperature derivatives with respect to the targeted parameters. The accuracy and efficacy of the proposed method are validated through thermal property inversion of a heterogeneous rocket nozzle composite, considering both single-peak and dual-peak heterogeneous distributions. Numerical results demonstrate that the CV-HROM achieves over tenfold acceleration in solving inverse heat conduction problems. The method enables stable, accurate estimation of multi-parameter, multi-peak heterogeneous thermal conductivity distributions, making it a promising tool for efficient thermal property inversion of large-scale composite structures.

KEYWORDS: Hyper reduced order model; complex variable derivative method; heterogeneous thermal property; inverse heat conduction problem

1 Introduction

Thermal properties of composite materials—thermal conductivity in particular—dictate heat transfer behavior across a broad spectrum of engineering disciplines, spanning microelectronic thermal management, chemical reactor thermal control, and aerospace propulsion systems [1–3]. The rapid advancement of additive manufacturing, or three-dimensional (3D) printing, has enabled the widespread adoption of heterogeneous composites in critical load-bearing and thermally loaded components such as rocket nozzles, where they exhibit thermal performance unattainable with homogeneous materials [4]. Accurate characterization of heterogeneous thermal properties is therefore indispensable for optimization designs, operational safety assurance, and service-life extension of such high-performance components. Nevertheless, the spatially varying nature of thermal parameters in heterogeneous composites presents formidable challenges for both experimental measurement and numerical identification [5,6].

Identification of heterogeneous thermal properties is conventionally formulated as an inverse heat conduction problem (IHCP), wherein unknown thermal parameters are reconstructed from available temperature measurements [7,8]. The IHCP is intrinsically ill-posed: minor perturbations in measured temperature data can propagate to substantial deviations in the identified parameter fields [9]. Compounding this ill-posedness, the forward heat conduction simulation at the IHCP solver typically relies on the finite element method (FEM) for geometrically complex structures. The large number of degrees of freedom inherent in FEM leads to prohibitive computational costs and severely degrades inversion efficiency, particularly in large-scale engineering applications or multi-parameter identification scenarios [10,11]. This computational bottleneck constitutes a major impediment to the practical application of the IHCP in real-time or large-scale thermal property identification tasks.

To alleviate the computational burden of FEM-based forward simulation, reduced order models (ROMs) have been extensively developed and integrated into IHCP frameworks [12–14]. ROMs project the original high-fidelity model onto a low-dimensional subspace, thereby drastically accelerating forward simulations. However, conventional ROMs—exemplified by Galerkin-projection-based formulations—remain constrained by the computational overhead associated with assembling parameter-dependent terms or performing global numerical integration, which limits their efficiency gains in heterogeneous heat conduction problems [15,16]. To overcome this limitation, hyper-reduced order models (HROMs) have been proposed. These combine ROMs with hyper-reduction techniques—most notably the discrete empirical interpolation method (DEIM)—to replace global numerical integration with localized discrete element integration, delivering additional efficiency gains for forward simulation [17–19].

In the IHCP solution procedure, the gradient of the loss function critically determines both the convergence rate and the identification accuracy of the optimization algorithm. Conventional gradient evaluation approaches, namely the finite difference method (FDM) and the adjoint method, each carry inherent limitations. The FDM necessitates additional forward simulations for every parameter perturbation, driving up computational cost and introducing numerical instability tied to the choice of perturbation step size [20,21]. The adjoint method, by contrast, requires solving an auxiliary adjoint equation, entailing substantial implementation complexity and non-negligible computational overhead, especially for high-dimensional parameter identification problems [22,23]. The complex-variable derivative method (CVDM) offers a compelling alternative: by introducing an infinitesimal complex perturbation, it recasts partial derivative evaluation of implicit functions as direct function-value computation [24]. This approach obviates both step-size selection and adjoint equation solving, thereby simultaneously preserving high gradient accuracy and computational efficiency.

While HROMs and CVDM have each been applied to IHCP independently, their combination for heterogeneous thermal property identification remains largely unexplored. Existing studies either leverage HROMs for forward acceleration without refining the gradient computation process, or employ CVDM within high-fidelity models without tackling the forward simulation bottleneck [25–27]. For structurally complex heterogeneous composites, the dual challenges of expensive forward simulation and inaccurate gradient evaluation remain unresolved, resulting in inefficient inversion and unstable parameter identification.

Accordingly, the present work proposes a complex-variable hyper-reduced order model (CV-HROM) for rapid, accurate identification of heterogeneous thermal properties. The framework synergistically integrates HROM and CVDM: the HROM accelerates forward heat conduction simulations, while the CVDM enables efficient, high-precision gradient evaluation of the objective function. Specifically, the HROM is constructed upon FEM and DEIM, replacing global numerical integration with localized discrete element integration to reduce forward simulation cost. During parameter identification, an infinitesimal

complex perturbation is injected into the HROM: the real part of the computed response forms the loss function, whereas the imaginary part directly yields the temperature derivatives with respect to the targeted parameters. The Levenberg–Marquardt (LM) algorithm is adopted to minimize the objective function, ensuring rapid convergence and robust identification performance. The proposed methodology is validated using a heterogeneous composite rocket engine nozzle case, demonstrating superior computational efficiency and identification accuracy relative to conventional approaches.

2 Methodology

2.1 Hyper Reduced Order Model for Heterogeneous Heat Conduction Problem

To reduce the prohibitive computational cost of the finite element method (FEM) in forward heat conduction simulation of heterogeneous composite materials, this section develops a hyper reduced order model (HROM) for heat conduction problem. The proposed HROM retains the accuracy of the high-fidelity FEM while substantially reducing computational overhead. It reconstructs global integrals via localized discrete elements, thereby establishing an efficient forward-simulation foundation for parameter identification in inverse heat conduction problems.

2.1.1 FEM of Heterogeneous Heat Conduction

For three-dimensional heterogeneous composite structures with spatially varying thermal conductivity, the governing equation of steady-state heat conduction problem can be expressed as:

$$\nabla \cdot (k(\mathbf{x}, \mu) \nabla T(\mathbf{x})) = 0 \quad (1)$$

where $\mathbf{x} \in \Omega$ denotes the spatial coordinate in the computational domain Ω , $T(\mathbf{x})$ is the temperature field, $k(\mathbf{x}, \mu)$ represents the heterogeneous thermal conductivity to be identified (with μ being the vector of identified parameters), and ∇ is the gradient operator.

The outer boundary Γ of the computational domain Ω is subject to the Dirichlet, Neumann and Robin boundary conditions respectively prescribed as:

$$T(\mathbf{x}) = \bar{T}(\mathbf{x}), \mathbf{x} \in \Gamma_1 \quad (2a)$$

$$\mathbf{q}(\mathbf{x}) = -k \frac{\partial T(\mathbf{x}, t)}{\partial \mathbf{n}_{\Gamma_2}} = \bar{q}(\mathbf{x}), \mathbf{x} \in \Gamma_2 \quad (2b)$$

$$\mathbf{q}(\mathbf{x}) = h(T(\mathbf{x}) - T_\infty(\mathbf{x})), \mathbf{x} \in \Gamma_3 \quad (2c)$$

where $\bar{T}$ is the prescribed temperature on Γ_1 . $\bar{q}$ is the imposed heat flux on Γ_2 , and $\mathbf{n}$ is the normal vector outside the boundary. h and T_∞ represent the convective heat transfer coefficient and ambient temperature on Γ_3 , respectively.

Applying the Galerkin finite element method to Eq. (1), yields the discretized heat conduction system:

$$\mathbf{K}(\mu) \mathbf{T} = \mathbf{F} \quad (3)$$

where $\mathbf{K}(\mu) \in \mathbb{R}^{m \times m}$ is the parameter-dependent thermal conductivity matrix, $\mathbf{F} \in \mathbb{R}^m$ is the load vector, $\mathbf{T} \in \mathbb{R}^m$ is the nodal temperature vector, and m denotes the total number of degree of freedom (DoF) of the FEM.

The thermal conductivity matrix in Eq. (3) is assembled via global domain integration, given by:

$$\mathbf{K}(\mu) = \int k(\mu) \mathbf{N}^T \mathbf{N} d\Omega + \int h \mathbf{N}^T \mathbf{N} d\Gamma_3 \quad (4)$$

The load vector is assembled via boundary integration over the Neumann and Robin boundaries, given by:

$$\mathbf{F} = \int \bar{q} \mathbf{N} d\Gamma_2 + \int h T_\infty \mathbf{N} d\Gamma_3 \quad (5)$$

where $\mathbf{N}$ denotes the shape function of the adopted element type.

2.1.2 HROM of Heterogeneous Heat Conduction

As evident from Eqs. (4) and (5), the dimensionality of the thermal conductivity matrix is a dominant factor governing the computational efficiency of heat conduction simulations. To reduce the dimensionality of the full-order FEM model, the discrete empirical interpolation method (DEIM) is employed to reconstruct the global domain integration of the thermal conductivity matrix, yielding:

$$\mathbf{K}(\mu) = \mathbf{\Psi}_K \tilde{\mathbf{\Psi}}_K \begin{bmatrix} \int k(\mu) \mathbf{N}^T \mathbf{N} d\Omega_1 \\ \vdots \\ \int k(\mu) \mathbf{N}^T \mathbf{N} d\Omega_r \end{bmatrix} + \int h \mathbf{N}^T \mathbf{N} d\Gamma_3 \quad (6)$$

where $\mathbf{\Psi}_K$ is the orthogonal basis corresponding to the thermal conductivity matrix. $\tilde{\mathbf{\Psi}}_K$ is the interpolation matrix of the orthogonal basis associated with the interpolation nodes $\Xi = [\xi_1, \xi_2 \dots \xi_r]$, defined as:

$$\tilde{\mathbf{\Psi}}_K = \begin{bmatrix} \Psi_K(\xi_1, \xi_1) & \dots & \Psi_K(\xi_1, \xi_r) \\ \vdots & \ddots & \vdots \\ \Psi_K(\xi_r, \xi_1) & \dots & \Psi_K(\xi_r, \xi_r) \end{bmatrix} \quad (7)$$

where $\Omega_1, \dots, \Omega_r$ represents the adjacent discrete elements of the interpolation nodes $\Xi = [\xi_1, \xi_2 \dots \xi_r]$.

