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ARTICLE

Finite Element Analysis of Micromorphic Electrodynamics

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ABSTRACT: The key points of micromorphic theory, including the balance laws and entropy principle, are briefly introduced. Maxwell's equations and the Lorentz Transformation of **E** and **B** fields in both relativistic and non-relativistic electromagnetic theory are discussed. The link between the thermomechanical part and the electromagnetic part of the micromorphic electromagnetic theory is established through the body force, body moment, and energy source. The constitutive theory for thermo-visco-elastic-plastic-electromagnetic (TVEP-EM) materials is formulated. Then the constitutive relations are reduced to the materially linear constitutive equations. *Onsager's postulate* is utilized for the derivation of viscosity. *Return-Mapping-Algorithm* is invoked for plasticity. It is a well-known physical fact that the electric field **E** and the magnetic flux **B** are not independent of each other. To resolve this problem, the scalar potential ϕ and the vector potential **A** are introduced and derived, which are related to the electric field and magnetic flux as $\mathbf{B} \equiv \nabla \times \mathbf{A}$ and $\mathbf{E} \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$. Finite element formulations are rigorously derived. On each node, there are displacements, micromotions, temperature, scalar, and vector potentials. It is numerically impossible and physically meaningless to solve the five sets of finite element equations simultaneously. We propose to solve the problem of a hollow cylinder subjected to twist in two stages. In the first stage, the static or nearly static solutions for displacements, micromotions, plastic strains, and temperatures are obtained. In the second stage, the propagation of scalar and vector potentials under the influence of deformations and temperature gradients is investigated. The material of micromorphic theory can contain more complex substances, so it can be utilized to treat blood, bubbly fluids, liquid crystals, etc. Incorporating the coupling between thermomechanics and electromagnetics in micromorphic theory can further enhance the understanding and prediction of large classes of physical phenomena and provide many technological applications. Phenomenologically important cross-effects, such as Peltier, Seebeck, Hall, Ettingshausen, Righi-Leduc, and Nernst effects, can now be studied theoretically and numerically.

KEYWORDS: Micromorphic theory; thermomechanical-electromagnetic coupling; plasticity; finite element formulation; Maxwell equations

1 Introduction

Micromorphic theory envisions a material body as a continuous collection of deformable particles; each possesses finite size and inner structure. On the other hand, classical continuum mechanics envisions a material body as a continuous collection of material points, each with infinitesimal size and no inner structure. The purpose of going beyond the classical continuum mechanics is to take into account the microstructure of the material body in question while still keeping the advantages of continuum theory intact. Because micromorphic theory can incorporate more complex substances, it can be utilized to treat blood, bubbly fluids, liquid crystals [1–5], etc. Incorporating the coupling between thermomechanics and electromagnetics into micromorphic theory can further enhance the understanding and the prediction of large classes of physical phenomena and provide innumerable technological applications. To cite a few, piezoelectric crystals, magnetic memory devices, energy generation by means of controlled plasma, and the study of many natural phenomena, such as earth magnetism and stellar dynamics. Phenomenologically important cross effects such as electostriction, magnetostriction, Peltier, Seebeck, Hall, Ettingshausen, Righi-Leduc, and Nernst effects can now be studied theoretically and numerically [6–8].

Microcontinuum field theories constitute extensions of the classical field theories concerned with deformations, motions, and electromagnetic interactions of material media as continua. It is emphasized that in the classical continuum theory, a point is represented by a geometrical point, infinitesimal in size and without inner structure. Then the question becomes *How can one represent the intrinsic deformation of a point particle in a microcontinuum?* Eringen settled this question by replacing the deformable point particle with a geometric point P and some vectors attached to P, which denote the orientations and intrinsic deformations of the material points in the deformable point particle [5,7,9]. Geometrically, a particle P is identified with its position vector $\mathbf{X}$ in a Lagrangian coordinate system and vectors Ξ^{α} $(\alpha = 1, 2, 3, ..., N)$ attached to P, representing the inner structure of P. Here N is the number of discrete material points in the particle. Now the motions may be expressed as

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{1}$$

$$\boldsymbol{\xi}^{\alpha} = \boldsymbol{\xi}^{\alpha}(\mathbf{X}, \Xi^{\alpha}, t), \qquad \alpha \in (1, 2, 3, ..., N) \tag{2}$$

where t is the time, $\boldsymbol{\xi}^{\alpha}$ is the Eulerian coordinates of α-th material point in the particle. A medium with such general motions was named by Eringen as the microcontinuum of grade N. Obviously, this is too complicated. Therefore, in a so-called two-level continuum model, first let the position vector of a material point be decomposed as the sum of the position vector of the centroid of the particle and the position vector of a material point relative to the centroid:

$$\mathbf{x}' = \mathbf{x} + \xi, \qquad \mathbf{X}' = \mathbf{X} + \Xi \tag{3}$$

Then let the motions be reduced to

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \qquad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X}, \Xi, t) \tag{4}$$

If the micromotions $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X}, \Xi, t)$ is further reduced to an affine motion, i.e.,

$$\boldsymbol{\xi} = \chi_K(\mathbf{X}, t)\,\Xi_K \quad \text{or} \quad \xi_k = \chi_{kK}(\mathbf{X}, t)\,\Xi_K \tag{5}$$

Then we arrive at the doorstep of the micromorphic theory.

The ultimate goal of this work is to formulate the finite element equations for a micromorphic electromagnetic continuum. To begin with, notice that the balance laws of micromorphic electromagnetic continuum consist of two parts: the thermomechanical (TM) part and the electromagnetic (EM) part. In Section 2, the electromagnetic (EM) balance laws, i.e., the famous Maxwell's equations, are introduced. We also introduce the Lorentz Transformation of $\mathbf{E}$ and $\mathbf{B}$ fields in relativistic electromagnetic theory [10], as well as in its special case, non-relativistic electromagnetic theory. In Section 3, the Eulerian description of the basic laws of micromorphic theory, including *Conservation of Mass*, *Conservation of Microinertia, Balance of Linear Momentum, Balance of Moments of Momentum, Conservation of Energy, and Entropy Principle*, is introduced. Also included in the basic laws of micromorphic theory are the electromagnetic (EM) part of the body force, body moment, and energy source [6,7,11–13]. It is emphasized that this inclusion establishes the link between the thermomechanical part (TM) and the electromagnetic part (EM) of the micromorphic electromagnetic theory. The constitutive theory for thermo-visco-elastic-plastic-electromagnetic (TVEP-EM) materials is formulated in Section 4. The formulation of constitutive theory, including plasticity, is unique in the sense that one needs to add a set of internal variables to the list of the independent constitutive variables and, of course, one needs to supply a set of governing equations for the newly added internal variables [5,14]. The materially linear constitutive equations are formulated in Section 5. It is emphasized that this work is based on *geometrically nonlinear*, or say large-strain, approaches. *Onsager's postulate* is utilized for the derivation of viscosity. *Return-Mapping-Algorithm* is invoked for plasticity.

It is a well-known physical fact that the electric field **E** and the magnetic flux **B** are not independent from each other, nor are the dielectric displacement **D** and the magnetic field **H**. To resolve this problem, scalar potential ϕ and vector potential **A** are introduced in Section 6. The scalar potential ϕ and the vector potential **A** are related to the electric field **E** and magnetic flux **B** [10] as

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \tag{6}$$

$$\mathbf{E} \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \tag{7}$$

Eventually, the governing equations for ϕ and **A**, not **E** and **B**, are obtained in Section 6. In Section 7, for finite element formulation, we link the displacement **U**, micromotion $\boldsymbol{\chi}$, temperature **T**, scalar potential $\boldsymbol{\varphi}$, and vector potential **A** at a generic point in a finite element to the corresponding nodal values associated with that finite element. Finally, the finite element equations for $U_{K\beta}$, $\chi_{kK\beta}$, T_β, ϕ_β, and $A_{k\beta}$ are obtained. It is noticed that the governing equations for displacements **u** and micromotions $\boldsymbol{\chi}$ are mainly acoustic wave equations; the governing equations for temperature **T** are mainly diffusion equations; The governing equations for scalar potential $\boldsymbol{\varphi}$ and vector potential **A** are mainly wave equations with wave speed in the order of the speed of light. The speed of acoustic waves and the speed of light are not in the same order of magnitude. To solve the five sets of differential equations, Eqs. (172)–(176), simultaneously is numerically impossible and physically meaningless. In Section 8, we propose to solve the problem in two stages. To the electromagnetic waves in terms of the scalar and vector potentials, everything else seems to stand still. Therefore, in $Stage\,I$, we solve Eqs. (172)–(174) dynamically and seek the static or nearly static solutions. In $Stage\,II$, based on the static or nearly static solutions obtained in $Stage\,I$, namely **u**, T, and $\boldsymbol{\chi}$, we calculate the forcing terms for the differential equations for ϕ and **A**. Discussions and conclusions are given in Section 9.

2 Maxwell's Equations

The balance laws of micromorphic electromagnetic continuum consist of two parts: the thermomechanical (TM) part and the electromagnetic (EM) part. The EM balance laws are the well-known Maxwell's equations that can be expressed in the Heaviside-Lorentz system [7,10,12] as

$$\nabla \cdot \mathbf{D} = q \quad \text{or} \quad D_{k,k} = q \tag{8}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \text{or} \quad \varepsilon_{ijk}E_{k,j} + \frac{1}{c}\frac{\partial B_i}{\partial t} = 0 \tag{9}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{or} \quad B_{k,k} = 0 \tag{10}$$

$$\nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{c}\mathbf{J} \quad \text{or} \quad \varepsilon_{ijk}H_{k,j} - \frac{1}{c}\frac{\partial D_i}{\partial t} = \frac{1}{c}J_i \tag{11}$$

where **D** is the dielectric displacement vector, **B** the magnetic induction vector, **E** the electric field vector, **H** the magnetic field vector, **J** the total current vector, and q the free charge density. Define the polarization vector **P** and the magnetization vector **M** as

$$\begin{aligned}&\mathbf{P} = \mathbf{D} - \mathbf{E} \quad \text{or} \quad P_k = D_k - E_k \\ &\mathbf{M} = \mathbf{B} - \mathbf{H} \quad \text{or} \quad M_k = B_k - H_k\end{aligned} \tag{12}$$

It is noticed that the quantities $q, \mathbf{J}, \mathbf{D}, \mathbf{E}, \mathbf{P}, \mathbf{B}, \mathbf{H}, \mathbf{M}$ are all referred to a fixed laboratory frame R_G. On the other hand, $q^*, \mathbf{J}^*, \mathbf{D}^*, \mathbf{E}^*, \mathbf{P}^*, \mathbf{B}^*, \mathbf{H}^*, \mathbf{M}^*$ are referred to a co-moving reference frame R_C.

In relativistic electromagnetic theory, the Lorentz Transformation of **E** and **B** fields can be expressed as [10]

$$\mathbf{E}^* = \gamma\left(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}\right) \tag{13}$$

$$\mathbf{B}^* = \gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}\right) \tag{14}$$

where $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}, \beta = \sqrt{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}, \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$, and $\mathbf{v}$ is the velocity of co-moving frame R_C relative to the fixed laboratory frame R_G, and c is the speed of light in vacuum. Jackson [10] also explained that the inverse can be found by interchanging the star and the no-star quantities and by changing $\boldsymbol{\beta}$ to $-\boldsymbol{\beta}$, i.e.,

$$\mathbf{E} = \gamma\left(\mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}^*\right) \tag{15}$$

$$\mathbf{B} = \gamma\left(\mathbf{B}^* + \boldsymbol{\beta} \times \mathbf{E}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}^*\right) \tag{16}$$

The proof of Jackson's formula, Eqs. (13)–(16), and further discussion of the Lorentz Transformation are included in Appendix A.

Following the same pattern of Eqs. (13)–(16), one may write and prove the validity of

$$\mathbf{D}^* = \gamma\left(\mathbf{D} + \boldsymbol{\beta} \times \mathbf{H}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{D}\right), \quad \mathbf{D} = \gamma\left(\mathbf{D}^* - \boldsymbol{\beta} \times \mathbf{H}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{D}^*\right) \tag{17}$$

$$\mathbf{H}^* = \gamma\left(\mathbf{H} - \boldsymbol{\beta} \times \mathbf{D}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{H}\right), \quad \mathbf{H} = \gamma\left(\mathbf{H}^* + \boldsymbol{\beta} \times \mathbf{D}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{H}^*\right) \tag{18}$$

$$\mathbf{M}^* = \gamma\left(\mathbf{M} + \boldsymbol{\beta} \times \mathbf{P}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{M}\right), \quad \mathbf{M} = \gamma\left(\mathbf{M}^* - \boldsymbol{\beta} \times \mathbf{P}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{M}^*\right) \tag{19}$$

$$\mathbf{P}^* = \gamma\left(\mathbf{P} - \boldsymbol{\beta} \times \mathbf{M}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{P}\right), \quad \mathbf{P} = \gamma\left(\mathbf{P}^* + \boldsymbol{\beta} \times \mathbf{M}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{P}^*\right) \tag{20}$$

Recall that $\gamma \equiv \left(1-\beta^2\right)^{-\frac{1}{2}}$, the Taylor series expansion of γ about $\beta = 0$ is obtained as

$$\gamma \approx 1 + \frac{1}{2}\beta^2 + \frac{3}{4}\beta^4 + \ldots. \tag{21}$$

Let's keep the Taylor series at most to β^2. Then, for non-relativistic electromagnetic theory, we have

$$\mathbf{E}^* = \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}, \quad \boldsymbol{E} = \mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^* \tag{22}$$

$$\mathbf{B}^* = \mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}, \quad \mathbf{B} = \mathbf{B}^* + \boldsymbol{\beta} \times \mathbf{E}^* \tag{23}$$

$$\mathbf{D}^* = \mathbf{D} + \boldsymbol{\beta} \times \mathbf{H}, \quad \mathbf{D} = \mathbf{D}^* - \boldsymbol{\beta} \times \mathbf{H}^* \tag{24}$$

$$\mathbf{H}^* = \mathbf{H} - \boldsymbol{\beta} \times \mathbf{D}, \quad \mathbf{H} = \mathbf{H}^* + \boldsymbol{\beta} \times \mathbf{D}^* \tag{25}$$

$$\mathbf{P}^* = \mathbf{P} - \boldsymbol{\beta} \times \mathbf{M}, \quad \mathbf{P} = \mathbf{P}^* + \boldsymbol{\beta} \times \mathbf{M}^* \tag{26}$$

$$\mathbf{M}^* = \mathbf{M} + \boldsymbol{\beta} \times \mathbf{P}, \quad \mathbf{M} = \mathbf{M}^* - \boldsymbol{\beta} \times \mathbf{P}^* \tag{27}$$

Also, it is noticed that Eringen [6] gave a detailed discussion and derivation and showed that

$$q^* = q, \quad \mathbf{J}^* = \mathbf{J} - q\mathbf{v} \tag{28}$$

3 Balance Laws of Micromorphic Theory

To begin with, let's briefly derive the balance laws of micromorphic theory [7]. There are three parts:

