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Torsional Wave Propagation in the Finitely Pre-Stretched Hollow Bi-Material Compound Circular Cylinder

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Abstract: This paper studies the torsional wave dispersion in the hollow bi-material compounded cylinder with finite initial strains. The investigations are carried out within the scope of the piecewise homogeneous body model with the use of the three-dimensional linearized theory of elastic waves in initially stressed bodies. The mechanical relations of the materials of the cylinders are described through the harmonic potential. The numerical results on the influence of the initial stretching or compression of the cylinders along the torsional wave propagation direction are presented and discussed

Keywords: finite initial strains, wave dispersion control, hollow bi-material compound cylinder, high elastic material, torsional wave.

1 Introduction

The study of propagation laws of the torsional waves in bi-material compounded circular cylinders have been made in many investigations, such as Armenakas (1967, 1971), Reuter (1969), Haines and Lee (1971a, 1971b), Thurston (1976), Kaul, Shaw and Muller (1981), Kleczevski and Parnes (1987) and others. The studies of the corresponding problems in the noted researches have been carried out within the scope of the piecewise homogeneous body model by employing the classical linear theory of elastodynamics. However, the present level of all the areas of natural sciences and engineering requires investigations of such problems with the nonlinear dynamical effects taken into account, one of which is the initial stresses in the cylinders. The initial stresses occur in the structural elements during their manufacture and assembly. Moreover, the initial

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stresses may appear in the structural elements as a result of the changing of the environmental conditions, for example, temperature.

The investigations of the influence of the initial (residual) stresses on the propagation (dispersion) of the elastic waves are very interesting and urgent.

Because, the results of these investigations are used for measuring residual as well as applied stresses in a member of elements of constructions. The laws of wave propagation bodies with initial stresses are the basic of the mentioned measurement system which is called as non-destructive (NDT) stress analysis. The results of the corresponding theoretical and experimental investigations are periodically discussed at NDT conferences, many related articles are published in The Journal of the Acoustical Society of America, NDT International, ets. Note that the mentioned theoretical investigations are usually made within the framework of the Three-dimensional Linearized Theory of Elastic Waves in Initially Stresses Bodies (TLTEWISB) constructed using the linearization principle of the general nonlinear theory of elasticity or its simplified modifications.

A systematic consideration of the results attained by employing TLTEWISB has been given in monographs Eringen and Suhubi (1975), Guz (2004). Moreover, a review of the mentioned investigations was made in papers Guz and Makhort (2000), Guz (2002), Akbarov (2007). It follows from the foregoing references that the investigations of the wave propagation in the pre-stressed cylinders were started by the works Green (1961), Suhubi (1965), Demiray and Suhubi (1970) and others, in which the subject of the study was the homogeneous circular cylinder.

Before the beginning of the 21th century there was not any investigation on the wave propagation problems in the pre-stressed bi-material compounded cylinders. The axisymmetric wave dispersion in the initially stressed bi-material compounded cylinder has been studied in such papers as Akbarov and Guz (2004), Akbarov and Guliev (2008, 2009a, 2009b, 2009c, 2010), Ozturk and Akbarov (2008, 2009a, 2009b) for the recent five years. When the subjects of these papers are viewed, it is seen that the paper Akbarov and Guz (2004) investigates the longitudinal axisymmetric wave propagation in the initially small pre-strained bi-material compounded cylinder. The works Akbarov and Guliev (2008, 2009a, 2009b, 2009c, 2010) develop the investigation started in Akbarov and Guz (2004) for the case where the initial strains in the cylinders are finite ones. In other words, the results attained in the papers Akbarov and Guliev (2008, 2009a, 2009b, 2009c, 2010) can be employed not only for the cases where the cylinders' materials are sufficiently stiff, but also for the cases where these materials are highly elastic.

The subject of the papers Ozturk and Akbarov (2008, 2009a, 2009b) is the investigation of the dispersion relations of the torsional waves in a pre-stressed bi-material compounded cylinder. In these works it is assumed that the initial strains in the constituents are small and these strains are calculated within the scope of the classical linear theory of elasticity. Consequently, the results of the investigations can be employed only for the compounded cylinders made from stiff materials. But these results do not suitable for the compounded cylinders fabricated from the high elastic materials such as elastomers, various type polymers and etc. Therefore in present paper the attempt is made for the development of the investigations carried out in the papers Ozturk and Akbarov (2008, 2009a, 2009b) for the hollow bi-material compound cylinder made from high elastic materials, in other words for the case where the initial strains in the components of the cylinder are finite ones and the magnitude of those are not restricted. In this case, as in Akbarov and Guz (2004), Akbarov and Guliev (2008, 2009a, 2009b, 2009c, 2010), Ozturk and Akbarov (2008, 2009a, 2009b), it is assumed that in each component of the compounded cylinder there exists only the homogenous normal stress acting on the areas which are perpendicular to the lying direction of the cylinders. The mechanical relations of the materials of the cylinders are described through the harmonic potential.

2 Formulation of the problem

We consider the compound (composite) circular cylinder shown in Fig. 1 and assume that in the natural state the radius of the internal circle of the inner hollow cylinder is$R$ and the thickness of the inner and outer cylinders are $h^{(1)}$ and $h^{(2)}$, respectively. In the natural state we determine the position of the points of the cylinders by the Lagrangian coordinates in the Cartesian system of coordinates $Oy_1y_2y_3$ as well as in the cylindrical system of coordinates $Or\theta z$.