The orthogonal basis matrix $\mathbf{\Psi}_K$ in Eq. (6) is constructed via the proper orthogonal decomposition (POD):

$$\mathbf{\Psi}_K = \mathbf{A} \cdot \left[\frac{\mathbf{v}_1}{\sigma_1}, \frac{\mathbf{v}_2}{\sigma_2} \dots \frac{\mathbf{v}_r}{\sigma_r} \right] \quad (8)$$

where $\mathbf{A}$ denotes the snapshots matrix of the thermal conductivity matrix $[K(\mu_1), \dots, K(\mu_n)]$, $[\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_n]$ and $[\sigma_1, \sigma_2 \dots \sigma_n]$ are the eigenvectors and eigenvalues, respectively, as follows:

$$\mathbf{A} = \begin{bmatrix} K(x_1, \mu_1) & \dots & K(x_1, \mu_n) \\ \vdots & \ddots & \vdots \\ K(x_m, \mu_1) & \dots & K(x_1, \mu_n) \end{bmatrix} \quad (9a)$$

$$\mathbf{A}^T \mathbf{A} [\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_n] = [\mathbf{v}_1, \mathbf{v}_2 \dots \mathbf{v}_n] \begin{bmatrix} \sigma_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_n \end{bmatrix} \quad (9b)$$

The number of reduced basis should satisfy:

$$\sum_{i=1}^r \frac{\sigma_i}{\sum_{j=1}^n \sigma_j} \geq 1 - \varepsilon_{tol} \quad (10)$$

where ε_{tol} is a small tolerance, such as 0.01.

The interpolation nodes $\Xi = [\xi_1, \xi_2 \dots \xi_r]$ in Eq. (6) are iteratively selected from the orthogonal basis Ψ via a greedy algorithm that minimizes the interpolation error (usually take 10^{-5}) across the snapshots ensemble, given by:

$$\mathbf{K}(\mu) \approx \Psi \Psi_{\Xi}^{-1} \mathbf{K}_{\Xi}(\mu) \quad (11)$$

where $\mathbf{K}_{\Xi}(\mu)$ is the reduced vector corresponding to the interpolation nodes, given by:

$$\mathbf{K}_{\Xi}(\mu) = \mathbf{E}^T \Psi \mathbf{K} = \begin{bmatrix} K_{\xi_1}(\mu) \\ \vdots \\ K_{\xi_r}(\mu) \end{bmatrix} \quad (12a)$$

$$\mathbf{E} = [\mathbf{e}_{\xi_1}, \mathbf{e}_{\xi_2} \dots \mathbf{e}_{\xi_r}] \quad (12b)$$

Consequently, numerical integration of the thermal conductivity matrix is performed exclusively over the local subdomains $\Omega_1, \dots, \Omega_r$ rather than the entire global domain Ω , giving rise to the hyper-reduction model. For heterogeneous heat conduction problems, the hyper-reduced order model achieves dramatic acceleration of the finite element forward simulation $O(\Omega_1, \dots, \Omega_r) \ll O(\Omega)$.

2.2 Inverse Heat Conduction Problem for Heterogeneous Thermal Conductivity

2.2.1 Parameterization Complex-Variable Heterogeneous Thermal Conductivity

The heterogeneous thermal conductivity is parameterized via inverse distance weighting (IDW) interpolation:

$$k(\mathbf{x}, \mu) = \sum_{i=1}^p \omega_i \cdot \mu(x_i, y_i) \quad (13)$$

where p is the number of the identified points, $\mu(x_i, y_i)$ is the thermal conductivity of the identified point (x_i, y_i) , ω_i is the weight, given by:

$$\omega_i = \frac{1/d_i}{\sum_{i=1}^p 1/d_i} \quad (14)$$

where d_i is the distance from the identified point to all points, given by:

$$d_i = \|(x, y) - (x_i, y_i)\| \quad (15)$$

For a two-dimensional problem with p interpolation control points, the total number of unknown parameters is $3p$ (two spatial coordinates plus one thermal conductivity value per control point).

The complex variable hyper reduced order model is constructed by introducing an infinitesimal imaginary perturbation δi into the identified parameters, given by:

$$(x_1, y_1, \mu_1 \cdots x_p, y_p, \mu_p) \rightarrow \begin{cases} (x_1 + \delta i, y_1, \mu_1 \cdots x_p, y_p, \mu_p) \\ (x_1, y_1 + \delta i, \mu_1 \cdots x_p, y_p, \mu_p) \\ \vdots \\ (x_1, y_1, \mu_1 \cdots x_p, y_p + \delta i, \mu_p) \\ (x_1, y_1, \mu_1 \cdots x_p, y_p, \mu_p + \delta i) \end{cases} \quad (16)$$

where δ denotes the perturbation magnitude (typically 1×10^{-10}).

2.2.2 Identification Procedure Based on the CV-HROM

The heterogeneous thermal conductivity is identified by minimizing a loss function defined as the L2 norm of the difference between the estimated temperatures and the reference temperatures, given by:

$$L = \|T^{ref} - \text{Re}(T^{esi})\|_2 \quad (17)$$

where T^{obs} denotes the reference temperatures, T^{esi} is the estimated complex variable temperatures predicted by the CV-HROM, $\text{Re}(\cdot)$ is the operator for taking the real part, $\|\cdot\|_2$ is the L2 norm operator.

According to the complex-variable differentiation principle, the complex-valued temperature field can be evaluated for any given set of complex-valued heterogeneous thermal conductivity parameters from Eq. (16):

$$\begin{aligned} \text{Re}[T^{esi}(x_1, y_1, \mu_1 \cdots x_p, y_p, \mu_p)] &= \text{Re}[T^{esi}(x_1 + \delta i, y_1, \mu_1 \cdots x_p, y_p, \mu_p)] \\ &= \text{Re}[T^{esi}(x_1, y_1, \mu_1 \cdots x_p, y_p + \delta i, \mu_p)] = \text{Re}[T^{esi}(x_1, y_1, \mu_1 \cdots x_p, y_p, \mu_p + \delta i)] \end{aligned} \quad (18)$$

To efficiently minimize the loss function in Eq. (17), a gradient-based iterative algorithm is adopted, given by:

$$k_i^{j+1} = k_i^j + \Delta k_i \quad (19)$$

where k_i^{j+1} , k_i^j denote the identified heterogeneous thermal conductivity parameters at the $(j+1)$ -th and j -th iteration step, and the initial data k_i^0 could be the spatial coordinates and thermal conductivity parameters of each control points. The increment Δk_i at the j th iteration step is generated by the Levenberg-Marquardt (LM) method, given by:

$$\Delta k_i = [J^T J + \lambda \cdot \text{diag}(J^T J)] J^T (T^{obs} - \text{Re}(T^{esi})) \quad (20)$$

where $\text{diag}(\cdot)$ denotes the operator for constructing a diagonal matrix, λ is a positive damping coefficient, and J is the gradient matrix, defined as:

$$\lambda_i = \begin{cases} 0.1 & i = 1 \\ \lambda_{i-1}/10 & L_i < L_{i-1} \\ 10\lambda_{i-1} & L_i \geq L_{i-1} \end{cases} \quad (21a)$$

$$J = \begin{bmatrix} \frac{\partial T_1^{esi}}{\partial k_1} & \dots & \frac{\partial T_1^{esi}}{\partial k_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_m^{esi}}{\partial k_1} & \dots & \frac{\partial T_m^{esi}}{\partial k_n} \end{bmatrix} \quad (21b)$$

To obtain the gradient $\left(\frac{\partial T_1^{esi}}{\partial k_{3p-2}}, \dots, \frac{\partial T_i^{esi}}{\partial k_{3p-2}}, \dots, \frac{\partial T_m^{esi}}{\partial k_{3p-2}} \right)$, $3p-2 \in (1, \dots, n)$ in Eq. (21), a complex variable hyper-reduced order model is solved with the heterogeneous thermal conductivity $(x_1, y_1, \mu_1 \dots x_p + \delta i, y_p, \mu_p)$, thus the gradient can be solved through complex variable derivative method, defined as:

$$\frac{\partial T_i^{esi}}{\partial k_{3p-2}} = \frac{\text{Im}[T_i^{esi}(x_1, y_1, \mu_1 \dots x_p + \delta i, y_p, \mu_p)]}{\delta} \quad (22)$$

The gradient computed by the complex variable derivative method avoids the numerical dissipation and step-size dependence that are common in finite difference methods, thus providing stable, high-precision search directions for each LM iteration. The mapping from perturbed parameters to model response should be continuously differentiable, to ensure the existence and uniqueness of the sensitivity solution.

3 Results and Discussion

3.1 Verification of CV-HROM for Heterogeneous Heat Conduction

To verify the accuracy and computational efficiency of the proposed hyper-reduced order model (HROM) for solving heat conduction problems involving heterogeneous thermal properties, a rocket engine nozzle is selected as the verification case. As schematically illustrated in Fig. 1, high-temperature gas from the combustion chamber enters through the left inlet, undergoes compression and expansion, and exits through the right outlet. The nozzle has an overall length of 380 mm, a maximum external width of 250 mm, and an inlet diameter of 240 mm.