(I) The concepts of mass and inertia require a finite volume of the material body. Before taking a limiting process of volume densities, consider a particle P having volume element ΔV in the reference state and its image p in the spatial frame at time t. The total mass of these particles is the sum of the masses of microelements, i.e.,

$$\rho_o \Delta V = \int_{\Delta V} \rho'_o dV', \quad \rho \Delta v = \int_{\Delta v} \rho' dv' \tag{29}$$

where the primed quantities refer to microelements of P and p. The relative position vector $\boldsymbol{\Xi}$ and $\boldsymbol{\xi}$ of the microelements are taken with respect to the centroids of ΔV and Δv, so that

$$\int_{\Delta V} \rho'_o \boldsymbol{\Xi} dV' = 0, \quad \int_{\Delta v} \rho' \boldsymbol{\xi} dv' = 0 \tag{30}$$

However, the second moments of $\rho'_o dV'$ and $\rho' dv'$ do not vanish and they are given by

$$\rho_o I_{KL} \Delta V = \int_{\Delta V} \rho'_o \Xi_K \Xi_L dV', \quad \rho i_{kl} \Delta v = \int_{\Delta v} \rho' \xi_k \xi_l dv' \tag{31}$$

We assume the mass of the microelement is conserved during the motion

$$\rho'_o dV' = \rho' dv' \tag{32}$$

This implies that, in the limit as $\Delta V \to 0$ and $\Delta v \to 0$,

$$\rho_o dV = \rho dv \tag{33}$$

$$i_{kl} = I_{KL} \chi_{kK} \chi_{lL}, \quad I_{KL} = i_{kl} \overline{\chi}_{Kk} \overline{\chi}_{Ll} \tag{34}$$

These two balance laws can be re-written as

$$\frac{d}{dt} \int_{v} \rho dv = 0 \tag{35}$$

$$\frac{d}{dt} \int_{v} \rho i_{kl} \overline{\chi}_{Kk} \overline{\chi}_{Ll} dv = 0 \tag{36}$$

(II) In classical continuum mechanics the energy associated with stresses can be written as $\int\limits_{\Delta a} t_{kl} v_l da_k$. Now in micromorphic theory, follow Eqs. (4) and (5), we write

$$\int\limits_{\Delta a} t'_{kl} \left(v_l + \dot{\xi}_l\right) da'_k = \left(t_{kl} v_l + m_{klm} \upsilon_{lm}\right) \Delta a_k \tag{37}$$

where the microgyration tensor υ_{kl}, moment stress tensor m_{klm}, and body moment ρl_{kl} are defined through

$$\dot{\xi}_k = \upsilon_{kl} \xi_l \tag{38}$$

$$m_{klm} \Delta a_k = \int\limits_{\Delta a} t'_{kl} \xi_m da'_k \tag{39}$$

$$\rho l_{lm} \Delta v = \int\limits_{\Delta v} \rho' f'_l \xi_m dv' \tag{40}$$

The kinetic energy per unit mass can now be expressed as

$$K = \frac{1}{2} \left\{v_k v_k + i_{kl} \upsilon_{mk} \upsilon_{ml}\right\} \tag{41}$$

Therefore, for micromorphic theory, the conservation of energy can be expressed as

$$\begin{aligned}\frac{d}{dt} \int\limits_v \rho \left(e + K\right) dv = &\oint\limits_s \left(t_{kl} v_l + m_{klm} \upsilon_{lm} - q_k\right) da_k \\ &+ \int\limits_v \rho \left(f_k v_k + l_{kl} \upsilon_{kl} + h\right) dv\end{aligned} \tag{42}$$

Consider a time-dependent rigid motion of the reference frame

$$\tilde{x}_k = Q_{kl}\left(t\right) x_l + b_k\left(t\right) \tag{43}$$

where Q_{kl} is an orthogonal tensor, i.e.,

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}, \quad \det\left(\mathbf{Q}\right) = 1 \tag{44}$$

Then the velocity, angular velocity, microinertia tensor, and its material time rate can be expressed as

$$\tilde{v}_k = Q_{kl} v_l + \dot{Q}_{kl} x_l + \dot{b}_k \tag{45}$$

$$\Omega_{kl} = \dot{Q}_{km} Q_{lm} \tag{46}$$

$$\tilde{i}_{kl} = Q_{km} i_{mp} Q_{lp} \tag{47}$$

$$\frac{d\tilde{i}_{kl}}{dt} = Q_{km} \frac{di_{mp}}{dt} Q_{lp} + 2\Omega_{km} i_{ml} \tag{48}$$

Suppose that at time t, the body is brought back to the original orientation (i.e., $\mathbf{Q} = \mathbf{I}$, having only translational and angular velocities, i.e., Ω and $\dot{b}$ are constants). Then we have

$$\tilde{v}_k = v_k + \Omega_{kl} x_l + \dot{b}_k \tag{49}$$

$$\Omega_{kl} = \dot{Q}_{kl} \tag{50}$$

$$\tilde{i}_{kl} = i_{kl} \tag{51}$$

$$\frac{d\tilde{i}_{kl}}{dt} = \frac{di_{kl}}{dt} + 2\Omega_{km} i_{ml} \tag{52}$$

and also, it is noticed that

$$\tilde{\upsilon}_{kl} = \Omega_{kl} + \upsilon_{kl} \tag{53}$$

$$\Omega_{kl} = -\Omega_{lk} \tag{54}$$

Under these transformations, the mass density ρ, the internal energy density e, Cauchy stress t_{kl}, moment stress m_{klm}, and heat flux q_k are not affected since the motion is rigid. However, the body force f_k and body moment l_{kl} must be accommodated by corresponding linear and micro-accelerations (spin inertia), i.e.,

$$\tilde{f}_k - \tilde{v}_k = f_k - v_k, \quad \tilde{l}_{kl} - \tilde{\sigma}_{kl} = l_{kl} - \sigma_{kl} \tag{55}$$

Also, notice that

$$\tilde{K} = \frac{1}{2}\left\{\tilde{v}_k \tilde{v}_k + \tilde{i}_{kl} \tilde{\upsilon}_{mk} \tilde{\upsilon}_{ml}\right\} \tag{56}$$

Then subtracting the energy balance law in the $\mathbf{x} - frame$ from that in the $\tilde{\mathbf{x}} - frame$ gives

$$\frac{d}{dt}\int_{v} \rho\left(\tilde{K} - K\right) dv = \oint_{s} \left\{t_{kl}\left(\tilde{v}_l - v_l\right) + m_{klm}\left(\tilde{\upsilon}_{lm} - \upsilon_{lm}\right\} da_k\right. + \int_{v} \rho\left\{\tilde{f}_k \tilde{v}_k - f_k v_k + \tilde{l}_{kl}\tilde{\upsilon}_{kl} - l_{kl}\upsilon_{kl}\right\} dv \tag{57}$$

After lengthy but straightforward derivation one obtains

$$\int_{v} \left\{\left[\Omega_{lm} x_m + \dot{b}_l\right]\overline{f}_l + \Omega_{lm}\overline{l}_{lm}\right\} dv - \int_{v} \left\{\hat{\rho}\left(\tilde{K} - K\right) + \tilde{i}_{kl}\left(\upsilon_{mk}\Omega_{ml} + \frac{1}{2}\Omega_{mk}\Omega_{ml}\right)\right\} = 0 \tag{58}$$

where

$$\overline{f}_l \equiv t_{kl,k} + \rho\left(f_l - \dot{v}_l\right), \overline{l}_{lm} \equiv m_{klm,k} + t_{ml} + \rho\left(l_{lm} - \sigma_{lm}\right) \tag{59}$$

For arbitrary and independent variations of $\dot{b}_l$ and $\Omega_{kl} = -\Omega_{lk}$, the coefficients these quantities must vanish. Hence

$$\overline{f}_l = 0 \Rightarrow t_{kl,k} + \rho\left(f_l - \dot{v}_l\right) = 0 \tag{60}$$

$$\overline{l}_{lm} = \overline{l}_{ml} \Rightarrow m_{klm,k} + t_{ml} - s_{lm} + \rho\left(l_{lm} - \sigma_{lm}\right) = 0 \tag{61}$$

where the introduction of a symmetric $s_{ml} = s_{lm}$ is due to the antisymmetric $\Omega_{kl} = -\Omega_{lk}$, and it is named microstress.

The law of conservation of energy can simply be derived from

$$\frac{d}{dt}\int_{v} \rho\left(e+K\right)dv = \oint_{s}\left(t_{kl}v_{l}+m_{klm}\upsilon_{lm}-q_{k}\right)da_{k} + \int_{v}\rho\left(f_{k}v_{k}+l_{kl}\upsilon_{kl}+h\right)dv \tag{62}$$

where e is the internal energy density, q_k is the heat flux and h is the energy source density. The conservation of energy in local form can be obtained as

$$\rho\dot{e} = m_{klm}\upsilon_{lm,k} + t_{kl}\left(v_{l,k}-\upsilon_{lk}\right) + s_{kl}\upsilon_{lk} - q_{k,k} + \rho h \tag{63}$$

(III) The second law of thermodynamics can be derived as

$$\frac{d}{dt}\int_{v}\rho\eta dv \geq -\oint_{s}\frac{1}{\theta}q_{k}da_{k} + \int_{v}\frac{\rho h}{\theta}dv \tag{64}$$

where η is the entropy density per unit mass, θ is the absolute temperature. The local form can now be written as

$$\rho\dot{\eta} + \left(\frac{q_i}{\theta}\right)_{,i} - \frac{\rho h}{\theta} \geq 0 \tag{65}$$

Now the Eulerian description of the basic laws of micromorphic thermomechanical (TM)-electromagnetic (EM) theory can be expressed as [7]:

Conservation of Mass

$$\dot{\rho} + \rho v_{k,k} = 0 \tag{66}$$

Conservation of Microinertia

$$\frac{di_{kl}}{dt} = i_{km}\upsilon_{lm} + i_{lm}\upsilon_{km} \tag{67}$$

Balance of Linear Momentum

$$t_{ji,j} + \rho\left(f_i - \dot{v}_i\right) + F_i = 0 \tag{68}$$

Balance of Moments of Momentum

$$m_{klm,k} + t_{ml} - s_{ml} + \rho\left(l_{lm} - \sigma_{lm}\right) + L_{lm} = 0 \tag{69}$$

Conservation of Energy

$$\rho\dot{e} = m_{klm}\upsilon_{lm,k} + t_{kl}\left(v_{l,k}-\upsilon_{lk}\right) + s_{kl}\upsilon_{kl} - q_{k,k} + \rho h + W \tag{70}$$

Entropy Principle (Second Law of Thermodynamics)

$$\rho\dot{\eta} + \left(\frac{q_i}{\theta}\right)_{,i} - \frac{\rho h}{\theta} \geq 0 \tag{71}$$

where $\rho, i_{kl} = i_{lk}, t_{kl} \neq t_{lk}, s_{kl} = s_{lk}, m_{klm}, e, q_k, \eta, \theta, \nu_i, \upsilon_{lm}, f_i, F_i, l_{lm}, L_{lm}, h$, and W are the mass density, microinertia, Cauchy stress, microstress, moment stress, internal energy density, hear flux, entropy density, absolute temperature, velocity, microgyration, body force (TM), body force (EM), body moment (TM), body moment (EM), energy source (TM), and energy source (EM), respectively; the spin inertia $\boldsymbol{\sigma}$ is defined as

$$\sigma_{kl} \equiv i_{ml}\left(\dot{\upsilon}_{km} + \upsilon_{kn}\upsilon_{nm}\right) \tag{72}$$

The EM part of the body force, body moment, and energy source are given as [6,7,11–13]

$$\begin{aligned} \mathbf{F} &= q\mathbf{E} + (\nabla\mathbf{E})\cdot\mathbf{P} + (\nabla\mathbf{B})\cdot\mathbf{M} \\ &\quad + \frac{1}{c}\left\{\mathbf{J}\times\mathbf{B} + \nabla\cdot(\mathbf{vP}\times\mathbf{B}) + \frac{\partial}{\partial t}(\mathbf{P}\times\mathbf{B})\right\} \end{aligned} \tag{73}$$

$$\mathbf{L} = \mathbf{E}^* \otimes \mathbf{P} + \mathbf{B}\otimes\mathbf{M}^* \tag{74}$$

$$W = \mathbf{E}^*\cdot\left(\dot{\mathbf{P}} + \mathbf{P}\nabla\cdot\mathbf{v}\right) - \mathbf{M}^*\cdot\mathbf{B} + \mathbf{J}^*\cdot\mathbf{E}^* \tag{75}$$

Or they can be expressed as

$$\begin{aligned} F_k &= qE_k + E_{l,k}P_l + B_{l,k}M_l \\ &\quad + \frac{1}{c}\varepsilon_{klm}\left\{J_lB_m + \left(\nu_nP_lB_m\right)_{,n} + \frac{\partial}{\partial t}\left(P_lB_m\right)\right\} \end{aligned} \tag{76}$$

$$L_{kl} = E^*_kP_l + B_kM^*_l \tag{77}$$

$$W = E^*_k\left(\dot{P}_k + P_k\nu_{l,l}\right) - M^*_k\dot{B}_k + J^*_kE^*_k \tag{78}$$

Notice that, if A is a function of Eulerian coordinate x_i and time t, then $\dot{A} = \frac{dA}{dt} = \frac{\partial A}{\partial t} + A_{,i}\nu_i$.

Define the Helmholtz free energy density as

$$\psi \equiv e - \theta\eta - \rho^{-1}E^*_kP_k \tag{79}$$

Notice that the Helmholtz free energy density defined this way is quite different from its usual way. Then one obtains the Lagrangian description of the law of conservation of energy and the Clausius-Duhem (CD) inequality as (for detailed derivation, see Appendix B)

$$\begin{aligned} \rho^o\dot{e} &= M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + S^m_{KL}\dot{\beta}_{KL} - Q_{K,K} \\ &\quad - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_KE^*_K + \rho^oh \end{aligned} \tag{80}$$

$$\begin{aligned} &-\rho^o\left(\dot{\psi} + \eta\dot{\theta}\right) + M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + S^m_{KL}\dot{\beta}_{KL} \\ &-\frac{1}{\theta}Q_K\theta_{,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_KE^*_K \geq 0 \end{aligned} \tag{81}$$

where the mechanical part of the Cauchy stress and the microstress are defined as

$$t^m_{kl} \equiv t_{kl} + P_kE^*_l + M^*_kB_l, \quad T^m_{KL} \equiv Jt^m_{kl}X_{K,k}\chi_{lL} \tag{82}$$

$$s^m_{kl} \equiv s_{kl} + \frac{1}{2}\left(P_kE^*_l + P_lE^*_k + M^*_kB_l + M^*_lB_k\right), \quad S^m_{KL} \equiv \frac{1}{2}Js^m_{kl}\bar{\chi}_{Kk}\bar{\chi}_{Ll} \tag{83}$$

In other words, the electromagnetic parts are defined as

$$t^{em}_{kl} \equiv -P_kE^*_l - M^*_kB_l$$

$$s_{kl}^{em} \equiv -\frac{1}{2}\left(P_k E_l^* + P_l E_k^* + M_k^* B_l + M_l^* B_k\right) \tag{84}$$

If there is no coupling with electromagnetics, i.e., $\mathbf{P} = \mathbf{E}^* = \mathbf{M}^* = \mathbf{B} = 0$, then one has

$$\begin{aligned} t_{kl}^{m} &= t_{kl}, \quad t_{kl}^{em} = 0 \\ s_{kl}^{m} &= s_{kl}, \quad s_{kl}^{em} = 0 \end{aligned} \tag{85}$$

4 Constitutive Equations

In this section we are going to formulate the micromorphic constitutive theory for thermo-visco-elastic-plastic-electromagnetic (TVEP-TM) materials. The formulation of constitutive theory including plasticity is unique in the sense that one needs to add a set of internal variables to the list of independent constitutive variables and, of course, one needs to supply a set of governing equations for the newly added internal variables [14]. For micromorphic thermo-visco-elastic-plastic continuum, a set of internal variables is introduced as [5]

$$\mathbf{W} = \left\{\boldsymbol{\alpha}^P, \boldsymbol{\beta}^P, \boldsymbol{\gamma}^P, \mathbf{R}\right\} \tag{86}$$

where $\boldsymbol{\alpha}^p, \boldsymbol{\beta}^p, \boldsymbol{\gamma}^p$ are respectively the plastic strains corresponding to the generalized Lagrangian strains $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ defined as

$$\begin{aligned} \alpha_{KL} &\equiv x_{k,K}\overline{\chi}_{Lk} - \delta_{KL} \\ \beta_{KL} &\equiv \chi_{kK}\chi_{kL} - \delta_{KL} = \beta_{LK} \\ \gamma_{KLM} &\equiv \overline{\chi}_{Kk}\chi_{kL,M} \end{aligned} \tag{87}$$

and $\mathbf{R}$, named as the hardening parameters, is a generalized vector of internal variables.