Assume that the cylinders have infinite length in the direction of the Oy_3 axis and the initial stress state in each component of the considered body is axisymmetric with respect to this axis and homogeneous. Such ah initial stress field may be present with stretching or compressing of the considered body along the Oy_3 axis. The stretching or compressing may be conducted for the inner hollow cylinder and the external hollow cylinder separately before they are compounded. However, the results which will be discussed below can also be related to the case where the inner and external hollow cylinders are stretched together after the compounding. In this case as a result of the difference of Poisson's coefficients of the inner and external cylinders' materials the inhomogeneous initial stresses acting on the areas which are parallel to the Oy_3 axis arise. Nevertheless, according to the well known mechanical consideration, the mentioned inhomogeneous initial stresses can be neglected under consideration, because these stresses are less significantly than those acting on the areas which are perpendicular to the Oy_3 axis.

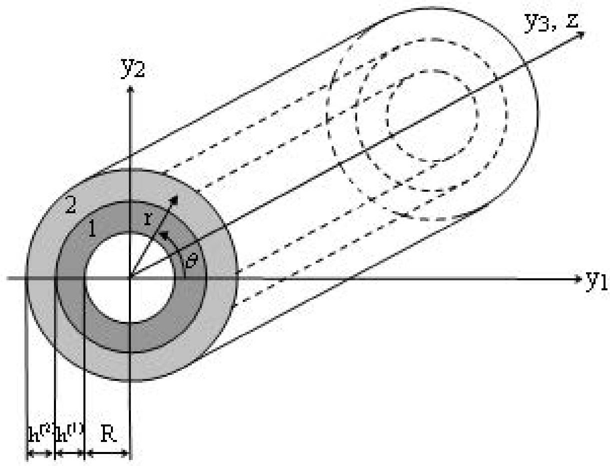

Figure 1: The geometry of the bi-material compound hollow cylinder.

With the initial state of the cylinders we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'z'$ and the Cartesian system of coordinates $O'y_1'y'_2y_3'$. The values related to the inner and external hollow cylinders will be denoted by the upper indices (1) and (2), respectively. Furthermore, we denote the values related to the initial state by an additional upper index, 0. Thus, the initial strain state in the inner and external hollow cylinders can be determined as follows.

$$u_m^{(k),0} = (\lambda_m^{(k)} - 1)y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = const,$$

$$m = 1,2,3; \quad k = 1,2, \tag{1}$$

where $u_m^{(k),0}$ is a displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m axis. We introduce the following notation:

$$y_i' = \lambda_i^{(k)} y_i, \quad r' = \lambda_1^{(k)} r, \quad R' = \lambda_1^{(2)} R. \tag{2}$$

The values related to the system of the coordinates associated with the initial state below, i.e. with $O'y_1'y'_2y_3'$, will be denoted by upper prime.

Within this framework, let us investigate the axisymmetric torsional wave propagation along the $O'y_3'$ axis in the considered body. We make this investigation by the use of coordinates r' and z' in the framework of the TLTEWISB. We will follow the style and notation used in the monograph Guz and Makhort (2000).

Thus, we write the basic relations of the TLTEWISB for the case considered. These relations are satisfied within each constituent of the considered body because we use the piecewise homogeneous body model.

The equations of motion are:

$$\frac{\partial}{\partial r'}Q'^{(k)}_{r'\theta}+\frac{\partial}{\partial z'}Q'^{(k)}_{\theta z}+\frac{1}{r'}\left(Q'^{(k)}_{r'\theta}+Q'^{(k)}_{\theta r'}\right)=\rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{\theta}. \tag{3}$$

The mechanical relations are:

$$Q'^{(k)}_{r'\theta}=\omega'^{(k)}_{1221}\frac{\partial u'^{(k)}_{\theta}}{\partial r'}-\omega'^{(k)}_{1212}\frac{u'^{(k)}_{\theta}}{r'},\quad Q'^{(k)}_{\theta' z'}=\omega'^{(k)}_{1331}\frac{\partial u'^{(k)}_{\theta}}{\partial z'}, \tag{4}$$

In (3) and (4) through the $Q'^{(k)}_{r'\theta}$, $Q'^{(k)}_{\theta z'}$ the perturbation of the components of Kirchhoff stress tensor are denoted. The notation $u'^{(k)}_{\theta}$ shows the perturbations of the components of the displacement vector. The constants $\omega'^{(k)}_{1221}, \omega'^{(k)}_{1212}$ and $\omega'^{(k)}_{3113}$ in (3), (4) are determined through the mechanical constants of the inner and outer cylinders' materials and through the initial stress state. $\rho'^{(k)}$ is a density of the k-th material.

As it has been noted above, in the present investigation we assume that the elasticity relations of the cylinders' materials are described by harmonic potential. This potential is given as follows:

$$\Phi=\frac{1}{2}\lambda s_1^2+\mu s_2 \tag{5}$$

where

$$\begin{aligned} s_1 &= \sqrt{1+2\varepsilon_1}+\sqrt{1+2\varepsilon_2}+\sqrt{1+2\varepsilon_3}-3\\ s_2 &= \left(\sqrt{1+2\varepsilon_1}-1\right)^2+\left(\sqrt{1+2\varepsilon_2}-1\right)^2+\left(\sqrt{1+2\varepsilon_3}-1\right)^2. \end{aligned} \tag{6}$$

In relation (6) λ, μ are material constants, $\varepsilon_i (i=1,2,3)$ are the principal values of the Green's strain tensor. The expressions (5) and (6) are supplied by the corresponding indices under solution procedure.