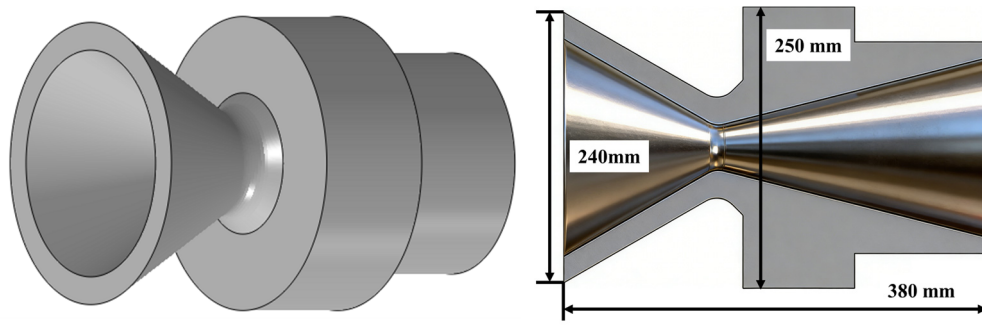


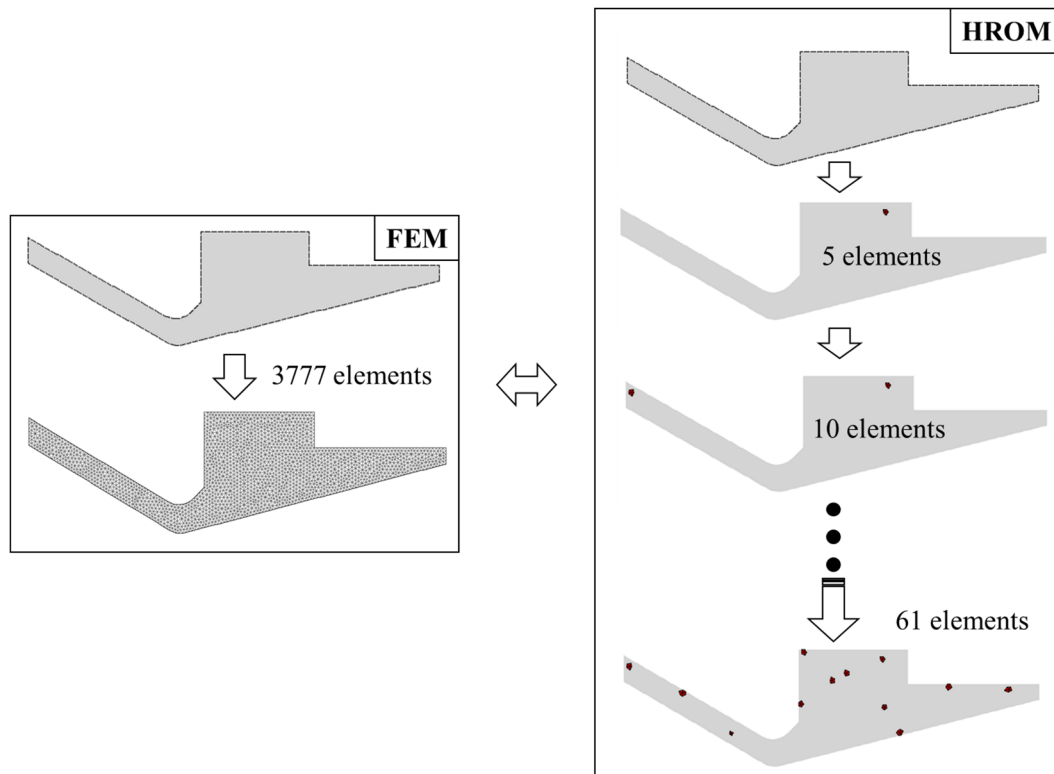
Figure 1: Geometry of rocket nozzle.

The inner wall of the nozzle is exposed to an extreme high-temperature thermal environment, whereas the outer wall dissipates heat via natural convection. The boundary conditions employed in this study are summarized in Table 1. Specifically, the inner surface is subjected to forced convective heat transfer with a reference temperature of 200°C and a convective heat transfer coefficient of 1000 W·m⁻²·°C⁻¹. The outer surface is subjected to natural air convection, with an ambient temperature of 20°C and a heat transfer coefficient of 20 W·m⁻²·°C⁻¹.

Table 1: Boundary conditions of the nozzle.

Boundary	Convection Heat Transfer Coefficient ($\text{W}\cdot\text{m}^{-2}\cdot\text{C}^{-1}$)	Ambient Temperature ($^{\circ}\text{C}$)
Inner Wall	1000	200
Outer Wall	20	20

The nozzle configuration shown in Fig. 1 is discretized as a two-dimensional axisymmetric model, and the heterogeneous heat conduction problem is solved using the finite element method. As depicted in Fig. 2, the computational mesh comprises 3777 finite elements. As elaborated in Section 2.1.2, the thermal conductivity matrix associated with the heterogeneous material field must be reassembled at each iteration step of the inverse identification process, constituting the primary computational bottleneck for both forward simulation and inverse identification efficiency. Consequently, the development of a hyper-reduced order model—also detailed in Section 2.1.2—is essential for alleviating this computational burden. Following the DEIM procedure, interpolation nodes are progressively selected and their adjacent elements are identified; the system equations are then reconstructed through numerical integration restricted to these local element subsets, rather than over the entire domain.

**Figure 2:** Discrete format of finite element model (FEM) and hyper reduced order model (HROM).

Based on the discretization schemes of both the full-order finite element model and the hyper reduced order model (HROM) outlined in the preceding section, Fig. 3 compares the finite element mesh and the HROM configurations with varying reduced degrees of freedom (DoFs). The full-order finite element model comprises 3777 elements; by contrast, the 6-DoF HROM employs 6 interpolation nodes and 30 adjacent elements, the 9-DoF HROM utilizes 9 interpolation nodes and 44 adjacent elements, and the 12-DoF HROM involves 12 interpolation nodes and 61 adjacent elements. This significant reduction in the number

of elements requiring numerical integration directly translates into substantially accelerated forward simulations for heterogeneous heat conduction problems.

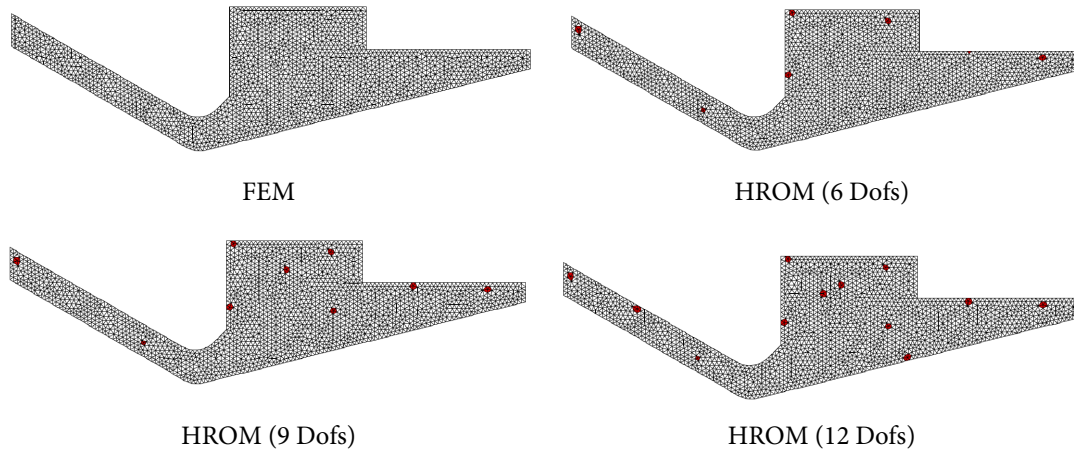


Figure 3: Discrete scheme of different models: finite element method (FEM), hyper reduced order model with 6 degree of freedoms (HROM 6 Dofs), hyper reduced order model with 9 degree of freedoms (HROM 9 Dofs) and hyper reduced order model with 12 degree of freedoms (HROM 12 Dofs).

To quantitatively assess the numerical accuracy of the proposed HROM for heterogeneous heat conduction simulation, the steady-state comparative analyses are performed using both the original full-order finite element model and the proposed HROM. The sample parameters are two spatial coordinates and one conductivity value, X coordinate in $[0 \text{ mm}, 120 \text{ mm}]$, Y coordinate in $[0 \text{ mm}, 400 \text{ mm}]$, conductivity value in $[5 \text{ W/(m·K)}, 40 \text{ W/(m·K)}]$, and 50 samples were randomly generated for POD. The boundary conditions for all test cases remain consistent with those specified in Table 1. As evidenced by the full-domain temperature contours in Fig. 4, the temperature field reconstructed by the HROM exhibits excellent agreement with the high-fidelity FEM reference solution in terms of spatial distribution, peak temperature magnitudes, and temperature gradient characteristics. These qualitative results provide preliminary confirmation of the reliable temperature-field prediction capability of the proposed hyper-reduction methodology.

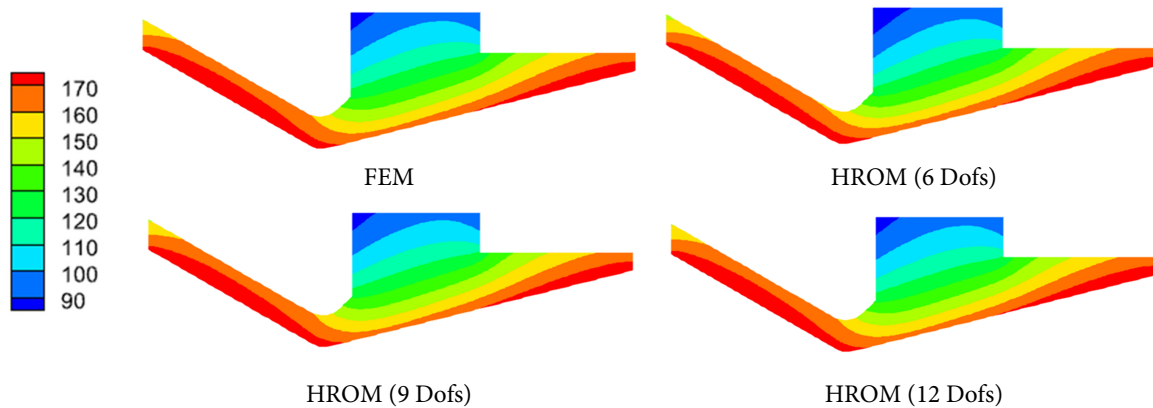


Figure 4: Steady-state temperature distribution predicted by different models: finite element method (FEM), hyper reduced order model with 6 degree of freedoms (HROM 6 Dofs), hyper reduced order model with 9 degree of freedoms (HROM 9 Dofs) and hyper reduced order model with 12 degree of freedoms (HROM 12 Dofs).