Remark 1:

In this work, the set of internal variables $\mathbf{W}$ includes the Lagrangian strains in micromorphic theory and generalized vector of hardening parameters $\mathbf{R}$. So far, it might be considered as the most general form for internal variables.

Now let the formulation of the micromorphic constitutive theory for the TVEP-EM materials begin with

$$\mathbf{Z} = \mathbf{Z}\left(\mathbf{Y}\right) \tag{88}$$

where

$$\begin{aligned} \mathbf{Z} &= \left\{\mathbf{T}^m, \mathbf{S}^m, \mathbf{M}, \mathbf{Q}, \psi, \eta, \mathbf{P}, \mathbf{M}^*, \mathbf{J}^*\right\} \\ &= \left\{T_{KL}^m, S_{KL}^m, M_{KLM}, Q_K, \psi, \eta, P_K, M_K^*, J_K^*\right\} \end{aligned} \tag{89}$$

$$\begin{aligned} \mathbf{Y} &= \left\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\beta}}, \dot{\boldsymbol{\gamma}}, \theta, \nabla\theta, \mathbf{E}^*, \mathbf{B}, \mathbf{W}\right\} \\ &= \left\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\beta}}, \dot{\boldsymbol{\gamma}}, \theta, \nabla\theta, \mathbf{E}^*, \mathbf{B}, \boldsymbol{\alpha}^p, \boldsymbol{\beta}^p, \boldsymbol{\gamma}^p, \mathbf{R}\right\} \\ &= \left\{\alpha_{KL}, \beta_{KL}, \gamma_{KLM}, \dot{\alpha}_{KL}, \dot{\beta}_{KL}, \dot{\gamma}_{KLM}, \theta, \theta_{,K}, E_K^*, B_K, \alpha_{KL}^p, \beta_{KL}^p, \gamma_{KLM}^p, \mathbf{R}\right\} \end{aligned} \tag{90}$$

The definitions of $E_K^*, B_K, P_K, M_K^*,$ and J_K^* are given in Appendix B.

To separate the material behavior into two distinct parts: thermo-visco-elastic-electromagnetic (TVE-EM) part and thermo-visco-elastic-plastic-electromagnetic (TVEP-EM) part, a scalar-valued yield function is introduced as

$$f = f\left(\mathbf{Y}\right) \tag{91}$$

For a set of fixed values of $\mathbf{W}$, a hyper surface, named yield surface, is determined by

$$f\left(\mathbf{Y}\right) = 0 \tag{92}$$

Define the loading rate λ as the scalar product between the outward normal to the yield surface and the tangent vector to the trajectory of $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\beta}}, \dot{\boldsymbol{\gamma}}, \nabla\theta, \mathbf{E}^*, \mathbf{B}$, i.e.,

$$\begin{aligned}\lambda = {} & \frac{\partial f}{\partial \alpha_{KL}}\dot{\alpha}_{KL} + \frac{\partial f}{\partial \beta_{KL}}\dot{\beta}_{KL} + \frac{\partial f}{\partial \gamma_{KLM}}\dot{\gamma}_{KLM} + \frac{\partial f}{\partial \theta}\dot{\theta} \\ & + \frac{\partial f}{\partial \dot{\alpha}_{KL}}\ddot{\alpha}_{KL} + \frac{\partial f}{\partial \dot{\beta}_{KL}}\ddot{\beta}_{KL} + \frac{\partial f}{\partial \dot{\gamma}_{KLM}}\ddot{\gamma}_{KLM} + \frac{\partial f}{\partial \theta_{,K}}\dot{\theta}_{,K} \\ & + \frac{\partial f}{\partial E^*_K}\dot{E}^*_K + \frac{\partial f}{\partial B_K}\dot{B}_K\end{aligned} \tag{93}$$

Three distinct cases, unloading, neutral loading, and loading, are defined as: (a) $f < 0$, (b) $f = \lambda = 0$, and (c) $f = 0, \lambda > 0$, respectively. The internal variables of plasticity $\mathbf{W}$ will remain unchanged in the cases of unloading and neutral loading. Following the axiom of equipresence, the governing equations for the internal variables of plasticity $\mathbf{W}$ are proposed to be

$$\dot{\mathbf{W}} = \lambda^* \boldsymbol{\Phi}\left(\mathbf{Y}\right) \tag{94}$$

where

$$\lambda^* = \begin{cases} 0 & if\, f < 0 \ \text{ or } \ \lambda < 0 \\ \lambda & if\, f = 0 \ \text{ and } \ \lambda \geq 0 \end{cases} \tag{95}$$

Now we substitute Eqs. (88) and (94) into the Clausius-Duhem inequality, Eq. (81). It leads to

$$\begin{aligned} & -\rho^o \left\{ \frac{\partial \psi}{\partial \alpha_{KL}}\dot{\alpha}_{KL} + \frac{\partial \psi}{\partial \beta_{KL}}\dot{\beta}_{KL} + \frac{\partial \psi}{\partial \gamma_{KLM}}\dot{\gamma}_{KLM} + \frac{\partial \psi}{\partial \dot{\alpha}_{KL}}\ddot{\alpha}_{KL} + \frac{\partial \psi}{\partial \dot{\beta}_{KL}}\ddot{\beta}_{KL} + \frac{\partial \psi}{\partial \dot{\gamma}_{KLM}}\ddot{\gamma}_{KLM} \right. \\ & \quad \left. + \frac{\partial \psi}{\partial \theta}\dot{\theta} + \frac{\partial \psi}{\partial \theta_{,K}}\dot{\theta}_{,K} + \frac{\partial \psi}{\partial E^*_K}\dot{E}^*_K + \frac{\partial \psi}{\partial B_K}\dot{B}_K + \eta\dot{\theta} + \frac{\partial \psi}{\partial \mathbf{W}} \cdot \lambda^* \boldsymbol{\Phi} \right\} \\ & \quad + M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + S^m_{KL}\dot{\beta}_{KL} - \frac{1}{\theta}Q_K\theta_{,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_K E^*_K \geq 0 \end{aligned} \tag{96}$$

Since this inequality is linear in $\ddot{\alpha}_{KL}, \ddot{\beta}_{KL}, \ddot{\gamma}_{KLM}, \dot{E}^*_K, \dot{B}_K, \dot{\theta}, \dot{\theta}_{,K}$, it leads to

$$\psi = \psi\left(\alpha_{KL}, \beta_{KL}, \gamma_{KLM}, \theta, E^*_K, B_K, \alpha^p_{KL}, \beta^p_{KL}, \gamma^p_{KLM}, \mathbf{R}\right) \tag{97}$$

$$\eta = -\frac{\partial \psi}{\partial \theta} \tag{98}$$

$$T^{me}_{KL} \equiv \rho^o \frac{\partial \psi}{\partial \alpha_{KL}}, \ T^m_{KL} = T^{me}_{KL} + T^{md}_{KL}\left(\mathbf{Y}\right) \tag{99}$$

$$S^{me}_{KL} \equiv \rho^o \frac{\partial \psi}{\partial \beta_{KL}}, S^{m}_{KL} = S^{me}_{KL} + S^{md}_{KL}(\mathbf{Y}) \tag{100}$$

$$M^{e}_{KLM} \equiv \rho^o \frac{\partial \psi}{\partial \gamma_{KLM}}, M_{KLM} = M^{e}_{KLM} + M^{d}_{KLM}(\mathbf{Y}) \tag{101}$$

$$P_K = -\rho^o \frac{\partial \psi}{\partial E^*_K} \tag{102}$$

$$M^*_K = -\rho^o \frac{\partial \psi}{\partial B_K} \tag{103}$$

$$\begin{aligned} &M^{d}_{KLM}\dot{\gamma}_{KLM} + T^{md}_{KL}\dot{\alpha}_{KL} + S^{md}_{KL}\dot{\beta}_{KL} \\ &\quad - \frac{1}{\theta} Q_K \theta_{,K} + J^*_K E^*_K - \rho^o \frac{\partial \psi}{\partial \mathbf{W}} \cdot \lambda^* \mathbf{\Phi} \geq 0 \end{aligned} \tag{104}$$

where the superscript d refers to the dissipative part and superscript e refers to the elastic part, or say, the reversible part.

Since the inequality, Eq. (104), must be satisfied for any value of the loading rate λ, it implies

$$\frac{\partial \psi}{\partial \mathbf{W}} \cdot \mathbf{\Phi} \leq 0 \tag{105}$$

$$M^{d}_{KLM}\dot{\gamma}_{KLM} + T^{md}_{KL}\dot{\alpha}_{KL} + S^{md}_{KL}\dot{\beta}_{KL} - \frac{1}{\theta} Q_K \theta_{,K} + J^*_K E^*_K \geq 0 \tag{106}$$

Also, it should be emphasized that, in the case of loading, a TVEP-EM state leads to another TVEP-EM state. In other words, the consistency condition of plasticity requires that $f = 0$ and $\lambda > 0$ lead to another state with $f = 0$ [15]. Therefore, in the case of loading, we have $\dot{f} = 0$, i.e.,

$$\dot{f} = \lambda^* + \frac{\partial f}{\partial \mathbf{W}} \cdot \dot{\mathbf{W}} = \lambda^* + \lambda^* \frac{\partial f}{\partial \mathbf{W}} \cdot \mathbf{\Phi} = \lambda^* \left(1 + \frac{\partial f}{\partial \mathbf{W}} \cdot \mathbf{\Phi}\right) = 0 \tag{107}$$

This gives another constitutive constraint to the plasticity

$$\frac{\partial f}{\partial \mathbf{W}} \cdot \mathbf{\Phi} = -1 \tag{108}$$

5 Linear Constitutive Equations

For micromorphic TVEP-EM solid, define the potential energy density function Σ as

$$\Sigma \equiv \rho^o \psi = \Sigma\left(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \boldsymbol{E}^*, \mathbf{B}, \boldsymbol{\alpha}^p, \boldsymbol{\beta}^p, \boldsymbol{\gamma}^p, \mathbf{R}\right) \tag{109}$$

Following the idea of Green and Naghdi [16], and assuming the hardening parameters $\mathbf{R}$ do not appear in the potential energy density function, Eq. (109) can now be written as

$$\begin{aligned} \Sigma &= \Sigma\left(\boldsymbol{\alpha} - \boldsymbol{\alpha}^p, \boldsymbol{\beta} - \boldsymbol{\beta}^p, \boldsymbol{\gamma} - \boldsymbol{\gamma}^p, T, \mathbf{E}^*, \mathbf{B}\right) \\ &\equiv \Sigma\left(\boldsymbol{\alpha}^e, \boldsymbol{\beta}^e, \boldsymbol{\gamma}^e, T, \mathbf{E}^*, \mathbf{B}\right) \\ &= \Sigma^o - \rho^o \eta^o T - \frac{1}{2}\rho^o \gamma \frac{T^2}{T^o} - \overline{A}_{KL}\alpha^e_{KL} T - \overline{B}_{KL}\beta^e_{KL} T \\ &\quad - \overline{C}_{KLM}\gamma^e_{KLM} T + a_K T E^*_K + b_K T B_K \\ &\quad + \left\{ \frac{1}{2} A_{KLMN}\alpha^e_{KL}\alpha^e_{MN} + \frac{1}{2} B_{KLMN}\beta^e_{KL}\beta^e_{MN} + \frac{1}{2} C_{KLMNPQ}\gamma^e_{KLM}\gamma^e_{NPQ} \right. \end{aligned}$$

$$
\begin{aligned}
&+D_{KLMN}\alpha^e_{KL}\beta^e_{MN}+E_{KLMNP}\alpha^e_{KL}\gamma^e_{MNP}+F_{KLMNP}\beta^e_{KL}\gamma^e_{MNP}\}\\
&+F^1_{KLM}\alpha^e_{KL}E^*_M+F^2_{KLM}\alpha^e_{KL}B_M+G^1_{KLM}\beta^e_{KL}E^*_M+G^2_{KLM}\beta^e_{KL}B_M\\
&-H^1_{KLMN}\gamma^e_{KLM}E^*_N-H^2_{KLMN}\gamma^e_{KLM}B_N\\
&-\frac{1}{2}\overline{D}^1_{KL}E^*_KE^*_L-\frac{1}{2}\overline{D}^2_{KL}B_KB_L-\overline{D}^3_{KL}E^*_KB_L
\end{aligned}
\tag{110}
$$

where T^o is the reference temperature and T is the temperature variation, and

$$
T\equiv\theta-T^o,\qquad T^o>0,\qquad |T|<<T^o \tag{111}
$$

Remark 2:

Green and Naghdi's idea is to replace $\alpha,\beta,\gamma,\alpha^p,\beta^p,\gamma^p$ by α^e,β^e, and γ^e, which are defined as $\alpha^e\equiv\alpha-\alpha^p$, $\beta^e\equiv\beta-\beta^p$, $\gamma^e\equiv\gamma-\gamma^p$. This simplifies a lot. Also, it shows the physical meaning clearly.