For the considered axisymmetric initial strain state the components of the Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{rr}=\frac{\partial u_r}{\partial r}+\frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2+\frac{1}{2}\left(\frac{\partial u_\theta}{\partial r}\right)^2+\frac{1}{2}\left(\frac{\partial u_z}{\partial r}\right)^2,$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{1}{2r^2}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right)^2 + \frac{1}{2r^2}\left(\frac{\partial u_\theta}{\partial \theta} - u_r\right)^2 + \frac{1}{2r^2}\left(\frac{\partial u_z}{\partial \theta}\right)^2,$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\frac{\partial u_r}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2,$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right),$$

$$\varepsilon_{r\theta} =$$

$$\frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial r}\left(\frac{\partial u_r}{\partial \theta} - u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial r}\left(\frac{\partial u_\theta}{\partial \theta} - u_\theta\right) + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_r}{\partial z}\left(\frac{\partial u_r}{\partial \theta} - u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial z}\left(\frac{\partial u_\theta}{\partial \theta} - u_\theta\right) + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\frac{\partial u_z}{\partial z}\right) \tag{7}$$

In this case the physical components $S_{(ij)}$ of the Lagrange stress tensor are determined as follows:

$$S_{(ij)} = \frac{1}{2}\left(\frac{\partial}{\partial \varepsilon_{(ij)}} + \frac{\partial}{\partial \varepsilon_{(ji)}}\right)\Phi, \tag{8}$$

where indices $(ij) = (11)$, (12), (13), (22), (23) and (33) indicate the symbols rr, $r\theta$, rz, $\theta\theta$, θz and zz respectively.

In this case the equation of motion is following one.

$$\nabla_i q^{ij} = \rho\frac{\partial^2 u^j}{\partial t^2}, \quad q^{ij} = S^{in}\left(g_n^j + \nabla_n u^j\right) \tag{9}$$

Here q^{ij} and S^{in} are the contravariant components of the Kirchhoff and Lagrange stress tensors, respectively, u^j is a contravariant component of the displacement vector, g_n^jmixed component of the metric tensor. Moreover, in (9) ∇_ishows the covariant derivative and ρ indicates the density of a material.

It is known that the following relation exists between the physical and contravariant components of tensors.

$$q_{(ij)} = q^{ij}H_iH_j \quad S_{(ij)} = S^{ij}H_iH_j, \quad u_{(i)} = u^iH_i \tag{10}$$

Using the expressions (5)- (10) it can be obtained the equation of motion in the cylindrical system of coordinates.

Note that the expressions (5)-(8) are written in the arbitrary system of cylindrical coordinate system without any restriction related to the association of this system to the natural or initial state of the considered compound cylinders. As well as the equations (9) and (10) are written in an arbitrary curvilinear system of coordinates.

According to the problem statement and according to the equations (1), (2), (5)-(8), we obtain the following expressions for the initial stress-strain state.

$$S_{zz}^{(k),0} = \left[\lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right) + 2\mu^{(k)}\left(\lambda_3^{(k)} - 1\right)\right]\left(\lambda_3^{(k)}\right)^{-1},$$

$$S_{(nm)}^{(k),0} = 0 \quad if\ (nm) \neq zz,$$

$$\lambda_2^{(k)} = \lambda_1^{(k)} = \left[2 - \frac{\lambda^{(k)}}{\mu^{(k)}}\left(\lambda_3^{(k)} - 3\right)\right]\left[2\left(\frac{\lambda^{(k)}}{\mu^{(k)}} + 1\right)\right]^{-1}, \tag{11}$$

By linearization of the equations (7)-(9) with respect to the perturbation of the rotational displacement of the cylinders (denoted by $u_\theta^{(k)}$ ($u'^{(k)}_\theta$) in the system of coordinate $Or\theta z$ ($O'r'\theta'z'$)) we attain the equations (3), (4) and the following expressions for the components $\omega'^{(k)}_{1221}$, $\omega'^{(k)}_{1212}$ and $\omega'^{(k)}_{1331}$.

$$\omega'^{(k)}_{1221} = \omega'^{(k)}_{1212} = \frac{\mu^{(k)}}{\lambda_3^{(k)}},$$

$$\omega'^{(k)}_{1331} = \frac{\lambda_1^{(k)}}{\lambda_1^{(k)} + \lambda_3^{(k)}}\left(2\mu^{(k)} - \lambda^{(k)}\left(2\lambda_1^{(k)} + \lambda_3^{(k)} - 3\right)\right) + \frac{1}{\lambda_3^{(k)}}S_{33}^{(k),0}. \tag{12}$$

Thus, the torsional wave propagation in the bi-material compounded cylinder will be investigated by the use of the equations (3), (4) and (12). In this case we will assume that the following complete contact conditions and boundary conditions are satisfied:

$$Q'^{(1)}_{r\theta}\Big|_{r'=\lambda_2^{(1)}R} = 0,$$

$$Q'^{(1)}_{r\theta}\Big|_{r'=\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)} = Q'^{(2)}_{r\theta}\Big|_{r'=\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)},$$

$$u'^{(1)}_{\theta}\Big|_{r'=\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)} = u'^{(2)}_{\theta}\Big|_{r'=\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)}$$

$$Q'^{(2)}_{r\theta}\Big|_{r'=\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(\lambda_2^{(1)}R\right)\right)}=0. \tag{13}$$

In this way, the investigation of the considered wave dispersion problem is reduced to the study of the eigen-value problem formulated through the equations (3), (4), (12) and condition (13). Note that in the case where the initial strains are absent in the constituents in the cylinder, i.e. in the case where $\lambda_1^{(k)}=\lambda_2^{(k)}=\lambda_3^{(k)}=1.0$, the foregoing formulation coincide with the corresponding one proposed in the scope of the classical linear elastodynamics.