Furthermore, the effect of the number of reduced DoFs on computational accuracy is investigated in Fig. 4. When the number of basis vectors is increased from 6 to 9, the temperature-field simulation error decreases appreciably, corresponding to a marked improvement in numerical prediction fidelity. Further increasing the reduced DoFs to 12 yields convergence of the full-domain temperature error, with no appreciable further gain in simulation accuracy. This observation indicates that the reduced basis has already captured the dominant thermal modes of the heterogeneous heat conduction problem at the 9-DoF level, and that additional basis vectors contribute only marginally to further accuracy improvement.

Fig. 5 presents the axial temperature distributions along the nozzle outer wall, as predicted by the hyper-reduced order models (HROMs) with varying reduced degrees of freedom (DoFs) and the full-order finite element model. It can be found that the temperature profiles reveal a pronounced inverse correlation between the outer wall temperature and the local wall thickness, which can be attributed to the radial heat transfer mechanism dominating the nozzle structure: thinner wall sections offer lower through-thickness thermal resistance under a given inner-wall condition, resulting in correspondingly elevated outer wall surface temperatures. Consequently, temperature peaks emerge at the nozzle inlet and outlet ends where the wall thickness reaches its minimum; conversely, the thickest throat-adjacent mid-section of the nozzle corresponds to the lowest outer wall temperature along the entire axial profile.

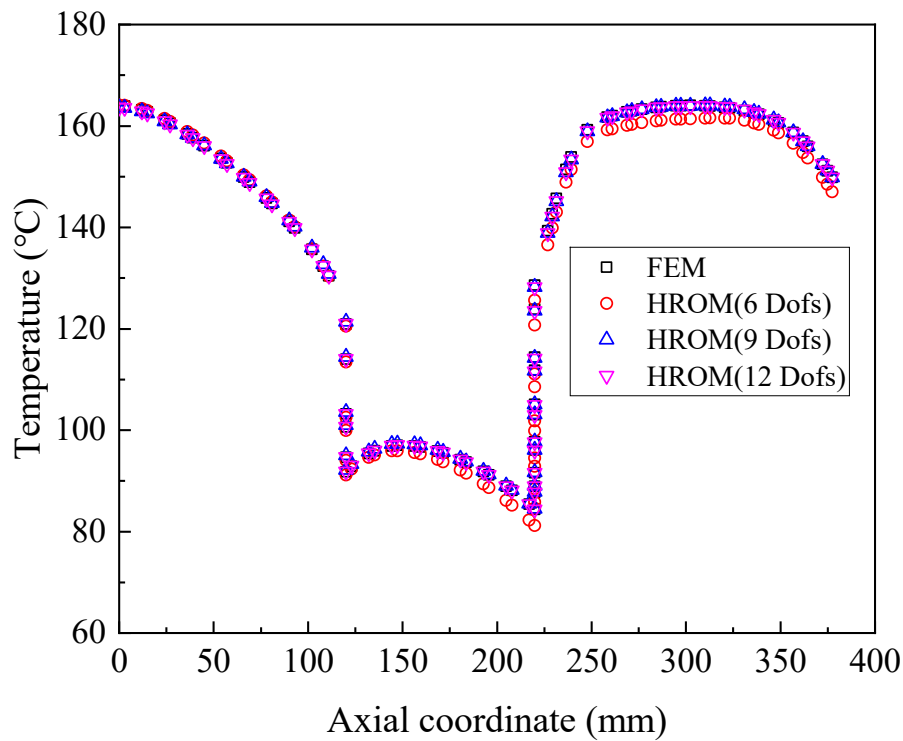


Figure 5: Simulated temperatures on the outer surface: finite element method (FEM), hyper reduced order model with 6 degree of freedoms (HROM 6 Dofs), hyper reduced order model with 9 degree of freedoms (HROM 9 Dofs) and hyper reduced order model with 12 degree of freedoms (HROM 12 Dofs).

A quantitative comparison of the three HROM predictions against the FEM reference solution reveals distinct accuracy levels across the different reduced basis dimensions. The 9-DoF and 12-DoF HROM temperature curves are in near-perfect agreement with the full-order finite element results: both configurations accurately reproduce the temperature peaks at the nozzle inlet and outlet, capture the steep temperature

gradients in the transition regions, and replicate the temperature variation across the mid-section. The maximum relative temperature errors for the 9-DoF and 12-DoF HROMs remain below 1% (Fig. 6). The 6-DoF HROM exhibits notable deviations from the FEM solution, with the most pronounced discrepancies localized in the peak temperature regions near the inlet and outlet. Specifically, the 6-DoF model underestimates the maximum wall temperature, and fails to accurately resolve the sharp temperature gradients in the vicinity of the inlet and outlet edges. This degradation in accuracy indicates that an insufficient number of basis vectors cannot adequately resolve the high-gradient thermal modes associated with the thermal property discontinuities at the nozzle ends.

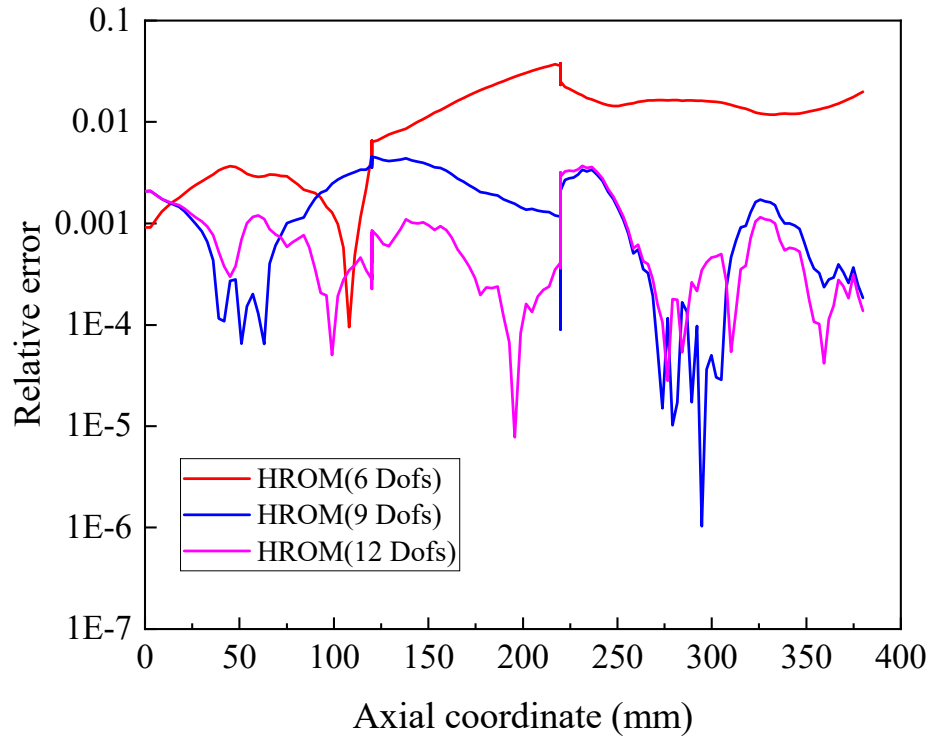


Figure 6: Relative errors of simulated temperature on the outer surface:: hyper reduced order model with 6 degree of freedoms (HROM 6 Dofs), hyper reduced order model with 9 degree of freedoms (HROM 9 Dofs) and hyper reduced order model with 12 degree of freedoms (HROM 12 Dofs).

Fig. 6 depicts the relative temperature error distributions of the 6-, 9-, and 12-DoF hyper-reduced order models (HROMs) with respect to the full-order finite element method along the nozzle axis. The results demonstrate slightly decreases in prediction error as the number of reduced degrees of freedom increases. The 6-DoF HROM exhibits an overall error level in the range of 0.1%–1%, with a peak error of approximately 3%. Increasing the basis size to 9 DoFs reduces the overall error to the 0.01%–0.1% range, representing an order-of-magnitude improvement. Further expanding to 12 DoFs drives the error down to the 0.001%–0.01% band, with the maximum error across the entire domain remaining below 0.01%.

The findings further verify the dependence of HROM accuracy on the reduced basis dimension. For the present heterogeneous nozzle heat conduction problem, the 6-DoF configuration is demonstrably inadequate. The 9-DoF configuration strikes a favorable balance between accuracy and computational efficiency. Further increasing the basis size to 12 DoFs yields only marginal accuracy gains, indicating that the reduced basis has effectively converged and that additional basis vectors contribute diminishing returns.

In particular, the 9-DoF HROM achieves error levels well below typical engineering tolerance thresholds, rendering it fully adequate for high-fidelity engineering simulation applications.

3.2 Identification of Single-Peak Heterogeneous Thermal Conductivity

The proposed hyper-reduced order model (HROM) substantially accelerates the solution of heterogeneous heat conduction problems. By extending the HROM formulation from the real domain into the complex domain, the computational efficiency can be further exploited for the inverse identification of heterogeneous thermal conductivity, as illustrated in Fig. 7.

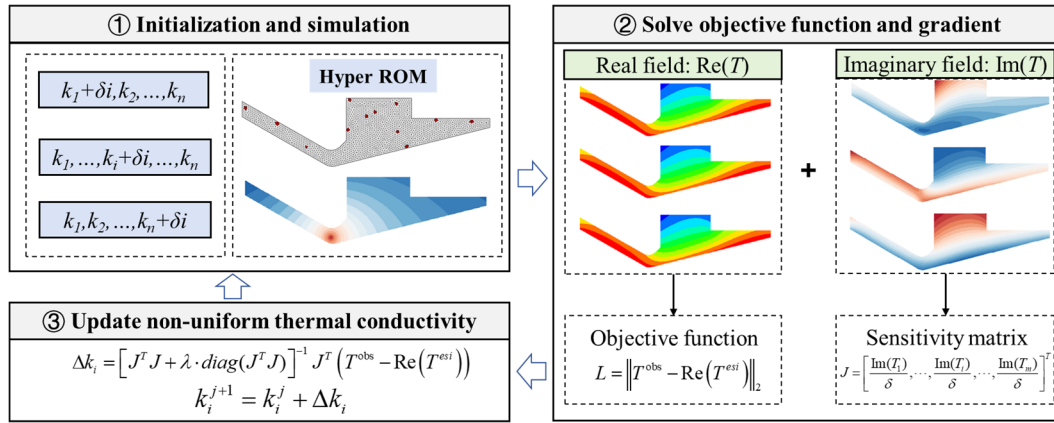


Figure 7: Identification process of the complex variable hyper reduced order model (Hyper ROM).