From now on we assume that the material is isotropic. Therefore, all the odd order material property tensors are vanishing. Therefore, Eq. (110) can be re-written as

$$
\begin{aligned}
\Sigma&=\Sigma\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}^p,\boldsymbol{\beta}-\boldsymbol{\beta}^p,\boldsymbol{\gamma}-\boldsymbol{\gamma}^p,T,\mathbf{E}^*,\mathbf{B}\right)\equiv\Sigma\left(\boldsymbol{\alpha}^e,\boldsymbol{\beta}^e,\boldsymbol{\gamma}^e,T,\mathbf{E}^*,\mathbf{B}\right)\\
&=\Sigma^o-\rho^o\eta^oT-\frac{1}{2}\rho^o\gamma\frac{T^2}{T^o}-\overline{A}_{KL}\alpha^e_{KL}T-\overline{B}_{KL}\beta^e_{KL}T\\
&\quad+\left\{\frac{1}{2}A_{KLMN}\alpha^e_{KL}\alpha^e_{MN}+\frac{1}{2}B_{KLMN}\beta^e_{KL}\beta^e_{MN}+D_{KLMN}\alpha^e_{KL}\beta^e_{MN}+\frac{1}{2}C_{KLMNPQ}\gamma^e_{KLM}\gamma^e_{NPQ}\right\}\\
&\quad-H^1_{KLMN}\gamma^e_{KLM}E^*_N-H^2_{KLMN}\gamma^e_{KLM}B_N-\frac{1}{2}\overline{D}^1_{KL}E^*_KE^*_L-\frac{1}{2}\overline{D}^2_{KL}B_KB_L-\overline{D}^3_{KL}E^*_KB_L
\end{aligned}
\tag{112}
$$

Then it is straightforward to obtain

$$
\begin{aligned}
\eta&=-\frac{\partial\psi}{\partial\theta}=-\frac{\partial\psi}{\partial T}\\
&=\eta^o+\gamma\frac{T}{T^o}+\left\{\overline{A}_{KL}\alpha^e_{KL}+\overline{B}_{KL}\beta^e_{KL}\right\}/\rho^o
\end{aligned}
\tag{113}
$$

$$
\begin{aligned}
T^{me}_{KL}&=\rho^o\frac{\partial\psi}{\partial\alpha^e_{KL}}=\frac{\partial\Sigma}{\partial\alpha^e_{KL}}\\
&=-\overline{A}_{KL}T+A_{KLMN}\alpha^e_{MN}+D_{KLMN}\beta^e_{MN}
\end{aligned}
\tag{114}
$$

$$
\begin{aligned}
S^{me}_{KL}&=\rho^o\frac{\partial\psi}{\partial\beta^e_{KL}}=\frac{\partial\Sigma}{\partial\beta^e_{KL}}\\
&=-\overline{B}_{KL}T+B_{KLMN}\beta^e_{MN}+D_{MNKL}\alpha^e_{MN}
\end{aligned}
\tag{115}
$$

$$
\begin{aligned}
M_{KLM}&=\rho^o\frac{\partial\psi}{\partial\gamma^e_{KLM}}\\
&=C_{KLMNPQ}\gamma^e_{NPQ}-H^1_{KLMN}E^*_N-H^2_{KLMN}B_N
\end{aligned}
\tag{116}
$$

$$
\begin{aligned}
P_K&=-\rho^o\frac{\partial\psi}{\partial E^*_K}=H^1_{LMNK}\gamma^e_{LMN}+\overline{D}^1_{KL}E^*_L+\overline{D}^3_{KL}B_L\\
&\equiv P^m_K+\overline{D}^1_{KL}E^*_L+\overline{D}^3_{KL}B_L
\end{aligned}
\tag{117}
$$

$$\begin{aligned} M^*_K &= -\rho^o \frac{\partial \psi}{\partial B_K} = H^2_{LMNK}\gamma^e_{LMN} + \overline{D}^2_{KL}B_L + \overline{D}^3_{LK}E^*_L \\ &\equiv M^{*m}_K + \overline{D}^2_{KL}B_L + \overline{D}^3_{LK}E^*_L \end{aligned} \tag{118}$$

Moreover, the 2nd order, 4th order, and 6th order material property tensors are made of a few material constants, for example,

$$\begin{aligned} &\overline{A}_{KL} = \overline{A}\delta_{KL}, \overline{B}_{KL} = \overline{B}\delta_{KL} \\ &A_{KLMN} = \lambda^1\delta_{KL}\delta_{MN} + \mu^{11}\delta_{KM}\delta_{LN} + \mu^{12}\delta_{KN}\delta_{LM} \\ &B_{KLMN} = \lambda^2\delta_{KL}\delta_{MN} + \mu^2\left(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}\right) \\ &D_{KLMN} = \lambda^3\delta_{KL}\delta_{MN} + \mu^3\left(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}\right) \\ &C_{KLMNPQ} = C^1\delta_{KL}\delta_{MN}\delta_{PQ} + C^2\delta_{KL}\delta_{MP}\delta_{NQ} + C^3\delta_{KL}\delta_{MQ}\delta_{NP} \\ &\qquad + C^4\delta_{KN}\delta_{LM}\delta_{PQ} + C^5\delta_{KM}\delta_{NL}\delta_{PQ} + C^6\delta_{KM}\delta_{LP}\delta_{NQ} \\ &\qquad + C^7\delta_{KN}\delta_{LP}\delta_{MQ} + C^8\delta_{KP}\delta_{LQ}\delta_{MN} + C^9\delta_{KN}\delta_{LQ}\delta_{MP} \\ &\qquad + C^{10}\delta_{KP}\delta_{NL}\delta_{MQ} + C^{11}\delta_{KQ}\delta_{LP}\delta_{MN} + C^{12}\delta_{KQ}\delta_{LM}\delta_{NP} \\ &\qquad + C^{13}\delta_{KM}\delta_{LQ}\delta_{NP} + C^{14}\delta_{KP}\delta_{LM}\delta_{NQ} + C^{15}\delta_{KQ}\delta_{NL}\delta_{MP} \end{aligned} \tag{119}$$

Onsager's Postulate

Now we follow Onsager's Postulate [17,18] and construct a potential of dissipation quadratic as

$$\begin{aligned} D = {} & \frac{1}{2}d^1_{KLMN}\dot{\alpha}_{KL}\dot{\alpha}_{MN} + d^2_{KLMN}\dot{\alpha}_{KL}\dot{\beta}_{MN} + \frac{1}{2}d^3_{KLMN}\dot{\beta}_{KL}\dot{\beta}_{MN} \\ & + \frac{1}{2}f^1_{IJKLMN}\dot{\gamma}_{IJK}\dot{\gamma}_{LMN} + f^2_{IJKM}\dot{\gamma}_{IJK}\theta_{,M} + f^3_{IJKM}\dot{\gamma}_{IJK}E^*_M \\ & + \frac{1}{2}g^1_{MN}\theta_{,M}\theta_{,N} + g^2_{MN}\theta_{,M}E^*_N + \frac{1}{2}g^3_{MN}E^*_M E^*_N \end{aligned} \tag{120}$$

where $\dot{\boldsymbol{\alpha}}, \dot{\boldsymbol{\beta}}, \dot{\boldsymbol{\gamma}}, \nabla\theta$, and $\mathbf{E}^*$ are the thermodynamic forces. Correspondingly, the thermodynamic fluxes $\mathbf{T}^{md}, \mathbf{S}^{md}, \mathbf{M}^d, \mathbf{Q}$, and $\mathbf{J}^*$ can be obtained as

$$T^{md}_{KL} = \frac{\partial D}{\partial \dot{\alpha}_{KL}} = d^1_{KLMN}\dot{\alpha}_{MN} + d^2_{KLMN}\dot{\beta}_{MN} \tag{121}$$

$$S^{md}_{KL} = \frac{\partial D}{\partial \dot{\beta}_{KL}} = d^2_{MNKL}\dot{\alpha}_{MN} + d^3_{KLMN}\dot{\beta}_{MN} \tag{122}$$

$$M^d_{IJK} = \frac{\partial D}{\partial \dot{\gamma}_{IJK}} = f^1_{IJKLMN}\dot{\gamma}_{LMN} + f^2_{IJKM}\theta_{,M} + f^3_{IJKM}E^*_M \tag{123}$$

$$-\frac{1}{\theta}Q_M = \frac{\partial D}{\partial \theta_{,M}} = f^2_{IJKM}\dot{\gamma}_{IJK} + g^1_{MN}\theta_{,N} + g^2_{MN}E^*_N \tag{124}$$

$$J^*_M = \frac{\partial D}{\partial E^*_M} = f^3_{IJKM}\dot{\gamma}_{IJK} + g^2_{NM}\theta_{,N} + g^3_{MN}E^*_N \tag{125}$$

One may verify that the CD inequality $\mathbf{M}^d\!:\!\dot{\boldsymbol{\gamma}} + \mathbf{T}^{md}\!:\dot{\boldsymbol{\alpha}} + \mathbf{S}^{md}\!:\dot{\boldsymbol{\beta}} - \frac{1}{\theta}\mathbf{Q}\cdot\nabla\theta + \mathbf{J}^*\cdot\mathbf{E}^* \geq 0$ is now reduced to

$$D \geq 0 \tag{126}$$

which is precisely what the Onsager's Postulate means.

Because the material is assumed to be isotropic, similar to Eq. (119), one could have

$$\begin{aligned}
&g^1_{KL} = \overline{g}^1 \delta_{KL}, g^2_{KL} = \overline{g}^2 \delta_{KL}, g^3_{KL} = \overline{g}^3 \delta_{KL} \\
&d^1_{KLMN} = \overline{\lambda}^1 \delta_{KL}\delta_{MN} + \overline{\mu}^{11} \delta_{KM}\delta_{LN} + \overline{\mu}^{12} \delta_{KN}\delta_{LM} \\
&d^2_{KLMN} = \overline{\lambda}^2 \delta_{KL}\delta_{MN} + \overline{\mu}^2 \left(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}\right) \\
&d^3_{KLMN} = \overline{\lambda}^3 \delta_{KL}\delta_{MN} + \overline{\mu}^3 \left(\delta_{KM}\delta_{LN} + \delta_{KN}\delta_{LM}\right) \\
&f^2_{KLMN} = \kappa^2 \delta_{KL}\delta_{MN} + \tau^{21} \delta_{KM}\delta_{LN} + \tau^{22} \delta_{KN}\delta_{LM} \\
&f^3_{KLMN} = \kappa^3 \delta_{KL}\delta_{MN} + \tau^{31} \delta_{KM}\delta_{LN} + \tau^{32} \delta_{KN}\delta_{LM} \\
&f^1_{KLMNPQ} = f^1 \delta_{KL}\delta_{MN}\delta_{PQ} + f^2 \delta_{KL}\delta_{MP}\delta_{NQ} + f^3 \delta_{KL}\delta_{MQ}\delta_{NP} \\
&\qquad + f^4 \delta_{KN}\delta_{LM}\delta_{PQ} + f^5 \delta_{KM}\delta_{NL}\delta_{PQ} + f^6 \delta_{KM}\delta_{LP}\delta_{NQ} \\
&\qquad + f^7 \delta_{KN}\delta_{LP}\delta_{MQ} + f^8 \delta_{KP}\delta_{LQ}\delta_{MN} + f^9 \delta_{KN}\delta_{LQ}\delta_{MP} \\
&\qquad + f^{10} \delta_{KP}\delta_{NL}\delta_{MQ} + f^{11} \delta_{KQ}\delta_{LP}\delta_{MN} + f^{12} \delta_{KQ}\delta_{LM}\delta_{NP} \\
&\qquad + f^{13} \delta_{KM}\delta_{LQ}\delta_{NP} + f^{14} \delta_{KP}\delta_{LM}\delta_{NQ} + f^{15} \delta_{KQ}\delta_{NL}\delta_{MP}
\end{aligned} \tag{127}$$

Return Mapping Algorithm for Micromorphic Plasticity

Define the generalized total strain tensor, elastic strain tensor, and plastic strain tensor as

$$\mathbf{E} \equiv \begin{vmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{vmatrix}, \mathbf{E}^e \equiv \begin{vmatrix} \boldsymbol{\alpha}^e \\ \boldsymbol{\beta}^e \\ \boldsymbol{\gamma}^e \end{vmatrix}, \mathbf{E}^p \equiv \begin{vmatrix} \boldsymbol{\alpha}^p \\ \boldsymbol{\beta}^p \\ \boldsymbol{\gamma}^p \end{vmatrix} \tag{128}$$

Notice that

$$\boldsymbol{\alpha}^e \equiv \boldsymbol{\alpha} - \boldsymbol{\alpha}^p, \quad \boldsymbol{\beta}^e \equiv \boldsymbol{\beta} - \boldsymbol{\beta}^p, \quad \boldsymbol{\gamma}^e \equiv \boldsymbol{\gamma} - \boldsymbol{\gamma}^p \tag{129}$$

Therefore

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p \tag{130}$$

Now, let the hardening parameters **R** be decomposed into two parts: the vector part **G**, which has the same dimension as **E**, and a scalar part g. We propose a yield function as [5,19]

$$f = \left\| \mathbf{E} - \mathbf{E}^p - \mathbf{G} \right\| - \left(E_Y + \kappa H g\right) = \left\| \mathbf{E}^e - \mathbf{G} \right\| - \left(E_Y + \kappa H g\right) \tag{131}$$

where $E_Y, H,$ and κ are material constants.

Remark 3:

This is a special form. One may want to improve it or to generalize it.

Let the evolution equations for the plastic strain tensor and the hardening parameters be

$$\dot{\mathbf{E}}^p = \Omega \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}, \quad \dot{\mathbf{G}} = \Omega C_1 \left(1 - \kappa\right) H \frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}, \quad \dot{g} = \Omega C_2 \tag{132}$$

where

$$\boldsymbol{\xi} \equiv \mathbf{E} - \mathbf{E}^p - \mathbf{G} = \mathbf{E}^e - \mathbf{G}, \quad \|\boldsymbol{\xi}\| \equiv \{\boldsymbol{\xi} \cdot \boldsymbol{\xi}\}^{\frac{1}{2}} \tag{133}$$

The trial value of the yield function at $t = t_{n+1}$, based on elastic prediction, is calculated as

$$\tilde{f}_{n+1} = \left\|\mathbf{E}_{n+1} - \mathbf{E}_n^p - \mathbf{G}_n\right\| - \left(E_Y + \kappa H g_n\right) \equiv \left\|\tilde{\boldsymbol{\xi}}_{n+1}\right\| - \left(E_Y + \kappa H g_n\right) \tag{134}$$

If $\tilde{f}_{n+1} \leq 0$, which means the elastic prediction is correct, then the plastic strain and hardening parameters at $t = t_{n+1}$ maintain their values at $t = t_n$, i.e.,

$$f_{n+1} = \left\|\mathbf{E}_{n+1} - \mathbf{E}_n^p - \mathbf{G}_n\right\| - \left(E_Y + \kappa H g_n\right) \tag{135}$$

It is worthwhile to mention that in this case $(\tilde{f}_{n+1} \leq 0, t = t_{n+1})$, $\mathbf{E}_{n+1}^p = \mathbf{E}_n^p$, $\mathbf{G}_{n+1} = \mathbf{G}_n$, and $g_{n+1} = g_n$. If $\tilde{f}_{n+1} > 0$, which means the elastic prediction is incorrect, then the values of plastic strain and hardening parameters at $t = t_{n+1}$ should be updated as follows:

$$\mathbf{E}_{n+1}^p = \mathbf{E}_n^p + \Delta\Omega\tilde{\boldsymbol{\xi}}_{n+1}/\left\|\tilde{\boldsymbol{\xi}}_{n+1}\right\| \tag{136}$$

$$\mathbf{G}_{n+1} = \mathbf{G}_n + \Delta\Omega C_1 \left(1 - \kappa\right) H\tilde{\boldsymbol{\xi}}_{n+1}/\left\|\tilde{\boldsymbol{\xi}}_{n+1}\right\| \tag{137}$$

$$g_{n+1} = g_n + \Delta\Omega C_2 \tag{138}$$

where

$$\Delta\Omega = \frac{\tilde{f}_{n+1}}{1 + C_1 H - \left(C_1 - C_2\right)\kappa H} \tag{139}$$

Now one can prove that yield function, based on the updated $\mathbf{E}_{n+1}^p, \mathbf{G}_{n+1}$, and g_{n+1}, returns to zero [5] i.e.,

$$f_{n+1} = \left\|\mathbf{E} - \mathbf{E}_{n+1}^p - \mathbf{G}_{n+1}\right\| - \left(E_Y + \kappa H g_{n+1}\right) = 0 \tag{140}$$

Remark 4:

The yield function based on the updated values returning to zero means the return mapping algorithm works. No need to worry theoretically.