3 Solution procedure and obtaining the dispersion relation

According to the monograph Guz (2004), for solution of the equations (3) and (4) we use the following presentation.

$$u'^{(m)}_{\theta}(r',z',t)=-\frac{\partial}{\partial r'}\Psi^{(m)}(r',z',t) \tag{14}$$

where the function $\Psi^{(m)}$ in (14) satisfies the equation written below.

$$\left[\Delta'+\left(\xi'^{(m)}\right)^2\frac{\partial^2}{\partial z'^2}-\frac{\rho'}{\omega'_{1221}}\frac{\partial^2}{\partial t^2}\right]\Psi=0, \tag{15}$$

where

$$\Delta'=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'},\quad \left(\xi'^{(m)}\right)^2=\frac{2\left(\lambda_3^{(m)}\right)^3}{\left(\lambda_2^{(m)}\right)^2\left(\lambda_2^{(m)}+\lambda_3^{(m)}\right)}. \tag{16}$$

It follows from the problem statement that the presentation

$$\Psi^{(m)}(r',z',t)=\psi^{(m)}\left(r'\right)\exp i(kz'-\omega t) \tag{17}$$

holds. Thus, we obtain from (15), (17) the following equation for unknown function $\psi^{(m)}\left(r'\right)$.

$$\left[\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'}-\left(k^2\left(\xi'^{(m)}_1\right)^2-\frac{\rho'^{(m)}\omega^2}{\omega'^{(m)}_{1221}}\right)\right]\psi^{(m)}\left(r'\right)=0. \tag{18}$$

Introducing the notation

$$\left(s^{(m)}\right)^2=\left(k^2\left(\xi'^{(m)}_1\right)^2-\frac{\rho'^{(m)}\omega^2}{\omega'^{(m)}_{1221}}\right) \tag{19}$$

The solution to the equation (18) can be written as follows.

$$\psi'^{(1)}(r') = \begin{cases} A^{(1)}J_0\left(s^{(1)}kr'\right) + B^{(1)}Y_0\left(s^{(1)}kr'\right) & if \quad \left(s^{(1)}\right)^2 > 0 \\ A^{(1)}I_0\left(s^{(1)}kr'\right) + B^{(1)}K_0\left(s^{(1)}kr'\right) & if \quad \left(s^{(1)}\right)^2 < 0 \end{cases}, \tag{20}$$

$$\psi'^{(2)}(r') = \begin{cases} A^{(2)}J_0\left(s^{(2)}kr'\right) + B^{(2)}Y_0\left(s^{(2)}kr'\right) & if \quad \left(s^{(2)}\right)^2 > 0 \\ A^{(2)}I_0\left(s^{(2)}kr'\right) + B^{(2)}K_0\left(s^{(2)}kr'\right) & if \quad \left(s^{(2)}\right)^2 < 0 \end{cases}. \tag{21}$$

Using the equations (21), (20), (17), (14) and (4) we obtain the following dispersion equation from the condition (13).

$$\det\left\|\alpha_{ij}\right\| = 0, \quad i;j = 1,2,3,4, \tag{22}$$

where

$$\alpha_{11} = \begin{cases} \frac{\mu^{(1)}}{\lambda_3^{(1)}}\left\{\frac{1}{2}\left[J_0\left(s^{(1)}k\lambda_2^{(1)}R\right) - J_2\left(s^{(1)}k\lambda_2^{(1)}R\right)\right] - \frac{J_1(s^{(1)}k\lambda_2^{(1)}R)}{s^{(1)}k\lambda_2^{(1)}R}\right\}, \\ \left(s^{(1)}\right)^2 > 0 \\ \frac{\mu^{(1)}}{\lambda_3^{(1)}}\left\{-\frac{1}{2}\left[\mathrm{I}_0\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\right) + I_2\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\right)\right] + \frac{\mathrm{I}_1(\left|s^{(1)}\right|k\lambda_2^{(1)}R)}{\left|s^{(1)}\right|k\lambda_2^{(1)}R}\right\}, \\ \left(s^{(1)}\right)^2 < 0 \end{cases}$$

$$\alpha_{12} = \begin{cases} \frac{\mu^{(1)}}{\lambda_3^{(1)}}\left\{\frac{1}{2}\left[Y_0\left(s^{(1)}k\lambda_2^{(1)}R\right) - Y_2\left(s^{(1)}k\lambda_2^{(1)}R\right)\right] - \frac{J_1(s^{(1)}k\lambda_2^{(1)}R)}{s^{(1)}k\lambda_2^{(1)}R}\right\}, \\ \left(s^{(1)}\right)^2 > 0 \\ \frac{\mu^{(1)}}{\lambda_3^{(1)}}\left\{-\frac{1}{2}\left[K_0\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\right) + K_2\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\right)\right] + \frac{K_1(\left|s^{(1)}\right|k\lambda_2^{(1)}R)}{\left|s^{(1)}\right|k\lambda_2^{(1)}R}\right\}, \\ \left(s^{(1)}\right)^2 < 0 \end{cases}$$

$$\alpha_{13} = 0, \quad \alpha_{14} = 0,$$

$$\alpha_{21} = \begin{cases} J_1\left(s^{(1)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right), & \left(s^{(1)}\right)^2 > 0 \\ -I_1\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right), & \left(s^{(1)}\right)^2 < 0 \end{cases},$$

$$\alpha_{22} = \begin{cases} Y_1\left(s^{(1)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right), & \left(s^{(1)}\right)^2 > 0 \\ -K_1\left(\left|s^{(1)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right), & \left(s^{(1)}\right)^2 < 0 \end{cases},$$

$$\alpha_{23} = \begin{cases} -\frac{s^{(2)}}{s^{(1)}} J_1\left(s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right), & \left(s^{(2)}\right)^2 > 0 \\ \frac{\left|s^{(2)}\right|}{\left|s^{(1)}\right|} I_1\left(\left|s^{(2)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right), & \left(s^{(2)}\right)^2 < 0 \end{cases},$$

$$\alpha_{24} = \begin{cases} -\frac{s^{(2)}}{s^{(1)}} \mathrm{Y}_1\left(s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right), & \left(s^{(2)}\right)^2 > 0 \\ \frac{\left|s^{(2)}\right|}{\left|s^{(1)}\right|} K_1\left(\left|s^{(2)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right), & \left(s^{(2)}\right)^2 < 0 \end{cases},$$