The heterogeneous thermal conductivity identification framework based on the complex-variable hyper-reduced order model (CV-HROM) employs an iterative architecture, comprising three stages:

- (1) Forward simulation with a small complex perturbation. The set of heterogeneous thermal conductivity parameters to be identified is first initialized. For each parameter, a complex perturbation is introduced in the imaginary part, generating multiple parameters conditions that are collectively fed into the HROM solver. The complex-valued temperature field is then computed efficiently, which drastically reduces the computational cost of the forward problem relative to full-order finite element analysis.
- (2) Loss function construction and sensitivity evaluation. The complex-variable temperature field is decomposed into its real and imaginary components. The 2-norm residual between the real-part temperature and the reference temperature is adopted as the loss function. Concurrently, the imaginary-part temperature is utilized to compute the parameter sensitivity matrix, which effectively eliminates the truncation and rounding errors.
- (3) Parameter update through the Levenberg–Marquardt algorithm. Combining the loss function and the sensitivity matrix, the parameter correction vector is solved using the Levenberg–Marquardt (LM) iterative scheme with adaptive regularization, and the thermal conductivity parameters are updated accordingly. The algorithm then returns to the forward simulation step to start a new cycle. This process repeats until the loss function converges below a predefined threshold ($L \leq 10^{-4}$ or $\Delta L \leq 10^{-4}$), ultimately delivering efficient and high-precision identification of the heterogeneous thermal property field.

To select outer-wall temperature measurement points with maximum information gain and thereby ensure the accuracy and stability of heterogeneous thermal conductivity parameter inversion, a normalized

sensitivity analysis is first performed for the nozzle outer-wall temperature with respect to the heterogeneous thermal conductivity with single peak (two spatial coordinates and one thermal conductivity magnitude). This yields the axial distribution of the sensitivity coefficients for each parameter along the nozzle axis. As shown in Fig. 8, the results reveal pronounced spatial variations in the high-sensitivity regions corresponding to different parameters: the inlet section exhibits elevated sensitivity to certain thermal conductivity parameters (Y coordinate), whereas the leading and trailing edges of the central thick-walled section emerge as concentrated peak zones for multi-parameter sensitivity. Notably, the sensitivity of all three parameters attains its global maximum at the trailing edge of the thick-walled section, where the observational information content is the richest.

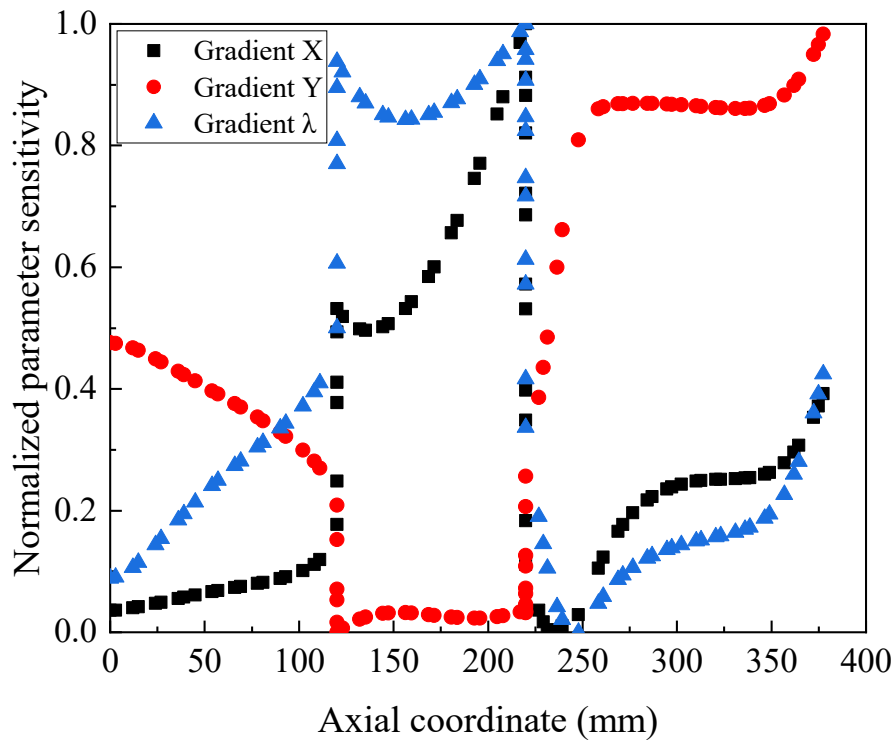


Figure 8: Normalized parameter sensitivity on the outer surface.

To guarantee that every parameter to be identified is paired with high-sensitivity observational data, and simultaneously accounting for the engineering feasibility of sensor placement, three sensitivity-extreme locations—namely the nozzle inlet (axial coordinate: 367.1 mm), the leading edge of the central thick-walled section (axial coordinate: 207.9 mm), and the trailing edge of the central thick-walled section (axial coordinate: 131.6 mm)—are finally selected as temperature measurement points.

Fig. 9 compares the iterative convergence behavior of heterogeneous thermal property parameter identification between the full-order finite element model and the complex-variable hyper-reduced order model (CV-HROM), and the initial parameters are 10 mm, 10 mm, $10 \text{ W} \cdot \text{m}^{-1} \cdot ^\circ\text{C}^{-1}$. Furthermore, the parameter coordinates are set within the range of $X \in [20 \text{ mm}, 120 \text{ mm}]$ and $Y \in [20 \text{ mm}, 300 \text{ mm}]$ to avoid unreasonable values. The parameter evolution trajectories of the two methods are in excellent agreement: all three parameters to be identified undergo rapid correction in the early iteration stage, and gradually converge to stable values after approximately 20 steps. The final parameter identification results obtained from the CV-HROM indicate no appreciable loss of identification accuracy.

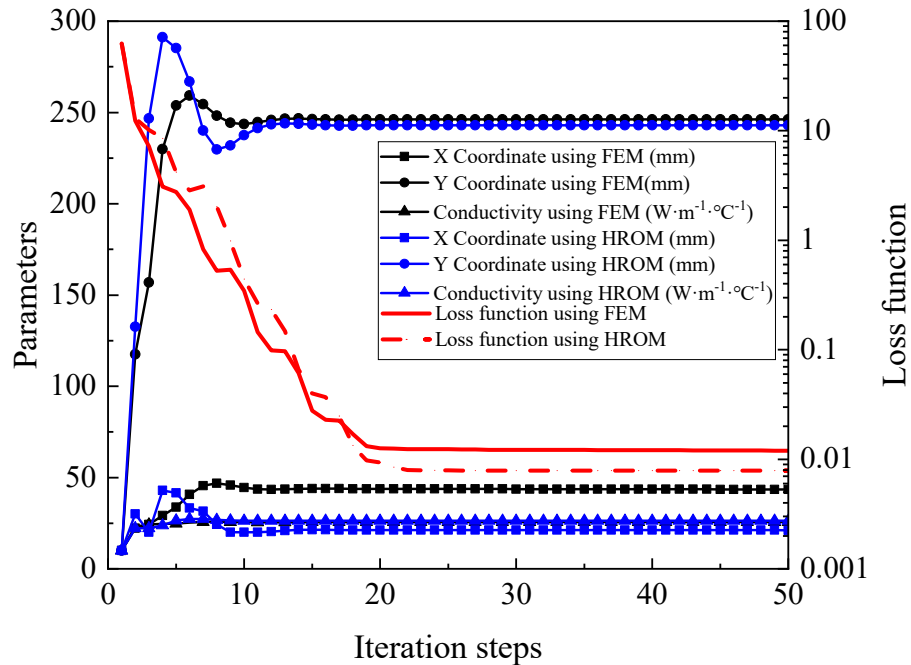


Figure 9: Iteration process of various parameters and loss functions using finite element method (FEM) and hyper reduced order model (HROM).

The loss functions of both methods exhibit the characteristic profile of “rapid initial descent followed by steady convergence”. Upon convergence, the residuals are reduced to below the 10^{-2} order of magnitude, confirming that both approaches achieve inversion accuracy at the same level. These comparative results validate the effectiveness of the proposed complex-variable hyper-reduced order identification framework: while fully preserving the identification accuracy of the full-order finite element method, the CV-HROM substantially reduces the computational cost of the forward simulation at each iteration through order-reduction acceleration, thereby enabling efficient and high-precision inversion of heterogeneous thermal conductivity parameters.

Table 2 summarizes the computational details for heterogeneous thermal conductivity identification, comparing the full-order finite element model against the complex-variable hyper-reduced order model (CV-HROM). It should be noted that the computational details in Table 2 is executed on a PC with a AMD Ryzen 9900X processor, 32 GB of RAM, and the CPU time is just the iteration process of the inverse problem, it does not include snapshot generation, POD construction, and DEIM selection. The results demonstrate that the CV-HROM compresses the original 3777 degrees of freedom of the full-order model to 44, achieving a significant reduction in problem dimensionality.

Table 2: Computational details of finite element method (FEM) and hyper reduced order model (HROM).

Methods	Dofs	CPU Time (s)	Loss Function
FEM	3777	139.2	1.20e−3
HROM	44	10.1	7.90e−4

Driven by this order-reduction acceleration, the total CPU time for a complete parameter identification run is substantially reduced from 139.2 s (full-order model) to 10.1 s (CV-HROM), delivering more than a 13-fold efficiency improvement and alleviating computational burden of the inverse iteration.