From Eqs. (114)–(118), it is noticed that the elastic parts of the stresses, $T_{KL}^{me}, S_{KL}^{me}, M_{KLM}^{e}$, and P_K, M_K^* depend on the elastic strains, not the total strains. Using the *Return Mapping Algorithm*, one may find the plastic strains exactly, and once the total strains are found, usually through finite element analysis (to be discussed later), then immediately elastic strains and elastic parts of the stresses as well are obtained. Also, it is seen that the *Return Mapping Algorithm* is strain-based, not stress-based. Because in this formulation the stresses have two parts: elastic parts and dissipative parts, which depend on strain rates. A strain-based *Return Mapping Algorithm* works independently of strain-rates.

6 Scalar and Vector Potentials

It is well-known that the electric field **E** and the magnetic flux **B** are not independent of each other, nor are the dielectric displacement **D** and the magnetic field **H**. Mathematically, this is noticed from Maxwell's equations, especially, Eqs. (9) and (11). To resolve this problem, especially in the finite element formulation presented in the next section, introduce the scalar potential ϕ and the vector potential **A**, which are related to the electric field **E** and magnetic flux **B** as

$$\mathbf{B} \equiv \nabla \times \mathbf{A} \quad \text{or} \quad B_i = e_{ijk} A_{k,j} \tag{141}$$

$$\mathbf{E} \equiv -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{or} \quad E_i \equiv -\phi_{,i} - \frac{1}{c}\frac{\partial A_i}{\partial t} \tag{142}$$

Now we follow the same pattern as Eqs. (117) and (118) and write the polarization P_k and magnetization M_k as

$$\begin{aligned} P_k &= P_k^m + D^1 E_k + D^3 B_k \\ M_k &= M_m^m + D^2 B_k + D^3 E_k \end{aligned} \tag{143}$$

where P_k^m and M_k^m are the polarization and magnetization induced by strains, i.e.,

$$\begin{aligned} P_k^m &\equiv H^1_{lmnk} \gamma^e_{lmn} \\ M_k^m &\equiv H^2_{lmnk} \gamma^e_{lmn} \end{aligned} \tag{144}$$

Admittedly the introduction of Eqs. (143) and (144) involves approximations. Without making these approximations, it is difficult, if not impossible, to derive the governing equations for scalar potential ϕ and vector potential **A**.

Now the dielectric displacement vector **D** and the magnetic field vector **H** can be expressed as

$$\begin{aligned} D_k &= E_k + P_k \\ &= E_k + P_k^m + D^1 E_k + D^3 B_k \\ &= \left(1 + D^1\right) E_k + D^3 B_k + P_k^m \\ &\equiv \varepsilon E_k + D^3 B_k + P_k^m \end{aligned} \tag{145}$$

$$\begin{aligned} H_k &= B_k - M_k^m \\ &= B_k - \left(M_k^m + D^2 B_k + D^3 E_k\right) \\ &= \left(1 - D^2\right) B_k - D^3 E_k - M_k^m \\ &\equiv \mu^{-1} B_k - D^3 E_k - M_k^m \end{aligned} \tag{146}$$

Substituting Eqs. (141) and (142) into the Maxwell's equations leads to

$$\begin{aligned} D_{k,k} &= \varepsilon E_{k,k} + D^3 B_{k,k} + P^m_{k,k} \\ &= -\varepsilon \left\{ \phi_{,kk} + \frac{1}{c} \dot{A}_{k,k} \right\} + P^m_{k,k} \\ &= q \end{aligned} \tag{147}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left\{ -\nabla\phi - \frac{1}{c}\dot{\mathbf{A}} \right\} + \frac{1}{c}\left\{ \nabla \times \dot{\mathbf{A}} \right\} = 0 \tag{148}$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{149}$$

$$\begin{aligned} \nabla \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \left[\mu^{-1}\mathbf{B} - D^3\mathbf{E} - \mathbf{M}^{*m} \right] - \frac{1}{c}\left\{ \varepsilon\dot{\mathbf{E}} + D^3\dot{\mathbf{B}} + \dot{\mathbf{P}}^m \right\} \\ &= \mu^{-1}\nabla \times \mathbf{B} - \nabla \times \mathbf{M}^{*m} - \frac{\varepsilon}{c}\dot{\mathbf{E}} - \frac{1}{c}\dot{\mathbf{P}}^m \end{aligned}$$

$$\begin{aligned} &= \mu^{-1}\nabla \times \nabla \times \mathbf{A} + \frac{\varepsilon}{c}\left[\nabla\dot{\phi} + \frac{1}{c}\ddot{\mathbf{A}}\right] - \frac{1}{c}\dot{\mathbf{P}}^m - \nabla \times \mathbf{M}^m \\ &= \frac{1}{c}\mathbf{J} \end{aligned} \tag{150}$$

It is seen that two of the four Maxwell's equations, Eqs. (148) and (149), are identically satisfied; the other two equations, Eqs. (147) and (150) can be re-written as

$$\phi_{,kk} + \frac{1}{c}\dot{A}_{k,k} = \left\{P^m_{k,k} - q\right\}/\varepsilon \tag{151}$$

$$\mu^{-1}\left[A_{j,ij} - A_{i,jj}\right] + \frac{\varepsilon}{c}\dot{\phi}_{,i} + \frac{\varepsilon}{c^2}\ddot{A}_i = \frac{1}{c}\left[J_i + \dot{P}^m_i\right] + e_{ijk}M^m_{k,j} \tag{152}$$

Gauge Transformation

The famous gauge transformation is represented by

$$\begin{aligned} \phi' &= \phi - \frac{1}{c}\dot{\psi} \\ \mathbf{A}' &= \mathbf{A} + \nabla\psi \end{aligned} \tag{153}$$

Substituting Eq. (153) into Eqs. (151) and (152) results

$$\begin{aligned} \nabla^2\phi' + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}}' &= \phi_{,ii} - \frac{1}{c}\dot{\psi}_{,ii} + \frac{1}{c}\left\{\nabla\cdot\dot{\mathbf{A}} + \nabla^2\dot{\psi}\right\} \\ &= \phi_{,ii} + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}} \end{aligned} \tag{154}$$

$$\begin{aligned} &-\mu^{-1}A'_{i,jj} + \mu^{-1}A'_{j,ji} + \varepsilon\frac{1}{c}\dot{\phi}'_{,i} + \frac{1}{c^2}\varepsilon\ddot{A}'_i \\ &= \mu^{-1}\left\{-A_{i,jj} - \psi_{,ijj} + A_{j,ji} + \psi_{,jji}\right\} + \varepsilon\left\{\frac{1}{c}\left[\dot{\phi}_{,i} - \frac{1}{c}\ddot{\psi}_{,i}\right] + \frac{1}{c^2}\left[\ddot{A}_i + \ddot{\psi}_{,i}\right]\right\} \\ &= -\mu^{-1}\left\{A_{i,jj} - A_{j,ji}\right\} + \varepsilon\left\{\frac{1}{c}\dot{\phi}_{,i} + \frac{1}{c^2}\ddot{A}_i\right\} = \frac{1}{c}\left[J_i + \dot{P}^m_i\right] + \varepsilon_{ijk}M^m_{k,j} \end{aligned} \tag{155}$$

In other words, the function ψ is arbitrary. We utilize this freedom to arrive at the Lorentz Condition.

Lorentz Condition

$$\varepsilon\frac{1}{c}\dot{\phi} + \mu^{-1}\nabla\cdot\mathbf{A} = 0 \tag{156}$$

Substituting the Lorentz condition into Eqs. (147) and (150) results

$$-\varepsilon\left\{\nabla^2\phi + \frac{1}{c}\nabla\cdot\dot{\mathbf{A}}\right\} = -\varepsilon\left\{\nabla^2\phi - \varepsilon\mu\frac{1}{c^2}\ddot{\phi}\right\} = q - \nabla\cdot\mathbf{P}^m \tag{157}$$

$$-\mu^{-1}\nabla^2 A_i + \frac{1}{c^2}\varepsilon\ddot{A}_i = \frac{J_i}{c} + \varepsilon_{ijk}M^m_{k,j} + \frac{1}{c}\frac{\partial P^m_i}{\partial t} \tag{158}$$

In other words, the Maxwell's equations are reduced to

$$\varepsilon\mu\frac{1}{c^2}\ddot{\phi} - \nabla^2\phi = a \tag{159}$$

$$\varepsilon\mu\frac{1}{c^2}\ddot{A}_i - \nabla^2 A_i = C_i \tag{160}$$

where

$$a \equiv \frac{1}{\varepsilon}\left(q - P^m_{k,k}\right) \tag{161}$$

$$C_k \equiv \mu\left\{\frac{J_k}{c} + \varepsilon_{kij}M^m_{j,i} + \frac{1}{c}\dot{P}^m_k\right\} \tag{162}$$

This means the Maxwell's equations become two wave equations with forcing terms and the speed of wave propagation is equal to

$$v = c\frac{1}{\sqrt{\varepsilon\mu}} \tag{163}$$

Since both the material constants ε and η are positive and greater than one, therefore Eq. (163) says the speed of EM wave propagation in material is smaller than that in vacuum or in air.

7 Finite Element Formulation

Since the Lagrangian stresses and the Eulerian stresses are related as

$$\begin{aligned} T_{KL} &= Jt_{kl}X_{K,k}\chi_{lL}, & t_{kl} &= J^{-1}T_{KL}x_{k,K}\overline{\chi}_{Ll} \\ S_{KL} &= \tfrac{1}{2}Js_{kl}\overline{\chi}_{Kk}\overline{\chi}_{Ll}, & s_{kl} &= 2J^{-1}S_{KL}\chi_{kK}\chi_{lL} \\ M_{KLM} &= Jm_{mkl}X_{M,m}\chi_{kK}\overline{\chi}_{Ll}, & m_{mkl} &= J^{-1}M_{KLM}x_{m,M}\overline{\chi}_{Kk}\chi_{lL} \end{aligned} \tag{164}$$

One may verify that the balance law of linear momentum and balance law of moment of momentum, can be written as

$$\left(T_{KL}\overline{\chi}_{Ll}\right)_{,K} + \rho^o\left(f_l - \dot{v}_l\right) + JF_l = 0 \tag{165}$$

$$\left(M_{LMK}\overline{\chi}_{Ll}\chi_{mM}\right)_{,K} + T_{ML}x_{m,M}\overline{\chi}_{Ll} - 2S_{LM}\chi_{lL}\chi_{mM} + \rho^o\left(l_{lm} - \sigma_{lm}\right) + JL_{lm} = 0 \tag{166}$$

Recall that the conservation of energy can be expressed as

$$\begin{aligned} \rho^o\dot{e} = {} & M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + \overline{S}^m_{KL}\dot{\beta}_{KL} \\ & - Q_{K,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_KE^*_K + \rho^o h \end{aligned} \tag{167}$$

Based on the constitutive equations, Eqs. (113)–(118) and (120)–(134), Eq. (167) can be further derived to be [5]

$$\begin{aligned} & \rho^o\gamma\frac{T + T^o}{T^o}\dot{T} + \left(T + T^o\right)\left(\overline{A}_{KL}\dot{\alpha}^e_{KL} + \overline{B}_{KL}\dot{\beta}^e_{KL}\right) \\ & = T^{me}_{KL}\dot{\alpha}^p_{KL} + S^{me}_{KL}\dot{\beta}^p_{KL} + M^e_{KLM}\dot{\gamma}^p_{KLM} \\ & + T^d_{KL}\dot{\alpha}_{KL} + S^d_{KL}\dot{\beta}_{KL} + M^d_{KLM}\dot{\gamma}_{KLM} \\ & - Q_{K,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_KE^*_K + \rho^o h \end{aligned} \tag{168}$$

Recall the governing equations of the scalar potential ϕ and the vector potential $\mathbf{A}$ as

$$\frac{\varepsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} - \phi_{,ll} = a \tag{169}$$

$$\frac{\varepsilon\mu}{c^2}\frac{\partial^2 A_k}{\partial t^2} - A_{k,ll} = C_k \tag{170}$$