$$\alpha_{31} = \begin{cases} \frac{\mu^{(1)}}{\lambda_3^{(1)}} \left\{ \frac{1}{2}\left[J_0\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) - J_2\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)\right] - \right. \\ \left. \frac{J_1\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(1)}\right)^2 > 0 \\ \frac{\mu^{(1)}}{\lambda_3^{(1)}} \left\{ -\frac{1}{2}\left[\mathrm{I}_0\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) + I_2\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)\right] + \right. \\ \left. \frac{I_1\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(1)}\right)^2 < 0 \end{cases}$$

$$\alpha_{32} = \begin{cases} \frac{\mu^{(1)}}{\lambda_3^{(1)}} \left\{ \frac{1}{2}\left[Y_0\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) - Y_2\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)\right] - \right. \\ \left. \frac{Y_1\left(s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{s^{(1)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(1)}\right)^2 > 0 \\ \frac{\mu^{(1)}}{\lambda_3^{(1)}} \left\{ -\frac{1}{2}\left[K_0\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) + K_2\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)\right] + \right. \\ \left. \frac{K_1\left(\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{\left|s^{(1)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(1)}\right)^2 < 0 \end{cases}$$

$$\alpha_{33} = \begin{cases} -\frac{\mu^{(2)}}{\lambda_3^{(2)}} \left\{ \frac{1}{2}\left[J_0\left(s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) - J_2\left(s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)\right] - \right. \\ \left. \frac{J_1\left(s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{s^{(2)} k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(2)}\right)^2 > 0 \\ -\frac{\mu^{(2)}}{\lambda_3^{(2)}} \left\{ -\frac{1}{2}\left[\mathrm{I}_0\left(\left|s^{(2)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right) + I_2\left(\left|s^{(2)}\right| \lambda_2^{(1)} kR\left(1 + h^{(1)}/R\right)\right)\right] + \right. \\ \left. \frac{I_1\left(\left|s^{(2)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)\right)}{\left|s^{(2)}\right| k\lambda_2^{(1)} R\left(1 + h^{(1)}/R\right)} \right\}, \quad \left(s^{(2)}\right)^2 < 0 \end{cases}$$

$$\alpha_{34}=\begin{cases}-\frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{\frac{1}{2}\left[Y_0\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right)-Y_2\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right)\right]-\right.\\ \left.\frac{Y_1\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right)}{s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)}\right\} \quad \left(s^{(2)}\right)^2>0\\ -\frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{-\frac{1}{2}\left[K_0\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right)+K_2\left(\left|s^{(2)}\right|\lambda_2^{(1)}kR\left(1+h^{(1)}/R\right)\right)\right]+\right.\\ \left.\frac{K_1\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)\right)}{\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R\right)}\right\}, \quad \left(s^{(2)}\right)^2<0\end{cases}$$

$$\alpha_{41}=0, \quad \alpha_{42}=0,$$

$$\alpha_{43}=\begin{cases}\frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{\frac{1}{2}\left[J_0\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)-\right.\right.\\ \left.J_2\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)\right]-\\ \left.\frac{J_1\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)}{s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)}\right\}, \quad \left(s^{(2)}\right)^2>0\\ \frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{-\frac{1}{2}\left[I_0\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)+\right.\right.\\ \left.I_2\left(\left|s^{(2)}\right|\lambda_2^{(1)}kR\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)\right]+\\ \left.\frac{I_1\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)\right)}{\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/\left(R\lambda_2^{(1)}\right)\right)}\right\}, \quad \left(s^{(2)}\right)^2<0\end{cases}$$

$$\alpha_{44}=\begin{cases}\frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{\frac{1}{2}\left[\mathrm{Y}_0\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)-\right.\right.\\ \left.Y_2\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)\right]+\\ \left.\frac{Y_1\left(s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)}{s^{(2)}k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)}\right\}, \quad \left(s^{(2)}\right)^2>0\\ \frac{\mu^{(2)}}{\lambda_3^{(2)}}\left\{-\frac{1}{2}\left[K_0\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)+\right.\right.\\ \left.K_2\left(\left|s^{(2)}\right|\lambda_2^{(1)}kR\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)\right]+\\ \left.\frac{K_1\left(\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)\right)}{\left|s^{(2)}\right|k\lambda_2^{(1)}R\left(1+h^{(1)}/R+\lambda_2^{(2)}h^{(2)}/R\lambda_2^{(1)}\right)}\right\}, \quad \left(s^{(2)}\right)^2<0\end{cases} \tag{23}$$

Thus, the dispersion equation for the considered torsional wave propagation problem has been derived in the form presented in (22) and (23).

4 Numerical Results and Discussions

We have found that the first lowest mode which is non-dispersive homogenous hollow cylinder, becomes dispersive for a compound one. The limiting value of the torsional wave speed for the case considered is determined from dispersion equation (22), (23) by using power series expansions of Bessel functions, retaining only the dominant term as $kR \to 0$:

$$\frac{c}{c_2^{(1)}} = \left[\frac{\frac{\mu^{(1)}}{\lambda_3^{(1)}}\left(\xi'^{(1)}\right)^2 + \frac{\mu^{(2)}}{\lambda_3^{(2)}}\alpha\left(\xi'^{(2)}\right)^2}{\mu^{(1)} + \mu^{(2)}\alpha\left(\frac{c_2^{(1)}}{c_2^{(2)}}\right)} \right]^{\frac{1}{2}}, \quad \alpha = \frac{\eta_2^4 - \eta_1^4}{\eta_1^4 - 1}, \tag{24}$$

where

$$\eta_2 = 1 + \frac{h^{(1)}}{R} + \frac{\lambda_2^{(2)}h^{(2)}}{\lambda_2^{(1)}R}, \quad \eta_1 = 1 + \frac{h^{(1)}}{R}, \quad c_2^{(1)} = \sqrt{\frac{\mu^{(1)}}{\rho^{(1)}}}, \quad c_2^{(2)} = \sqrt{\frac{\mu^{(2)}}{\rho^{(2)}}} \tag{25}$$

In equation (24) the values of $\xi'^{(1)}$ and $\xi'^{(2)}$ are determined through the expression (16).