Meanwhile, the final converged loss function value of the CV-HROM reaches 7.90×10^{-4} , which is marginally superior to the 1.20×10^{-3} obtained with the full-order finite element model, indicating a smaller inversion residual and marginally improved identification accuracy. This quantitative comparison comprehensively validates the multifaceted advantages of the proposed complex-variable hyper-reduced order identification method: while delivering substantial gains in computational efficiency, the method actually achieves improved convergence performance. These findings establish the CV-HROM as a promising and viable approach for the efficient inversion of large-scale heterogeneous thermal property fields.

Fig. 10 compares the distribution of the estimated heterogeneous thermal conductivity using the FEM and the hyper-reduced order model, and the real distribution is a assumed non-homogeneous distribution (three parameters: 30 mm, 245 mm, $26 \text{ W}\cdot\text{m}^{-1}\cdot^\circ\text{C}^{-1}$) in this case, and the reference temperature in the loss function is obtained by simulating the distribution of the real non-homogeneous thermal conductivity using the FEM. The thermal conductivity values span the range of $16.5\text{--}24 \text{ W}\cdot\text{m}^{-1}\cdot^\circ\text{C}^{-1}$. It can be found that both sets of identification results exhibit excellent agreement with the reference thermal conductivity field. The high-conductivity peak regions of all three fields are localized in the throat of the nozzle, with consistent peak magnitudes and overall distribution ranges. Furthermore, the annular gradient variation of thermal conductivity in the nozzle convergent section, as well as the low-conductivity morphology in the thick-walled section, are accurately reproduced by both methods. The results demonstrate that the identification performance of the HROM is virtually indistinguishable from that of the full-order finite element model: both recover the spatially heterogeneous distribution characteristics of the thermal conductivity field. By retaining identification accuracy on par with the full-order method while substantially reducing the computational cost of the inversion process, these results corroborate the reliability and efficiency of the proposed complex-variable hyper-reduced order identification framework for heterogeneous thermal property inversion.

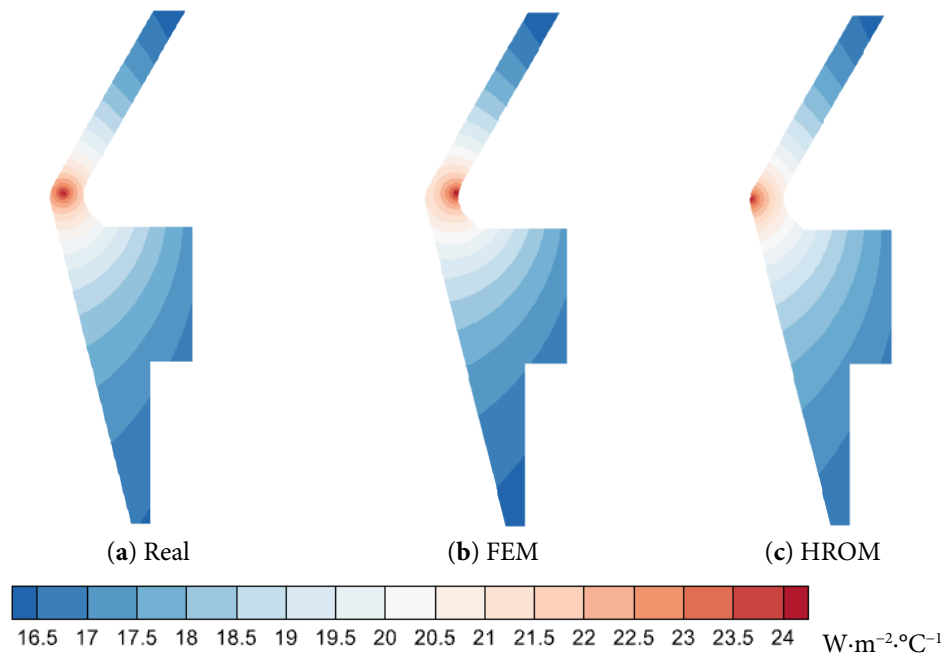


Figure 10: Heterogeneous thermal conductivity distribution of (a) real, (b) identification by finite element method (FEM), (c) identification by hyper reduced order model (HROM).

Fig. 11 presents the relative error contour for the thermal conductivity identification results obtained by the full-order finite element method and the hyper-reduced order model (HROM). In terms of the spatial distribution, the two identification methods exhibit highly consistent error characteristics. High-error regions are concentrated in the area of abrupt thermal conductivity gradient on the nozzle throat—this region is characterized by steep variations in thermal property parameters and pronounced inversion boundary effects. By contrast, the thermal conductivity distribution in the nozzle convergent section and the thick-walled section is relatively smooth, and the corresponding relative identification errors remain stable at a low level of approximately 1%. Consequently, the identification accuracy across the vast majority of the computational domain remains satisfactory.

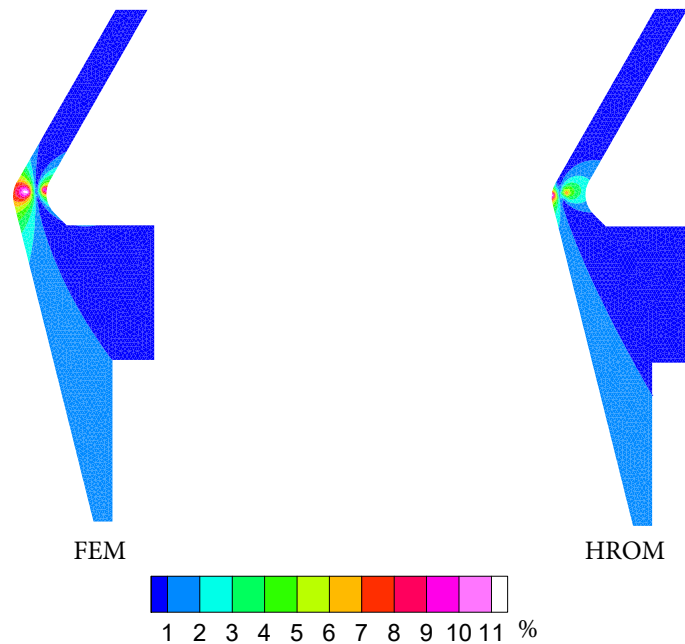


Figure 11: Relative error of finite element method (FEM) and hyper reduced order model (HROM) in identifying heterogeneous thermal conductivity.

A comparison between the two methods reveals that peak error magnitude of the HROM is on the same order as those of the full-order finite element model. No additional appreciable bias is introduced by the reduced-order approximation. This finding further corroborates the accuracy and reliability of the proposed complex-variable hyper-reduced order identification method, which achieves substantial reduction in inversion computational cost while preserving identification accuracy equivalent to that of the full-order finite element approach.

Fig. 12 compares the axial distributions of heterogeneous thermal conductivity along the nozzle inner wall. All three profiles exhibit a unimodal variation trend. The thermal conductivity peak is localized in the 240–260 mm axial segment, which corresponds to the region of concentrated thermal loading on the nozzle inner wall and is fully consistent with the inner-wall distribution characteristics.

Both methods faithfully reproduce the axial variation trend of thermal conductivity. The HROM identification results demonstrate good overall agreement with the real values, with only a slight overestimation in the peak region. The full-order finite element results are marginally lower than the real values overall, and exhibit a relatively more pronounced deviation within the peak range. In the smooth section near the nozzle outlet (axial position beyond 300 mm), the three curves are nearly superimposed, with identification

deviations at an extremely low level. The errors are predominantly concentrated in the peak region, where the thermal conductivity gradient is steepest.

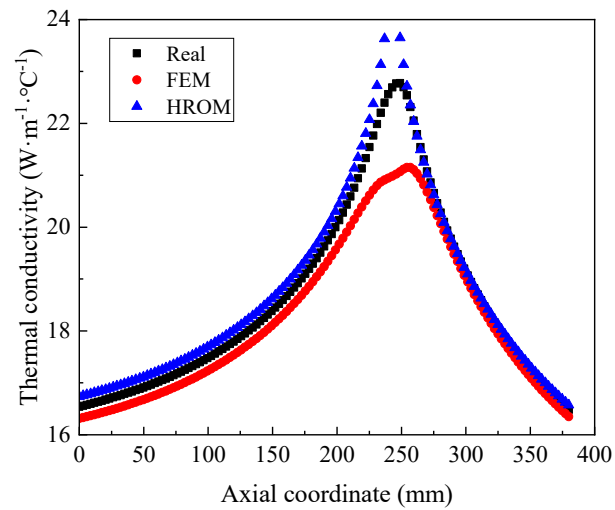


Figure 12: Identified thermal conductivity on the inner surface using finite element method (FEM) and hyper reduced order model (HROM).

To test the robustness with respect to initialization, different initial guesses (X Coordinate: 10 mm, Y Coordinate: 10 mm, Conductivity: $10 \text{ W}\cdot\text{m}^{-1}\cdot\text{°C}^{-1}$ and X Coordinate: 20 mm, Y Coordinate: 20 mm, Conductivity: $20 \text{ W}\cdot\text{m}^{-1}\cdot\text{°C}^{-1}$) have been applied. As shown in Fig. 13, heterogeneous thermal conductivity parameters can be accurately identified, with slightly larger identification errors in the X -coordinate parameters because the X -axis represents the thickness of the nozzle and has higher sensitivity than the Y -axis.