We now link the displacement, micromotion, temperature, scalar potential, and vector potential at a generic point in an element to the corresponding nodal values associated with this element as follows:

$$\begin{aligned}
&U_K = N_\alpha U_{K\alpha}\\
&\chi_{kK} = N_\alpha \chi_{kK\alpha}\\
&T = N_\alpha T_\alpha\\
&\phi = N_\alpha \phi_\alpha\\
&A_k = N_\alpha A_{k\alpha}
\end{aligned} \tag{171}$$

where $N_\alpha\left[\alpha = 1, 2, 3, \ldots., 8\right]$ are the 8 shape functions for a standard 8-noe 3D element. The finite element equations are obtained as (for detailed derivation, see Appendix C)

$$M_{\alpha\beta}\ddot{U}_{M\beta} = f^1_{M\alpha} + f^2_{M\alpha} + f^3_{M\alpha} \tag{172}$$

$$IM_{\alpha\beta}\ddot{\chi}_{kK\beta} = \phi^1_{kK\alpha} + \phi^2_{kK\alpha} + \phi^3_{kK\alpha} + \phi^4_{kK\alpha} + \phi^5_{kK\alpha} + \phi^6_{kK\alpha} \tag{173}$$

$$\gamma M_{\alpha\beta}\dot{T}_\beta = q^1_\alpha + q^2_\alpha + q^3_\alpha + q^4_\alpha + q^5_\alpha + q^6_\alpha + q^7_\alpha \tag{174}$$

$$m_{\alpha\beta}\ddot{\phi}_\beta + k_{\alpha\beta}\phi_\beta = \hat{\xi}_\alpha + \tilde{a}_\alpha \tag{175}$$

$$m_{\alpha\beta}\ddot{A}_{k\beta} + k_{\alpha\beta}A_{k\beta} = \hat{A}_{k\alpha} + \hat{C}_{k\alpha} \tag{176}$$

where

$$M_{\alpha\beta} \equiv \int\limits_V \rho^o N_\alpha N_\beta dV = M_{\beta\alpha} \tag{177}$$

$$IM_{\alpha\beta} \equiv \int\limits_V \rho^o I N_\alpha N_\beta dV = IM_{\beta\alpha} \tag{178}$$

$$\gamma M_{\alpha\beta} \equiv \int\limits_V \rho^o \gamma N_\alpha N_\beta dV = \gamma M_{\beta\alpha} \tag{179}$$

$$m_{\alpha\beta} \equiv \int\limits_v \frac{1}{\overline{c}^2} N_\alpha N_\beta dv \tag{180}$$

$$k_{\alpha\beta} \equiv \int\limits_v N_{\alpha,i}N_{\beta,i}dv \tag{181}$$

$$\begin{aligned}
&f^1_{M\alpha} \equiv \int\limits_{S_F} \hat{F}_l \delta_{lM} N_\alpha dS \equiv \int\limits_{S_F} \hat{F}_M N_\alpha dS\\
&f^2_{M\alpha} \equiv -\int\limits_V T_{KL}\overline{\chi}_{Ll}\delta_{lM}N_{\alpha,K}dV\\
&f^3_{M\alpha} \equiv \int\limits_V \left(\rho^o f_l + JF_l\right)\delta_{lM}N_\alpha dV
\end{aligned} \tag{182}$$

$$\phi^1_{kK\alpha} \equiv \int\limits_{S_M} \hat{M}_{kK}N_\alpha dS$$

$$\phi^2_{kK\alpha} \equiv -\int\limits_V M_{LMP}\overline{\chi}_{Lk}\chi_{mM}\overline{\chi}_{Km,P}N_\alpha dV$$

$$\phi^3_{kK\alpha} \equiv -\int\limits_V M_{LKM}\overline{\chi}_{Lk}N_{\alpha,M}dV$$

$$\phi^4_{kK\alpha} \equiv \int\limits_V T_{ML} x_{m,M} \overline{\chi}_{Lk} \overline{\chi}_{Km} N_\alpha dV$$

$$\phi^5_{kK\alpha} \equiv \int\limits_V -2 S_{LK} \chi_{kL} N_\alpha dV$$

$$\phi^6_{kK\alpha} \equiv \int\limits_V \left(\rho^o l_{km} + J L_{km}\right) \overline{\chi}_{Km} N_\alpha dV \tag{183}$$

$$q^1_\alpha \equiv \int\limits_V -\left(T + T^o\right) \left[\overline{A}_{KL} \dot{\alpha}^e_{KL} + \overline{B}_{KL} \dot{\beta}^e_{KL}\right] N_\alpha dV$$

$$q^2_\alpha \equiv \int\limits_V \left\{T^{me}_{KL} \dot{\alpha}^p_{KL} + S^{me}_{KL} \dot{\beta}^p_{KL} + M^e_{KLM} \dot{\gamma}^p_{KLM}\right\} N_\alpha dV$$

$$q^3_\alpha \equiv \int\limits_V \left\{T^d_{KL} \dot{\alpha}_{KL} + S^d_{KL} \dot{\beta}_{KL} + M^d_{KLM} \dot{\gamma}_{KLM}\right\} N_\alpha dV$$

$$q^4_\alpha \equiv \int\limits_V \left\{-P_K \dot{E}^*_K - M^*_K \dot{B}_K + J^*_K E^*_K\right\} N_\alpha dV$$

$$q^5_\alpha \equiv -\int\limits_{S_q} \hat{Q} N_\alpha dS$$

$$q^6_\alpha \equiv \int\limits_V Q_K N_{\alpha,K} dV$$

$$q^7_\alpha \equiv \int\limits_V \rho^o h N_\alpha dV \tag{184}$$

$$\hat{\xi}_\alpha = \int\limits_{s_\xi} \phi_{,i} n_i N_\alpha ds \equiv \int\limits_{s_\xi} \xi N_\alpha ds$$

$$\hat{a}_\alpha \equiv \int\limits_v a N_\alpha dv \tag{185}$$

$$\hat{A}_{k\alpha} = \int\limits_{s_\varsigma} A_{k,i} n_i N_\alpha ds \equiv \int\limits_{s_\varsigma} \tilde{A}_k N_\alpha ds$$

$$\hat{C}_{k\alpha} \equiv \int\limits_v C_k N_\alpha dv \tag{186}$$

In this work, for simplicity, we assume that the microinertia is isotropic, i.e.,

$$I_{KL} = I \delta_{KL} \tag{187}$$

which implies

$$i_{kl} = I_{KL} \chi_{kK} \chi_{lL} = I \delta_{KL} \chi_{kK} \chi_{lL} = I \chi_{kK} \chi_{lK} \tag{188}$$

Also, it is worthwhile to note that on the boundary the surface force, the surface moment and the heat flow are specified respectively as

$$\hat{F}_l = T_{KL} \overline{\chi}_{Ll} n_K, \hat{M}_{kK} = M_{LKM} \overline{\chi}_{Lk} n_M, \hat{Q} = Q_K n_K \tag{189}$$

where $n_K$is the unit outward normal to the boundary surface.

8 Two-Stage Solutions

It is noticed that Eqs. (165)–(170) are governing equations for $\mathbf{u}, \boldsymbol{\chi}, T, \phi$, and $\mathbf{A}$, which are coupled together—that is why it is coined with the name micromorphic thermomechanical–electromagnetic coupling. To solve the five sets of differential equations, Eqs. (165)–(170), simultaneously, is *numerically impossible and physically meaningless*. Also, *relative to the EM wave, everything else is standing still*. The strategy of a two-stage solution, in general, can be described as follows.

Consider that there are two sets of variables, Set A and Set B. Based upon their general governing equations, the variables in Set A move faster and the variables in Set B move slower. Then, in *Stage I*, we seek the static solutions of the variables in Set B. Once the solutions in Set B are obtained, one may find the forcing terms for those variables in Set A. Then, for *Stage II*, we solve for the dynamic solutions for variables in Set A.

In this work, in $Stage\ I$, we are seeking the static solutions of a hollow cylinder subjected to twisting. The finite element results $(T, \chi_{avg}, P_{avg}, T_{avg})$ of $Stage\ I$ are plotted on the deformed shape (cf. Fig. 1). Since in micromorphic theory the Cauchy stress tensor and the corresponding 2nd order Piola-Kirchhoff stress are not symmetric, the $L2\ norm$ of the 2nd order Piola-Kirchhoff stress is defined as

$$T_{avg} \equiv \|\mathbf{T}\| = \sqrt{T_{11}^2 + T_{22}^2 + T_{33}^2 + T_{23}^2 + T_{32}^2 + T_{31}^2 + T_{13}^2 + T_{12}^2 + T_{21}^2} \tag{190}$$

The $L2 - norm$ of the plastic strain $\boldsymbol{\alpha}^p$ is defined as

$$P_{avg} \equiv \left\{\alpha_{KL}^p \alpha_{KL}^p\right\}^{1/2} \tag{191}$$

Similarly, one may define the $L2 - norms$ of $\boldsymbol{\beta}^p$ and $\boldsymbol{\gamma}^p$ as

$$P_{avg}^2 \equiv \left\{\beta_{KL}^p \beta_{KL}^p\right\}^{1/2}, \quad P_{avg}^3 \equiv \left\{\gamma_{KLM}^p \gamma_{KLM}^p\right\}^{1/2} \tag{192}$$

The $L2 - norm$ of micromotion is defined as

$$\chi_{avg} \equiv \|\boldsymbol{\chi} - \boldsymbol{\delta}\| = \sqrt{(\chi_{11} - \delta_{11})^2 + (\chi_{22} - \delta_{22})^2 + (\chi_{33} - \delta_{33})^2 + \chi_{23}^2 + \chi_{32}^2 + \chi_{31}^2 + \chi_{13}^2 + \chi_{12}^2 + \chi_{21}^2} \tag{193}$$

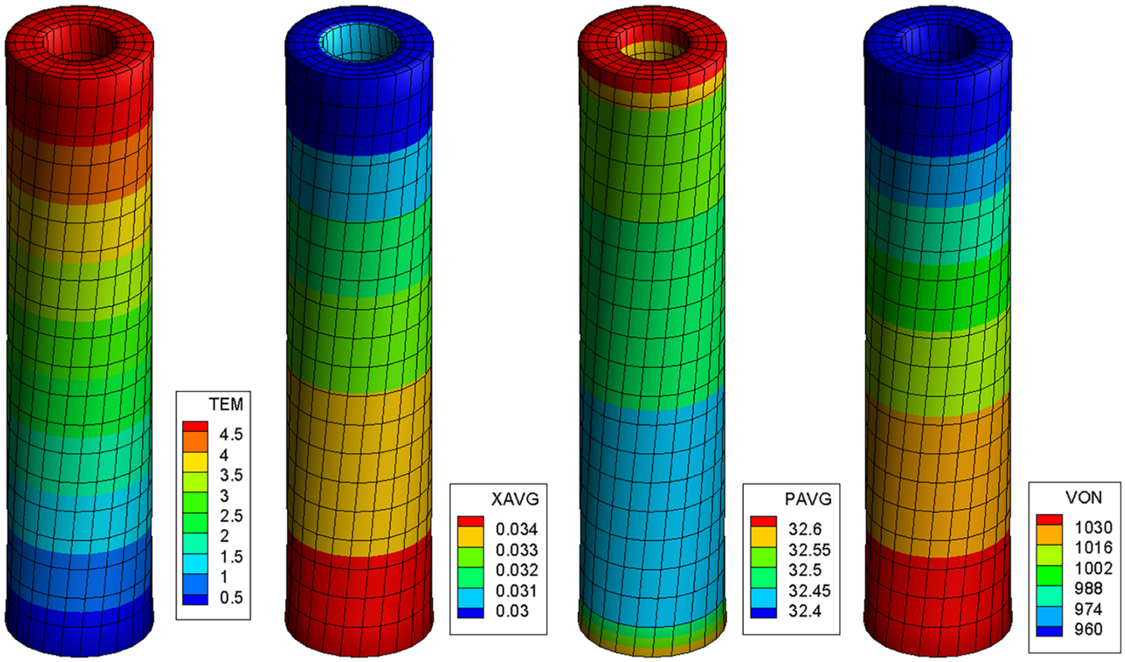
This means if there is no micromotion, then $\chi_{kK} = \delta_{kK}$ and δ_{kK} is the shifter. Therefore in Eq. (193) we have three terms like $\chi_{11} - \delta_{11}, \chi_{22} - \delta_{22}$, and $\chi_{33} - \delta_{33}$ and the other six terms like $\chi_{23}, \chi_{32}, \chi_{31}, \chi_{13}, \chi_{12}, \chi_{21}$.

In this work, we are using the MKS system, i.e., meter, kilogram, second are the units for length, mass, and time, respectively. Also, let the units of an electric charge, electric field, and temperature be $q, e,\ and\ {}^oK$, respectively. Some typical numerical values of the material constants common to both stages are specified as

$$\begin{aligned}
&\overline{A} = 0.002\, K/M/s^2/{}^oK, \quad \overline{B} = 0.002\, K/M/s^2/{}^oK\\
&\lambda^1 = 300\, K/M/s^2, \quad \mu^{11} = 225\, K/M/s^2, \quad \mu^{12} = 75\, K/M/s^2\\
&\lambda^2 = 3\, K/M/s^2, \quad \mu^2 = 3\, K/M/s^2\\
&\lambda^3 = 0.75\, K/M/s^2, \quad \mu^3 = 0.75\, K/M/s^2\\
&C^i = 10\, KM/s^2, \quad i \in [1, 15]\\
&\overline{\lambda}^1 = 0.06\, KM/s, \quad \overline{\mu}^{11} = 0.045\, KM/s, \quad \overline{\mu}^{12} = 0.015\, KM/s\\
&\overline{\lambda}^2 = 0.0006\, KM/s, \quad \overline{\mu}^2 = 0.0004\, KM/s
\end{aligned}$$

$$
\begin{aligned}
&\overline{\lambda}^3 = 0.00015\,KM/s, \quad \overline{\mu}^3 = 0.0001\,KM/s\\
&g^1 = 2000\,KM/s^3/\,(^oK)^2\\
&g^2 = 10\,K/\,(^oK)\,/s^3/e\\
&g^3 = 10\,K/M/s^3/e^2\\
&E_Y = 1, \quad \kappa = 0.2, \quad H = 5, \quad C_1 = 0.6, \quad C_2 = 0.4\\
&\gamma = 200000\,M^2/s^2/^oK, \quad T^o = 300^oK\\
&\rho^o = 1000\,K/M^3, \quad I = 0.002\,M^2\\
&\varepsilon = 2, \quad \mu = 2
\end{aligned}
\tag{194}
$$

These numerical values are our trial values, to the most are our educated guessing. Because, to the best of our knowledge, we don't know whether there exits a testing machine which can measure microstresses, moment stresses, and micromotions.



(a) (b) (c) (d)

Figure 1: (**a**) Temperature variation, (**b**) $L2-norm$ of the micromotion, (**c**) $L2-norm$ of the plastic strain, (**d**) $L2-norm$ of the 2nd order Piola-Kirchhoff stress. The color bars indicate the magnitudes of those variables.

In $Stage\,I$, we solve Eqs. (165), (166) and (168) dynamically and seek the static or nearly static solutions with $\mathbf{E}^* = \mathbf{B} = \mathbf{P} = \mathbf{M}^* = 0$. The assumption $\mathbf{E}^* = \mathbf{B} = \mathbf{P} = \mathbf{M}^* = 0$ implies that $\mathbf{F} = \mathbf{L} = W = 0$ (cf. Eqs. (36)–(38)) and also $\mathbf{t}^m = \mathbf{t}, \mathbf{s}^m = \mathbf{s}, \mathbf{T}^m = \mathbf{T}, \mathbf{S}^m = \mathbf{S}$ (cf. Eqs. (42)–(44)). In $Stage\,II$, based on the static or nearly static solutions obtained in $Stage\,I$, namely $\mathbf{U}$, T, and χ, we calculate the forcing terms, a and $\mathbf{C}$, and solve the differential equations, Eqs. (169) and (170), for ϕ and $\mathbf{A}$. Of course, one may generalize the procedure a little bit: assume $\mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{M}$ are constants, not necessarily vanishing. This mean $\mathbf{F}, \mathbf{L}$, and W are not vanishing either. They should be calculated according to Eqs. (36)–(38), respectively.

In a simpler case with the assumption that there are no free charge and no current, the forcing terms are reduced to

$$a = -\frac{1}{\varepsilon} P^m_{k,k} \tag{195}$$

$$C_k = \mu \left\{ \varepsilon_{kij} M^m_{j,i} + \frac{1}{c} \dot{P}^m_k \right\} \tag{196}$$

Also, because P^m_k and M^m_k are the polarization and magnetization induced by strains (cf. Eq. (144)) and from *Stage I* the strains are static or nearly static solutions, Eq. (196) is further reduced to

$$C_k = \mu \varepsilon_{kij} M^m_{j,i} \tag{197}$$

What if we would like to interchange the order of treatment for Set A and Set B. Of course, the variables in Set A move fast, but if we seek static solutions for those variables in Set A and seek dynamic solutions for those variables in Set B, then the order is reversed. For example, now in *Stage I*, we seek static solutions of scalar potential and vector potentials—they are mutually independent of each other. Then, in *Stage II*, we seek the dynamic solutions of temperature, displacements, and micromotions. Of course, they are moving faster than the static scalar and vector potential—their velocities are zero now.