In the case where $\lambda_3^{(m)} = \lambda_2^{(m)} = 1.0$, the expression (24) transforms to the following one.

$$\left(\frac{c}{c_2^{(1)}}\right)^2 = \frac{\mu^{(1)} + \mu^{(2)}\alpha}{\mu^{(1)} + \mu^{(2)}\alpha\left(\frac{c_2^{(1)}}{c_2^{(2)}}\right)}. \tag{26}$$

The expression (26) coincides with the corresponding one attained in the paper Armenakas (1971). Moreover, the expression (24) is a generalization of the corresponding one attained in the paper Ozturk and Akbarov (2008) for the finite initial strain state. Note that in the paper Ozturk and Akbarov (2008) this type expression was obtained for the small initial strain state.

It follows from the expression (24) and (16) that the limit values of $c/c_2^{(1)}$, where $c_2^{(1)} = \sqrt{\mu^{(1)}/\rho^{(1)}}$ decrease with $\mu^{(1)}/\mu^{(2)}$ and increase with $\lambda\left(=\lambda_3^{(1)} = \lambda_3^{(2)}\right)$. Consequently, the initial stretching (compression) of the compound cylinder along the torsional wave propagation direction causes to increase (to decrease) of the limit velocity of this wave as $\kappa R \to 0$. According to the known physical-mechanical consideration, the other limit value of the velocity of the considered wave, i.e. the limit velocity as $\kappa R \to \infty$ must be equal to min $\left\{c_2^{(1)}\left(\lambda_3^{(1)}\right), c_2^{(2)}\left(\lambda_3^{(2)}\right)\right\}$, i.e. the following relation must be hold.

$$c \to \min\left\{c_2^{(1)}\left(\lambda_3^{(1)}\right), c_2^{(2)}\left(\lambda_3^{(2)}\right)\right\} \text{ as } \kappa R \to \infty \tag{27}$$

where

$$c_2^{(\kappa)}\left(\lambda_3^{(\kappa)}\right) = \left\{\frac{2\mu^{(\kappa)}}{\rho^{(\kappa)}}\left(\lambda_1^{(\kappa)}\right)^{-2}\left(\lambda_3^{(\kappa)}\right)^2\left(\lambda_3^{(\kappa)}+\lambda_3^{(\kappa)}\right)^{-1}\right\}^{-1/2}. \tag{28}$$

The expression (28) is called the accusto-elastic relation for initially stressed elastic body and is attained from the equation:

$$\left(c_2^{(\kappa)}\left(\lambda_3^{(\kappa)}\right)\right)^2 = \left[\rho'^{(\kappa)}/\left(\left(\xi'^{(\kappa)}\right)^2 \omega_{1221}^{'(\kappa)}\right)\right]^{-1/2}, \quad \rho'^{(\kappa)} = \rho^{(\kappa)}\lambda_1^{(\kappa)}\lambda_2^{(\kappa)}\lambda_3^{(\kappa)} \tag{29}$$

where $\omega_{1221}^{'(\kappa)}$ and $\left(\xi'^{(\kappa)}\right)^2$ are determined through the expressions (12) and (16), respectively.

It follows from the foregoing results that the limit values of the torsional wave propagation velocity depend not only on the initial strain state and not only the ratio $\mu^{(\kappa)}/\rho^{(\kappa)}\,(\kappa = 1,2)$, but also depend on the rate $\lambda^{(\kappa)}/\mu^{(\kappa)}$. Because, the expression (11) through which the values of $\lambda_2^{(\kappa)}\left(=\lambda_1^{(\kappa)}\right)$ are determined, contains this rate.

Now we consider the results obtained by the numerical solution to the dispersion equation (22) under $\rho^{(2)}/\rho^{(1)} = 1$. This solution is made by employing the "bisection method" algorithm by the use of PC. We verify the validation of this algorithm. For this purpose, as in the paper Armenakas (1971), we assume that $\mu^{(1)}/\mu^{(2)} = 1$, $\rho^{(2)}/\rho^{(1)} = 0.5, h^{(1)}/R = 0.2$, $h^{(2)}/R = 0.2$ and consider the dependence between $\omega h^{(2)}/\left(\pi c_2^{(2)}\right)$ and $2h^{(2)}/\Lambda$, where Λ is the wavelength. Thus, within the framework of the foregoing assumptions, the graphs of these dependencies for the lowest three modes are given in Fig.2 for various values of $\lambda\left(=\lambda_3^{(1)}=\lambda_3^{(2)}\right)$.

Note that the graphs constructed in the case where $\lambda_3^{(2)} = 1.0$ coincide with the corresponding ones given in paper Armenakas (1971). At the same time, it can be seen from Fig.2 that the initial stretching (compression) of the compounded bi-material hollow cylinder causes an increase (a decrease) in the torsional wave propagation velocity.

The results discussed above indicate that the analytical and numerical solution method used in the present investigation is correct. Now we consider the numerical results regarding the dependence between $c/c_2^{(1)}$ and $\kappa R$for various values of the problem parameters.