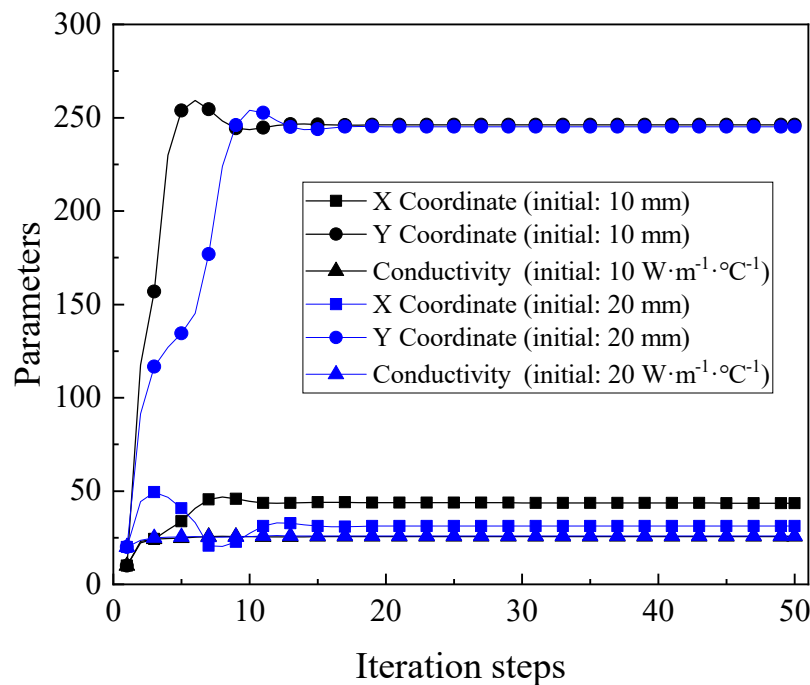


Figure 13: Iteration process of various parameters with different initial parameters.

To demonstrate the robustness against measurement noise of the proposed method, 0.5%, 1% and 3% random noise levels is introduced, and the identified results is listed in Table 3. It can be seen that despite measurement errors, accurate results can still be identified, and the proposed method shows certain robustness against noise. As the measurement noise increases, the identification relative error also increases. When the noise is 3%, the maximum identification error is 8.06%. Of course, excessive noise can lead to distorted identification results.

Table 3: Identified results with different measurement noise.

Noise	X Coordinate (mm)	Y Coordinate (mm)	Conductivity ($\text{W}\cdot\text{m}^{-1}\cdot^{\circ}\text{C}^{-1}$)	Relative Error of Conductivity
0%	43.49	246.14	25.56	0%
0.5%	43.34	244.84	25.74	0.70%
1%	42.71	259.16	26.86	5.09%
3%	48.43	260.31	27.62	8.06%

3.3 Identification of Dual-Peak Heterogeneous Thermal Conductivity

The foregoing validation on single-peak heterogeneous thermal conductivity identification has demonstrated the accuracy and efficiency of the proposed CV-HROM framework for relatively simple heterogeneous fields. However, practical engineering composites—particularly those employed in extreme thermal environments such as rocket nozzles—frequently exhibit multi-peak heterogeneous characteristics with spatially segregated material regions, each governed by distinct thermal property parameters and geometric boundaries. In such scenarios, a single-peak representation is insufficient to capture the full complexity of the material distribution, and the inverse problem expands to simultaneous identification of multiple sets of geometric coordinates and corresponding thermal conductivity values. This multi-parameter coupled inversion presents additional challenges, aggravated ill-posedness, and convergence stability. To further assess the generality and robustness of the proposed method under more demanding conditions, this section extends the validation to a dual-peak heterogeneous thermal conductivity identification problem, wherein six parameters—two sets of geometric coordinates and two distinct thermal conductivity values—are recovered simultaneously. And six reference temperature are located on the outer surface at the axial coordinate of 207.9 mm, 367.1 mm, 131.6 mm, 120 mm, 5.9 mm, 108 mm.

Fig. 14 illustrates the iterative convergence process for multi-parameter identification of dual-peak heterogeneous thermal conductivity. There are six parameters to be identified—comprising the geometric coordinates (X_1/Y_1 , X_2/Y_2) of the dual-peak heterogeneous zones and their respective thermal conductivities (μ_1 , μ_2), and the initial parameters are 10 mm, 10 mm, 50 mm, 50 mm, $10 \text{ W}\cdot\text{m}^{-1}\cdot^{\circ}\text{C}^{-1}$, $10 \text{ W}\cdot\text{m}^{-1}\cdot^{\circ}\text{C}^{-1}$. The real distribution is a assumed non-homogeneous distribution (six parameters: 30 mm, 245 mm, 75 mm, 150 mm, $26 \text{ W}\cdot\text{m}^{-1}\cdot^{\circ}\text{C}^{-1}$; $26 \text{ W}\cdot\text{m}^{-1}\cdot^{\circ}\text{C}^{-1}$) in this case, and the reference temperature in the loss function is obtained by simulating the distribution of the real non-homogeneous thermal conductivity using the FEM. All identified parameters exhibit a typical three-stage convergence pattern: rapid initial correction, intermediate minor fluctuations, and stable late-stage convergence. Specifically, parameter correction amplitudes are largest during the first 30 iterations, during which both the geometric positions and thermal conductivity values rapidly approach their real values. Between 30 and 80 iterations, the parameters undergo fine adjustments before gradually stabilizing. After approximately 80 iterations, all parameters converge to steady values, with no obvious fluctuations in subsequent iterations.

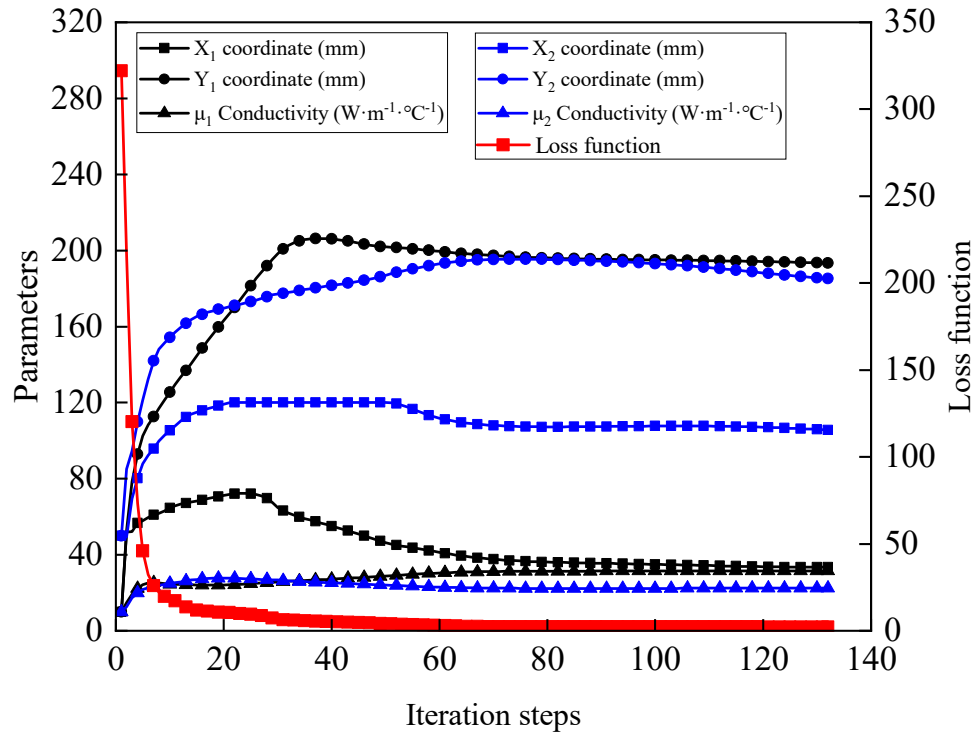


Figure 14: Iteration process of various parameters and loss functions in dual-peak heterogeneous thermal conductivity.

The objective loss function undergoes a steep, cliff-like decrease within the first 10 iterations, followed by a gradual deceleration in its decay rate. Beyond 80 iterations, it converges to an extremely low level and remains stable thereafter; no oscillations or divergence are observed throughout the entire process, indicating a smooth and robust convergence trajectory. These results demonstrate the favorable robustness of the proposed identification framework for multi-peak coupled heterogeneous thermal property inversion problems. The method is capable of simultaneously and stably recovering multiple sets of zonal geometric positions and thermal property parameters (X_1 : 33.4 mm, Y_1 : 193.6 mm, μ_1 : 31.6 W·m⁻¹·°C⁻¹, X_2 : 105.9 mm, Y_2 : 185.6 mm, μ_2 : 22.5 W·m⁻¹·°C⁻¹), exhibiting reliable convergence performance.

Fig. 15 compares the real heterogeneous thermal conductivity distribution against the identification results, with thermal conductivity values spanning the range of 20–25.5 W·m⁻¹·°C⁻¹. The identification results exhibit excellent agreement with the reference thermal conductivity field. Both fields display characteristic bimodal heterogeneous distribution features: a localized high-conductivity zone on the left side of the nozzle inlet, and an annular core high-conductivity zone in the central portion of the nozzle. The positions, spatial extents, and peak magnitudes of the two high-conductivity regions are in close agreement. Furthermore, the gradient transition trend of the global thermal conductivity field and the spatial morphology of the low-conductivity zones are highly consistent; there are only minor visual differences in the gradient transition bands at the peak peripheries, with no evident zonal displacement or morphological distortion. These findings confirm that the proposed identification method accurately recovers the spatial distribution characteristics of heterogeneous thermal conductivity fields and delivers high inversion fidelity for complex heterogeneous thermal property parameters.

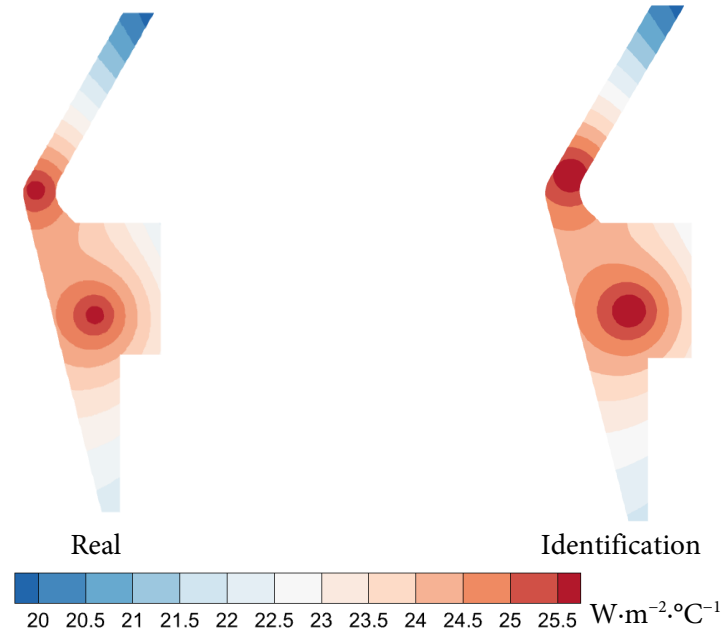


Figure 15: Comparison of dual-peak heterogeneous thermal conductivity.