Numerical Results of the Torsion Problem

Let the specimen be a hollow cylinder occupying

$$V = \{R, \theta, Z \mid \quad 1.0 \le R \le 2.0, \quad 0 \le \theta \le 2\pi, \quad 0 \le Z \le 30\} \tag{198}$$

The finite element mesh has 2520 nodes and 1920 8-node solid element. In *Stage I*, each node has 13 unknowns (three for **U**, nine for $\boldsymbol{\chi}$, and one for T), therefore this sample problem has 32,760 degrees of freedom. Here central difference method is employed to solve Eqs. (165)–(168) which are two wave equations and one diffusion equation. To ensure the numerical stability, the time step Δt cannot be too large. For this sample problem, in *Stage I*, we use $\Delta t = 0.001$ sec and have verified that such Δt not only provides stability but also accuracy.

The boundary conditions are specified as: the surface force, surface moment, and heat flow are vanishing (cf. Eq. (189)), i.e., $\hat{F}_M = \hat{M}_{kK} = \hat{Q} = 0$, except

(I) At the bottom surface, $Z = 0$, the displacements and temperature are specified as

$$U_X = U_Y = U_Z = T = 0 \tag{199}$$

(II) At the top surface, $Z = 30$, a torsion and an elongation are applied, and temperature is specified as

$$U_x = r\cos(\theta + \Delta\theta) - X, U_y = r\sin(\theta + \Delta\theta) - Y, U_z = 3.0, T = 5.0 \tag{200}$$

where $\Delta\theta = 0.25\pi$ and

$$r \equiv \sqrt{X^2 + Y^2}, \theta = \tan^{-1}(Y/X) \tag{201}$$

The corresponding system of equations, Eqs. (165)–(168), statically can be expressed as

$$f^1_{M\alpha} + f^2_{M\alpha} + f^3_{M\alpha} = 0 \quad \text{or} \quad \mathbf{f}(\mathbf{U}, \boldsymbol{\chi}, \mathbf{T}) = 0 \tag{202}$$

$$\phi^1_{kK\alpha} + \phi^2_{kK\alpha} + \phi^3_{kK\alpha} + \phi^4_{kK\alpha} + \phi^5_{kK\alpha} + \phi^6_{kK\alpha} = 0 \quad \text{or} \quad \boldsymbol{\varphi}(\mathbf{U}, \boldsymbol{\chi}, \mathbf{T}) = 0 \tag{203}$$

$$q^1_\alpha + q^2_\alpha + q^3_\alpha + q^4_\alpha + q^5_\alpha + q^6_\alpha + q^7_\alpha = 0 \quad \text{or} \quad \mathbf{q}(\mathbf{U}, \boldsymbol{\chi}, \mathbf{T}) = 0 \tag{204}$$

Usually we would solve Eqs. (202)–(204) by using Newton-Raphson method. Since, in the finite element analysis of micromorphic theory each node has 13 degrees of freedom, this simple model has 32,760 degrees of freedom. In this case, Newton-Raphson method involves a stiffness matrix of size 32,760 × 32,760. This means it is practically impossible to solve it by Newton-Raphson method. Another way is to solve Eqs. (165)–(168) dynamically with proper damping coefficients so that, with reasonable number of time steps, one may obtain nearly static solutions.

The solutions for $Stage\ I$ are presented in Fig. 1.

Fig. 1 are the nearly static solutions of temperature variation, L2-norms of micromotion, plastic strain, and 2nd order Piola-Kirchhoff stress. One may observe the axis symmetry, the twist of the hallow cylinder through the finite element mesh, and the magnitude through the color bars.

In $Stage\ II$, we investigate four cases. In general, we have

$$\phi = A_x = A_y = A_z = 0 \tag{205}$$

Except for the following four cases:

$$\text{Case 1: } \frac{\varepsilon\mu}{c^2}\ddot{A}_1 - \nabla^2 A_1 = C_1 \tag{206}$$

Initial Conditions: $A_1 = \dot{A}_1 = 0$

Boundary Conditions: at $Z = 0$ $A_1 = \begin{cases} -\sin(\omega t) & \text{if } 0 \le \omega t \le 4\pi \\ 0 & \text{if } \omega t \ge 4\pi \end{cases}$

At $Z = 30$, $A_1 = 0$

$$\text{Case 2: } \frac{\varepsilon\mu}{c^2}\ddot{A}_2 - \nabla^2 A_2 = C_2 \tag{207}$$

Initial Conditions: $A_2 = \dot{A}_2 = 0$

Boundary Conditions: at $Z = 0$ $A_2 = \begin{cases} -\sin(\omega t) & \text{if } 0 \le \omega t \le 4\pi \\ 0 & \text{if } \omega t \ge 4\pi \end{cases}$

At $Z = 30$, $A_2 = 0$

$$\text{Case 3: } \frac{\varepsilon\mu}{c^2}\ddot{A}_3 - \nabla^2 A_3 = C_3 \tag{208}$$

Initial Conditions: $A_3 = \dot{A}_3 = 0$

Boundary Conditions: at $Z = 0$ $A_3 = \begin{cases} -\sin(\omega t) & \text{if } 0 \le \omega t \le 4\pi \\ 0 & \text{if } \omega t \ge 4\pi \end{cases}$

At $Z = 30$, $A_3 = 0$

$$\text{Case 4: } \frac{\varepsilon\mu}{c^2}\ddot{\phi} - \nabla^2 \phi = a \tag{209}$$

Initial Conditions: $\phi = \dot{\phi} = 0$

Boundary Conditions: at $Z = 0$ $\phi = \begin{cases} -\sin(\omega t) & \text{if } 0 \le \omega t \le 4\pi \\ 0 & \text{if } \omega t \ge 4\pi \end{cases}$

At $Z = 30$, $\phi = 0$

Notice that, in Eqs. (206)–(209), $\omega = 4\pi/(400\Delta t)$.

It is worthwhile to mention that the time step in $Stage\ II$ is $\Delta t = 0.25 \times 10^{-9}$ s, comparing with $\Delta t = 0.001$ s, in $Stage\ I$. This huge difference justifies the treatment of the coupling problem by a two-stage solution procedure. Also, the boundary conditions, associated with Eqs. (206)–(209), show that there is a half sine wave started at the bottom surface ($Z = 0$) and then bounced back and forth between the top and bottom surfaces (cf. Figs. 2–5)—this is indicated through boundary conditions at $Z = 0$ after $\omega t \geq 4\pi$ and at $Z = 30$.

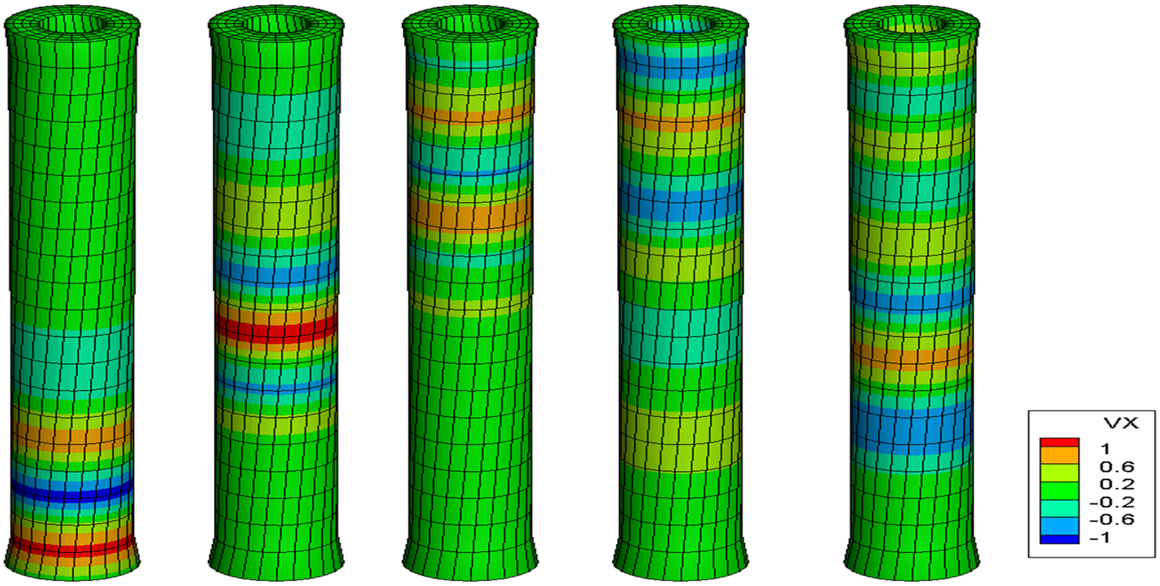


Figure 2: Numerical results of A_x for Case 1: The first, second, third, fourth, and fifth column are referred to time at $400\,\Delta t, 800\,\Delta t, 1200\,\Delta t, 1600\,\Delta t,$ and $2000\,\Delta t$, respectively. The color bars indicate the magnitudes of the variables.

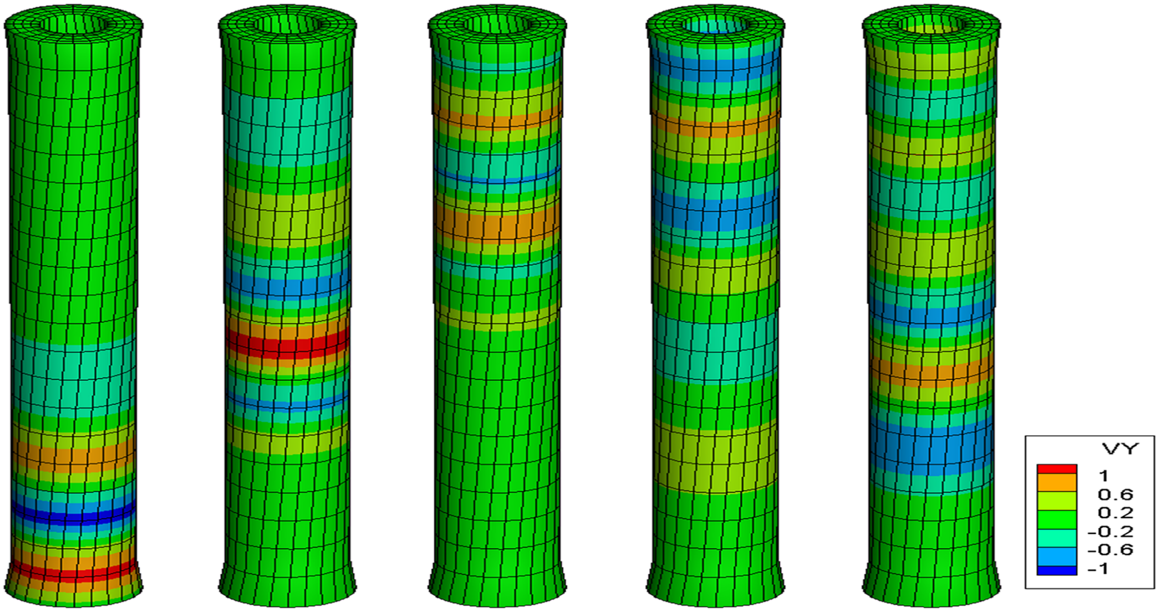


Figure 3: Numerical results of A_y for Case 2: The first, second, third, fourth, and fifth column are referred to time at $400\,\Delta t, 800\,\Delta t, 1200\,\Delta t, 1600\,\Delta t,$ and $2000\,\Delta t$, respectively. The color bars indicate the magnitudes of the variables.

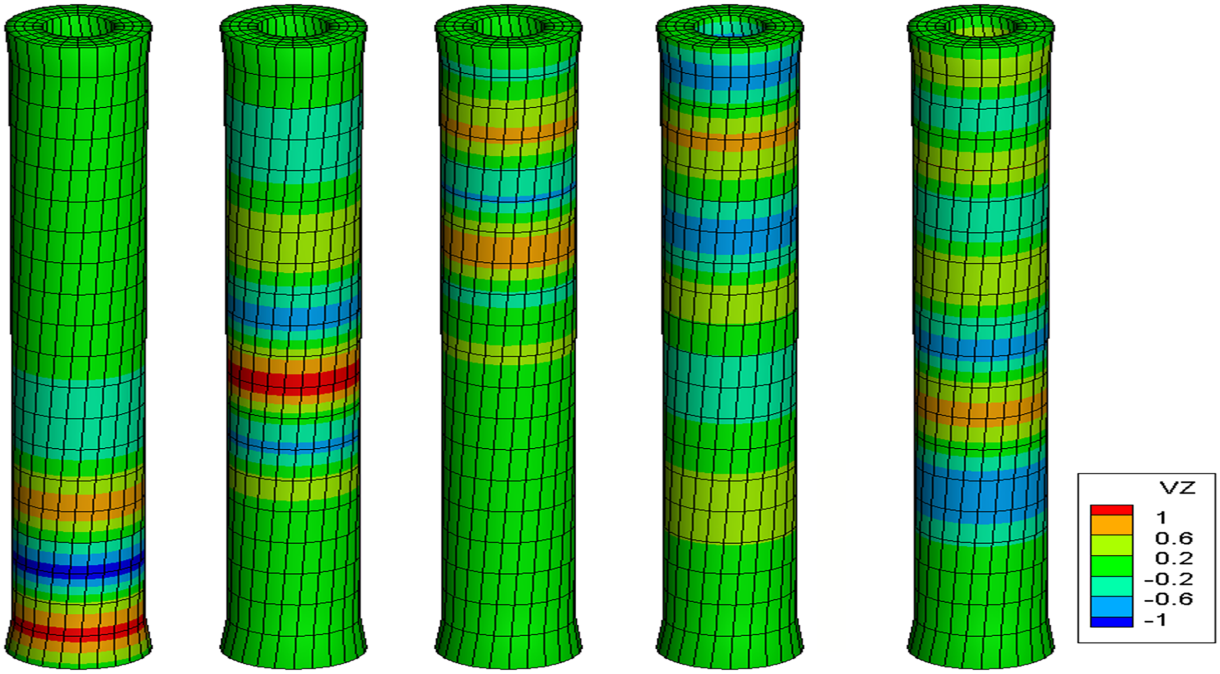


Figure 4: Numerical results of A_z for Case 3: The first, second, third, fourth, and fifth column are referred to time at $400\ \Delta t, 800\ \Delta t, 1200\ \Delta t, 1600\ \Delta t$ and $2000\ \Delta t$, respectively. The color bars indicate the magnitudes of the variables.