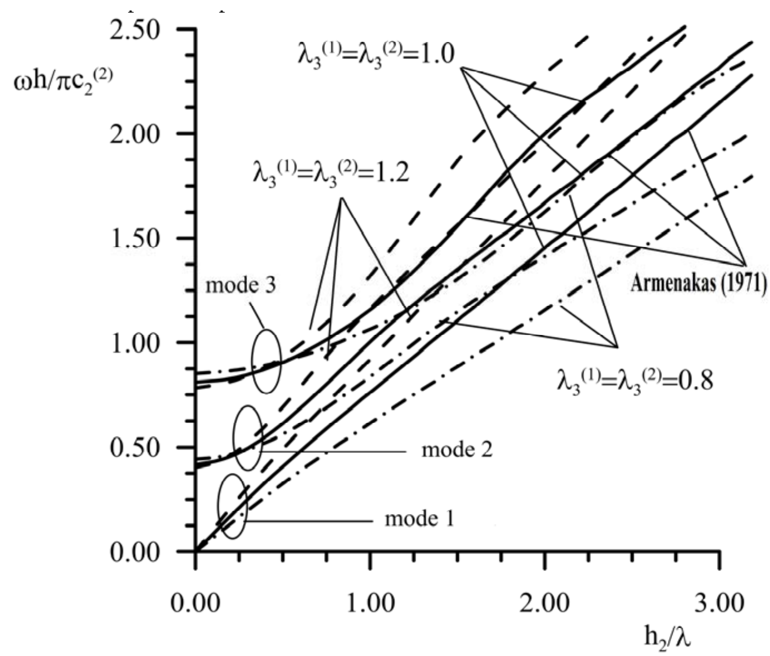

Figure 2: The dispersion diagram for the lowest three modes for various values of $\lambda_3^{(1)}\left(=\lambda_3^{(2)}\right)$.

Fig 3. shows the graphs of the dependencies between $c/c_2^{(1)}$and $\kappa R$for various values $h^{(2)}/R$ under $\mu^{(1)}/\mu^{(2)}=5$, $h^{(1)}/R=0.1$.

But the graphs of the same dependencies constructed for values of the ratio $h^{(1)}/R$ under $\mu^{(1)}/\mu^{(2)}=5$, $h^{(2)}/R=0.1$ are given in Fig.4.

The influence of the ratio $\mu^{(1)}/\mu^{(2)}$ on the behavior of the considered dispersion curves is illustrated by the results given in Fig.5, which are obtained under $h^{(1)}/R=h^{(2)}/R=0.1$.

The conclusions followed from these numerical results are given in the following section.

5 Conclusions

Within the framework of the piecewise homogenous body model with the use of the TLTEWISB, the torsional wave dispersion in the finite pre-strained bi-material

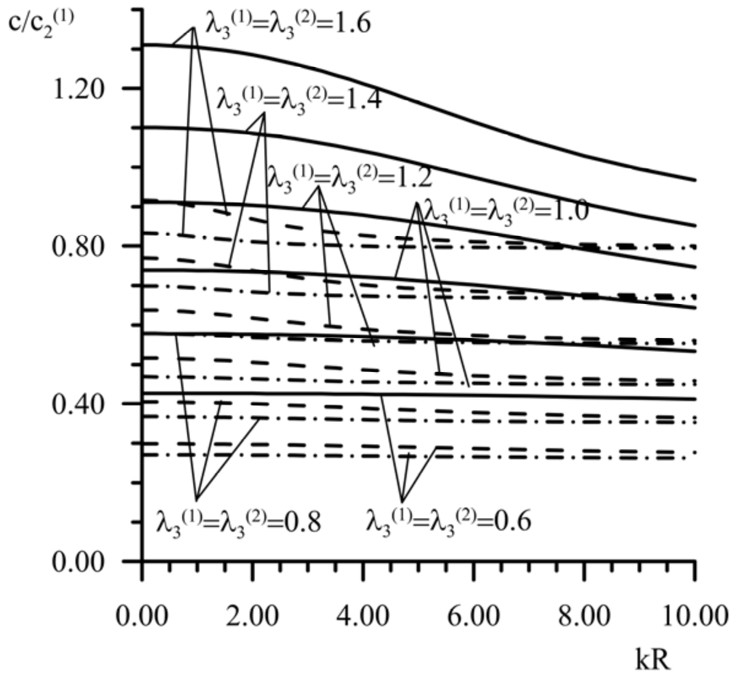

Figure 3: The influence of the initial strains on the dispersion curves constructed for the various values of $h^{(2)}/R$ under $\mu^{(1)}/\mu^{(2)} = 5$, $h^{(1)}/R = 0.1$; $h^{(2)}/R = 0.1; h^{(2)}/R = 0.5; h^{(2)}/R = 1.0$

compounded hollow cylinder made from high elastic materials was investigated. In this case, it was assumed that the mechanical relations for the components of the cylinder are described through the harmonic potential.

The corresponding dispersion equation was derived and analytical expressions (24) and (27) are found for the limiting value of the velocity of the lowest dispersive mode from this dispersion equation.

The algorithm was developed for doing numerical investigations and this algorithm, first, was tested on the known problem which had been investigated by the other authors. The basic numerical investigations were made for the case where the material of the inner hollow cylinder is stiffer than that of the external

hollow cylinder, i.e. for the case where $\mu^{(1)}/\mu^{(2)} > 1$. Concrete numerical results are presented for the case where the initial strains in the cylinders are equal to each other, i.e. $\lambda_3^{(1)} = \lambda_3^{(2)}$. According to these results the following concrete conclusions are indicated.