4 Conclusions

This study develops a complex-variable hyper-reduced order model (CV-HROM) for efficient, high-fidelity identification of spatially heterogeneous thermal conductivity in composite structures. By integrating the discrete empirical interpolation method (DEIM)-based hyper-reduction technique with the complex-variable derivative method (CVDm), the proposed framework effectively addresses two core bottlenecks of inverse heat conduction problems: the excessive computational overhead of full-order forward simulation and the cumbersome and error-prone gradient evaluation process. Numerical validations are conducted on a typical composite rocket nozzle structure to verify the effectiveness and superiority of the developed method. The results demonstrate that the CV-HROM successfully reduces the 3777 degrees of freedom of the full-order finite element model to only 44, achieving more than a 13-fold acceleration in the complete parameter inversion process while maintaining accuracy consistent with the full-order benchmark. Notably, the dimensionality reduction operation does not induce any accuracy degradation; instead, the CV-HROM yields slightly lower converged residuals than the conventional finite element method. The proposed method accurately reconstructs both single-peak and dual-peak heterogeneous thermal conductivity fields, with most regional relative identification errors maintained below 1%. Driven by complex-variable gradient calculation, the Levenberg–Marquardt optimization algorithm presents a stable and smooth iterative convergence characteristic, enabling reliable convergence of all unknown parameters within approximately 80 iterations.

The proposed CV-HROM framework provides an efficient and robust alternative for large-scale, multi-parameter inversion of heterogeneous thermal properties. It possesses great application potential for high-precision thermal characterization of aerospace thermal protection structures such as rocket nozzles, where both computational efficiency and identification fidelity are critically required. Future research will extend the current framework to transient heat conduction scenarios and real composites inversion. Furthermore, the broader applicability of the framework to data-driven and multiphysics material-property identification should be further investigated. In addition, real composites often exhibit filler agglomeration,

interface effects, connectivity/percolation, discontinuous domains, and spatially irregular microstructures, and some realistic microstructures would be systematically investigated in the future work.

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Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

A	snapshots matrix
d	distance
F	load vector
h	convective heat transfer coefficient
J	gradient matrix
k	thermal conductivity
K	thermal conductivity matrix
L	loss function
N	shape function
q	heat flux
x	spatial coordinate
T	temperature field
T	nodal temperature vector
v	eigenvectors
<i>Greek symbols</i>	
Γ	outer boundary
δ	imaginary perturbation
ε	tolerance
λ	damping coefficient
μ	identified parameters
σ	eigenvalues
ψ	orthogonal basis
ω	weight
Ω	computational domain
μ	identified parameters

References

1. He M, Zhuang Y, Xu W, Feng JC. Experimental study on thermal performance regulation of composited phase change system using non-uniform magnetic field generated by a Helmholtz coil. Appl Therm Eng. 2025;269:126081. [[CrossRef](#)].

2. Wang J, Zhuang Y, Feng JC. Coupling regulation of variable gravity and magnetic field on the heat transfer dynamics of nano-enhanced phase change material: an experimental study. *Appl Therm Eng.* 2025;279:127614. [[CrossRef](#)].
3. Almheidat M, Alqudah M, Imran A, Allehiany FM, Alqarni MM, Alqurashi MS, et al. Thermal analysis using Prandtl nanofluid in a non-uniform channel: computational paradigm focusing on complex cilia waves. *Case Stud Therm Eng.* 2025;68:105872. [[CrossRef](#)].
4. Sciti D, Vinci A, Zoli L, Galizia P, Failla S, Mungiguerra S, et al. Propulsion tests on ultra-high-temperature ceramic matrix composites for reusable rocket nozzles. *J Adv Ceram.* 2023;12(7):1345–60. [[CrossRef](#)].
5. Lin XW, Zhou ZF, Zhu XG, Chen B, Zhu H, Ben NY, et al. Non-uniform thermal characteristics investigation of three-dimensional electrochemical-thermal coupled model for pouch lithium-ion battery. *J Clean Prod.* 2023;417:137912. [[CrossRef](#)].
6. Chu L, Shi J, Long X. Heterogeneity-induced thermal mismatch in BGA interconnects: insights from mechanical-thermal finite element modeling. *Microelectron Reliab.* 2025;168:115703. [[CrossRef](#)].
7. Bergagio M, Li H, Anglart H. An iterative finite-element algorithm for solving two-dimensional nonlinear inverse heat conduction problems. *Int J Heat Mass Transf.* 2018;126:281–92. [[CrossRef](#)].
8. Somasundharam S, Reddy KS. Estimation of spatially distributed thermal properties of heterogeneous media with non-intrusive measurement. *Heat Transf Eng.* 2021;42(1):61–87. [[CrossRef](#)].
9. Zhao X, Jin K, Yan M, Nan P, Zhou F, Xin G, et al. Inverse heat transfer for real-time thermal evaluation of aircraft thermal protection structure with embedded FBG sensors. *Appl Therm Eng.* 2025;260:124869. [[CrossRef](#)].
10. Meethal RE, Kodakkal A, Khalil M, Ghantasala A, Obst B, Bletzing KU, et al. Finite element method-enhanced neural network for forward and inverse problems. *Adv Model Simul Eng Sci.* 2023;10:6. [[CrossRef](#)].
11. Berger J, Orlande HRB, Mendes N. Proper Generalized Decomposition model reduction in the Bayesian framework for solving inverse heat transfer problems. *Inverse Probl Sci Eng.* 2017;25(2):260–78. [[CrossRef](#)].
12. Amooie MA, Gilbert HB, Wilson PJ, Khonsari MM. Application of 2D inverse heat transfer to analyze mechanical fatigue. *Int J Fatigue.* 2025;198:109030. [[CrossRef](#)].
13. Jiang G, Wang J, Cheng C. A novel complex variable hyper-reduction model for efficient determination of convective heat transfer coefficient at the inlet of a steam turbine. *Int Commun Heat Mass Transf.* 2025;162:108587. [[CrossRef](#)].
14. Chen Y, Chen Q, Ma H, Chen S, Fei Q. Transfer machine learning framework for efficient full-field temperature response reconstruction of thermal protection structures with limited measurement data. *Int J Heat Mass Transf.* 2025;242:126785. [[CrossRef](#)].
15. Parish E, Yano M, Tezaur I, Iliescu T. Residual-based stabilized reduced-order models of the transient convection–diffusion–reaction equation obtained through discrete and continuous projection. *Arch Comput Methods Eng.* 2025;32(3):1885–929. [[CrossRef](#)].
16. Choi Y, Carlberg K. Space–time least-squares petrov–Galerkin projection for nonlinear model reduction. *SIAM J Sci Comput.* 2019;41(1):A26–58. [[CrossRef](#)].
17. Lauzon JT, Cheung SW, Shin Y, Choi Y, Copeland DM, Huynh K. S-OPT: a points selection algorithm for hyper-reduction in reduced order models. *SIAM J Sci Comput.* 2024;46(4):B474–501. [[CrossRef](#)].
18. Jiang G, Wang W, Cheng C. A hyper reduction model for fast thermo-mechanical coupling analysis of high-temperature components. *J Therm Stress.* 2025;48(6):707–32. [[CrossRef](#)].
19. Fleres D, De Gregoriis D, Atak O, Naets F. Non-intrusive parametric hyper-reduction for nonlinear structural finite element formulations. *Comput Methods Appl Mech Eng.* 2025;434:117532. [[CrossRef](#)].
20. Roy AD, Dhiman SK. Solutions of one-dimensional inverse heat conduction problems: a review. *Trans Can Soc Mech Eng.* 2023;47(3):271–85. [[CrossRef](#)].
21. Dinmohammadi A, Jafarabadi A. Inverse heat conduction problem with a nonlinear source term by a local strong form of meshless technique based on radial point interpolation method. *Comput Appl Math.* 2023;42(6):284. [[CrossRef](#)].
22. Cherikh MB, Bauzin JG, Hocine A, Peter ZA, Laraqi N. Detection of surface moving heat source using experimental temperature measurements on the opposite surface and inverse techniques. *Int J Heat Mass Transf.* 2024;219:124840. [[CrossRef](#)].

23. Rashedi K. Finding a time-dependent reaction coefficient of a nonlinear heat source in an inverse heat conduction problem. *J Math Model.* 2024;12(1):17–32. [[CrossRef](#)].
24. Zhou ZY, Ruan B, Jiang GH, Xu BB, Liu HY, Zheng YT, et al. A new method to identify non-steady thermal load based on element differential method. *Int J Heat Mass Transf.* 2023;213:124352. [[CrossRef](#)].
25. Zhang C, Li Y, Cui M, Sun C, Gao XW. A novel method for simultaneous determination of thermophysical properties and boundary conditions of phase change problems based on element differential method. *Eng Anal Bound Elem.* 2023;157:1–11. [[CrossRef](#)].
26. Yang Z, Luo X, Liu P, Qiao J, Liu M. Estimation of the total heat exchange factor for the reheating furnace based on the first-optimize-then-discretize approach and an improved hybrid conjugate gradient algorithm. *Mathematics.* 2022;10(21):4074. [[CrossRef](#)].
27. Tan CH, Jiang WW, Zhou YT, Zhang SQ, Yang K, Gao XW. A new method for simultaneous identification of thermal-mechanical loading and thermophysical properties in dynamic coupled thermoelasticity problems based on Levenberg-Marquardt method. *Int Commun Heat Mass Transf.* 2025;169:109869. [[CrossRef](#)].