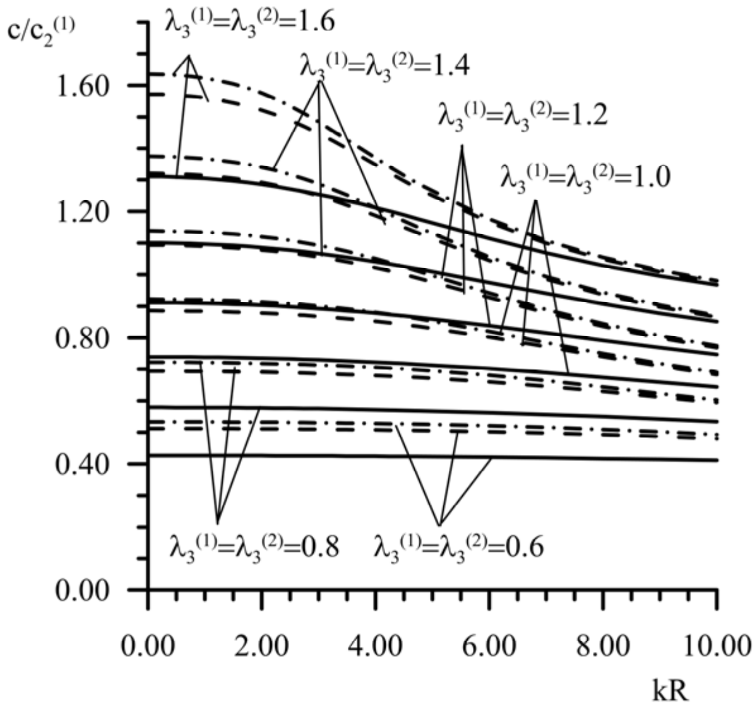

Figure 4: The influence of the initial strains on the dispersion curves constructed for the various values of $h^{(1)}/R$ under $\mu^{(1)}/\mu^{(2)} = 5$, $h^{(2)}/R = 0.1$; $h^{(1)}/R = 0.1; h^{(1)}/R = 0.5 h^{(1)}/R = 1.0$

The velocity of the torsional wave propagation $\mu^{(1)}/\mu^{(2)}$, i.e. with decreasing of the stiffness of the outer hollow cylinder material in the considered body decrease with i.e. the values of $c/c_2^{(1)}$ ($c_2^{(1)} = \sqrt{\mu^{(1)}/\rho^{(1)}}$)

The values of $c/c_2^{(1)}$increase (decrease) with initial stretching (compression) of the cylinder;

The influence of the initial strains of the cylinders on the torsional wave propagation velocity in that decrease (increase) $h^{(2)}/R\left(h^{(1)}/R\right)$. Because the material of the inner hollow cylinder is stiffer than that of the outer hollow cylinder.

Moreover, the foregoing results agree qualitatively with the results attained in the paper Ozturk and Akbarov (2008).

The results and the method of the present investigation can be used for determination and controlling of the noise of the bi-layered polymer pipes which are used for transmitting of various type liquids. Because the initial strains in these tubes may

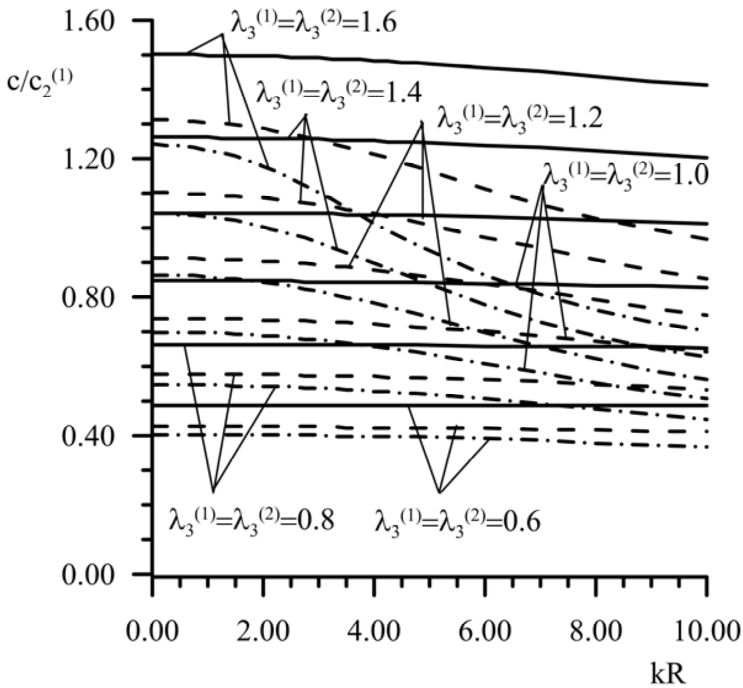

Figure 5: The dispersion curves constructed for various values of $\mu^{(1)}/\mu^{(2)}$ under $h^{(1)}/R = h^{(2)}/R = 0.1$, $h^{(1)}/R = 0.1$; $\mu^{(1)}/\mu^{(2)} = 2$; $\mu^{(1)}/\mu^{(2)} = 5$; $\mu^{(1)}/\mu^{(2)} = 10$

be arise as a result of corresponding manufacturing procedures, as well as, as a result of the change of the environmental temperature. Consequently, the knowledge on the influence of the considered initial strains on the wave propagation velocity can be used, for example, for decaying of the noises which arise under fluid flowing in the bi-layered pipes. At the same time, the results of these investigations can be used for the determination of the residual and applied stress in the bi-layered high elastic materials under non-destructive stress analyses Guz and Makhort (2000), Guz (2004), Rose (2004). In this reason, the results obtained in the present paper can be also taken as a little contribution to the theoretical bases of the non-destructive stress analyses of the many-layered polymer (high-elastic) materials. In addition, the expression (24) obtained in the present paper for asymptotic values of the torsional wave propagation velocity can be directly used for determination of the influence of the considered initial strains on the values of the effective shear modulus of the composite material consisting of the layers of the bi-layered hollow cylinder.

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