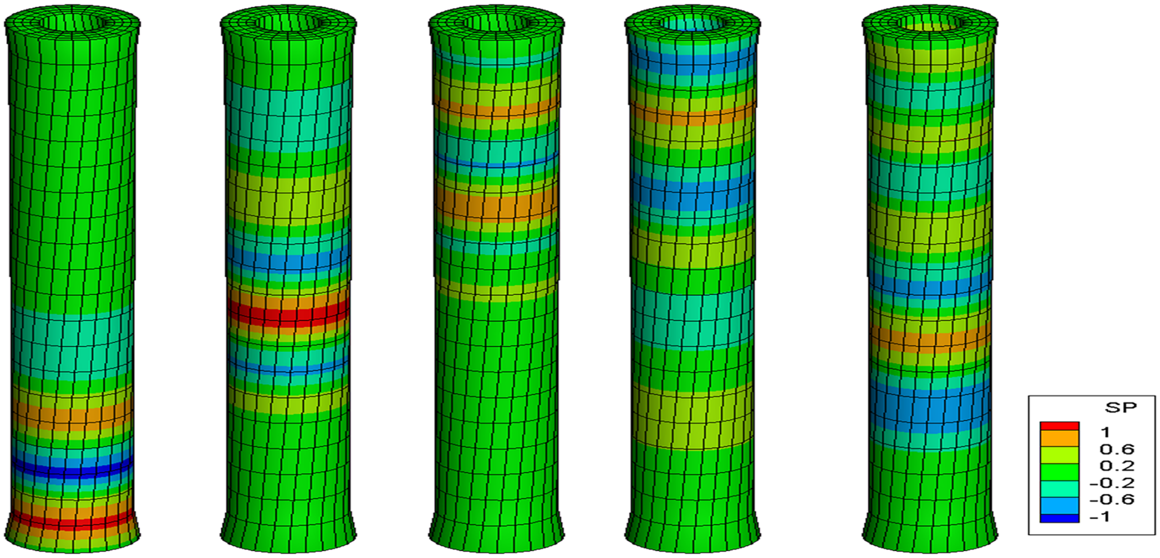


Figure 5: Numerical results of ϕ for Case 4: The first, second, third, fourth, and fifth column are referred to time at $400\ \Delta t, 800\ \Delta t, 1200\ \Delta t, 1600\ \Delta t,$ and $2000\ \Delta t$, respectively. The color bars indicate the magnitudes of the variables.

To understand the *Stage II* solutions about electromagnetic wave propagation in deformed micromorphic medium, we present the following discussion. For example, in Case 1, additional to the forcing term (cf. Eqs. (161) and (162)), and boundary condition, Eq. (206), one will obtain $\phi = A_y = A_z = 0$ for all the time and a wave propagation of A_x which will induce a wave propagation of **E** and **B** according to $\mathbf{B} = \nabla \times \mathbf{A}, \mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}/c$. In other words, we obtain

$$\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & 0 & 0 \end{vmatrix} = \frac{\partial A_x}{\partial z}\mathbf{j} - \frac{\partial A_x}{\partial y}\mathbf{k}$$

$$E_x = -\frac{1}{c}\frac{dA_x}{dt} \tag{210}$$

Similarly, for case 2, Case 3, and Case 4, one obtains the followings, respectively.

$$\mathbf{B} = -\frac{\partial A_y}{\partial z}\mathbf{i} + \frac{\partial A_y}{\partial x}\mathbf{k}$$

$$E_y = -\frac{1}{c}\frac{dA_y}{dt} \tag{211}$$

$$\mathbf{B} = \frac{\partial A_z}{\partial y}\mathbf{i} - \frac{\partial A_z}{\partial x}\mathbf{j}$$

$$E_z = -\frac{1}{c}\frac{dA_z}{dt} \tag{212}$$

$$\mathbf{B} = 0$$

$$E_x = -\frac{\partial\phi}{\partial x}, E_y = -\frac{\partial\phi}{\partial y}, E_z = -\frac{\partial\phi}{\partial z} \tag{213}$$

The numerical results for the four cases as functions of time are shown in Figs. 2–5.

9 Conclusion

First, we notice that the balance laws of micromorphic electromagnetic continuum consist of two parts: the thermomechanical (TM) part and the electromagnetic (EM) part. The TM part of the balance laws are the *Conservation of Mass*, *Conservation of Microinertia, Balance of Linear Momentum, Balance of Moments of Momentum, Conservation of Energy, and Entropy Principle* in Eringen's micromorphic theory. The EM part of the balance laws is the Maxwell's equations. The two parts are linked together by the electromagnetic body force, body moment, and energy source. We formulated a constitutive theory for thermo-visco-elastic-plastic-electromagnetic (TVEP-TM) materials. This work is based on *geometrically nonlinear*, or say large-strain, approaches. *Onsager's postulate* is utilized for the derivation of viscosity. *Return-Mapping-Algorithm* is invoked for plasticity. It is well-known that the electric field **E** and the magnetic flux **B** are not independent of each other. To proceed with finite element analysis, one has to utilize the scalar potential ϕ and the vector potential **A** from which the electric field **E** and magnetic flux **B** can be derived.

Finally, five sets of differential equations are obtained for displacements, micromotions, temperature, scalar potential, and vector potential (cf. Eqs. (172)–(176)). The governing equations for displacements **u** and micromotions $\boldsymbol{\chi}$ are mainly acoustic wave equations; the governing equations for temperature **T** are mainly diffusion equations; The governing equations for scalar potential $\boldsymbol{\varphi}$ and vector potential **A** are mainly wave equations with a wave speed in the order of the speed of light. The huge difference between the acoustic wave speed and the speed of light dictates that solving these differential equations simultaneously is numerically impossible and physically meaningless. We solved this problem in two stages. To the electromagnetic waves in terms of the scalar and vector potentials, everything else seems to stand still.

Therefore, in $Stage\, I$, we solve the differential equations for temperature, displacements, and micromotions dynamically and seek the static or nearly static solutions. In $Stage\, II$, based on the static or nearly static solutions obtained in $Stage\, I$, we calculate the forcing terms for the differential equations of ϕ and $\mathbf{A}$. Once ϕ or $\mathbf{A}$ is solved, the induced electric field $\mathbf{E}$ and magnetic flux $\mathbf{B}$ can be derived and obtained. For future study, it is worthwhile to recommend some papers related to micromorphic theory [20–24].

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Conflicts of Interest: The authors declare no conflicts of interest.

Nomenclature

$\mathbf{A}$	A vector A_k, vector potential
A_k	In finite element analysis, $A_k = N_\beta A_{k\beta}$
$\mathbf{B}$	A vector B_k, magnetic flux, $\mathbf{B} = \nabla \times \mathbf{A}$
C	Speed of light
$\mathbf{D}$	A vector D_k, dielectric displacement vector
e	Internal energy density
$\mathbf{E}$	A vector E_k, electric field, $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$
$\mathbf{E}, \mathbf{E}^e, \mathbf{E}^p$	Total strains, elastic strains, and plastic strains, $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$
$f(\mathbf{Y})$	A scalar-valued yield function of independent constitutive variables
$f(\mathbf{Y}) = 0$	A hyper surface, named yield surface which separates the TVE-EM part and the TVEP-EM part of the material
$\mathbf{f}$	A vector f_i, body force (Thermomechanical)
$\mathbf{F}$	A vector F_i, body force (Electromagnetic)
$\mathbf{H}$	A vector H_k, magnetic field
h	Energy source (Thermomechanical)
i_{kl}	A symmetric microinertia tensor, $i_{kl} = i_{lk}$
$\mathbf{J}$	A vector J_k, total current vector
l_{lm}	A 2nd order tensor, body moment (Thermomechanical)
L_{lm}	A 2nd order tensor, body moment (Electromagnetic)
$\mathbf{M}$	Magnetization vector M_k, $\mathbf{M} \equiv \mathbf{B} - \mathbf{H}$
m_{klm}	A third order moment stress tensor
$\mathbf{P}$	Polarization vector P_k, $\mathbf{P} \equiv \mathbf{D} - \mathbf{E}$
q	Free charge density
q_k	A vector, heat flux
R_G	Fixed laboratory frame to where q, $\mathbf{J}, \mathbf{D}, \mathbf{E}, \mathbf{P}, \mathbf{B}, \mathbf{H}, \mathbf{M}$ are referred
R_C	Co-moving reference frame to where q, $\mathbf{J}, \mathbf{D}, \mathbf{E}, \mathbf{P}, \mathbf{B}, \mathbf{H}, \mathbf{M}$ are referred
$\mathbf{R}$	A generalized vector of internal variables, named as the hardening parameters i.e., $\mathbf{R} = \{\mathbf{G}, g\}$. **G has the same dimension as the strain**, g is a scalarfunction.

s_{kl}	A symmetric microstress tensor, $s_{kl} = s_{lk}, s_{kl} = s_{kl}^{m} + s_{kl}^{em}$
s_{kl}^{m}	The mechanical part of the microstress tensor, $s_{kl}^{m} = s_{kl} - \frac{1}{2}\left(P_k E_l^* + P_l E_k^* + M_k^* B_l + M_l^* B_k\right)$
s_{kl}^{em}	The electromagnetic part of the microstress tensor, $s_{kl}^{em} = -\frac{1}{2}\left(P_k E_l^* + P_l E_k^* + M_k^* B_l + M_l^* B_k\right)$
t_{kl}	A non-symmetric Cauchy stress tensor, $t_{kl} \neq t_{lk}, t_{kl} = t_{kl}^{m} + t_{kl}^{em}$
t_{kl}^{m}	The mechanical part of the Cauchy stress tensor, $t_{kl}^{m} = t_{kl} + P_k E_l^* + M_k^* B_l$
t_{kl}^{em}	The electromagnetic part of the Cauchy stress tensor, $t_{kl}^{em} = -P_k E_l^* - M_k^* B_l$
T	Temperature
T_β	In finite element analysis, $T = N_\beta T_\beta$
$\mathbf{U}$	A vector U_K, displacement
$U_{K\alpha}$	In finite element analysis, $U_K = N_\beta U_{K\beta}$
$\mathbf{v}$	Velocity vector v_k
W	Energy source (Electromagnetic)
$\mathbf{W}$	The set of internal variables, $\mathbf{W} = \left\{\boldsymbol{\alpha}^p, \boldsymbol{\beta}^p, \boldsymbol{\gamma}^p, \mathbf{R}\right\}$
X	Eulerian coordinate of a particle P
X	Lagrangian coordinate of a particle P
$\mathbf{Y}$	Set of independent objective constitutive variables
$\mathbf{Z}$	Set if dependent objective constitutive variables
β	In finite element analysis, β means $\beta - th$ shape function
$\boldsymbol{\alpha}$	First strain tensor, $\alpha_{KL} \equiv x_{k,K}\overline{\chi}_{Lk} - \delta_{KL}$
$\boldsymbol{\beta}$	Second strain tensor (symmetric), $\beta_{KL} = \chi_{kK}\chi_{kL} - \delta_{KL} = \beta_{LK}$
$\boldsymbol{\gamma}$	Third strain tensor (third order), $\gamma_{KLM} \equiv \overline{\chi}_{Kk}\chi_{kL,M}$
$\mathbf{E} \equiv \begin{vmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{vmatrix}$	Total strains
$\mathbf{E}^e \equiv \begin{vmatrix} \boldsymbol{\alpha}^e \\ \boldsymbol{\beta}^e \\ \boldsymbol{\gamma}^e \end{vmatrix}$	Elastic strains
$\mathbf{E}^p \equiv \begin{vmatrix} \boldsymbol{\alpha}^p \\ \boldsymbol{\beta}^p \\ \boldsymbol{\gamma}^p \end{vmatrix}$	Plastic strains
$\alpha^p, \beta^p, \gamma^p$	The corresponding plastic strains of α, β, γ
ρ	Mass density
ϕ	Scalar potential
ϕ_β	In finite element analysis, $\phi = N_\beta \phi_\beta$
η	Entropy density
ψ	Helmholtz free energy density, $\psi \equiv e - \theta\eta - \rho^{-1}E_k^* P_k$
θ	Absolute temperature
υ_{lm}	Microgyration tensor
σ_{kl}	Spin inertia tensor, $\sigma_{kl} \equiv i_{ml}\left(\dot{\upsilon}_{km} + \upsilon_{kn}\upsilon_{nm}\right)$
χ_{kK}	The affine micromotion which links ξ_k and Ξ_K, i.e., $\xi_k = \chi_{kK}\left(\mathbf{X}, t\right)\Xi_K$
$\overline{\chi}_{Kk}$	The inverse of χ_{kK}, i.e., $\chi_{kK}\overline{\chi}_{Kl} = \delta_{kl}, \overline{\chi}_{Kk}\chi_{kL} = \delta_{KL}$
$\chi_{kK\beta}$	In finite element analysis, $\chi_{kK} = N_\beta \chi_{kK\beta}$
$\boldsymbol{\xi}^{\alpha}$	The Eulerian coordinates of $\alpha - th$ material point in the particle
Ξ^{α}	Vector of $\alpha - th$ material point that attached to P, representing the inner structure of P

Appendix A

It is noticed that the quantities $\mathbf{D}, \mathbf{E}, \mathbf{P}, \mathbf{B}, \mathbf{H}, \mathbf{M}$ are all referred to a fixed laboratory frame R_G. On the other hand, $\mathbf{D}^*, \mathbf{E}^*, \mathbf{P}^*, \mathbf{B}^*, \mathbf{H}^*, \mathbf{M}^*$ are referred to a co-moving frame R_C. In relativistic electromagnetic theory, Jackson [10] indicated and explained that the Lorentz Transformation of $\mathbf{E}$ and $\mathbf{B}$ fields can be expressed as

$$\mathbf{E}^* = \gamma\left(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{E}\right), \quad \mathbf{E} = \gamma\left(\mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{E}^*\right) \tag{A1}$$

$$\mathbf{B}^* = \gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{B}\right), \mathbf{B} = \gamma\left(\mathbf{B}^* + \boldsymbol{\beta} \times \mathbf{E}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{B}^*\right) \tag{A2}$$

where

$$\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c}, \beta = \sqrt{\boldsymbol{\beta} \cdot \boldsymbol{\beta}}, \gamma \equiv \frac{1}{\sqrt{1 - \boldsymbol{\beta} \cdot \boldsymbol{\beta}}} = \frac{1}{\sqrt{1 - \beta^2}} \tag{A3}$$

This means the inverse can be found by interchanging the star and the no-star quantities and by changing $\boldsymbol{\beta}$ to $-\boldsymbol{\beta}$.

To prove Jackson's formula, Eqs. (A1) and (A2), first notice that

$$\gamma^2\left(1 - \beta^2\right) = 1, -2\frac{\gamma^3}{\gamma+1} + \gamma^2 + \frac{\gamma^4}{\left(\gamma+1\right)^2}\beta^2 = 0 \tag{A4}$$

$$\begin{aligned}
&\boldsymbol{\beta} \times \left(\boldsymbol{\beta} \times \mathbf{E}\right) = 0\\
&\boldsymbol{\beta} \times \left(\boldsymbol{\beta} \times \mathbf{B}\right) = 0\\
&\boldsymbol{\beta} \times \left(\boldsymbol{\beta} \times \mathbf{E}\right) = \boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{E}\right) - \beta^2\mathbf{E}\\
&\boldsymbol{\beta} \times \left(\boldsymbol{\beta} \times \mathbf{B}\right) = \boldsymbol{\beta}\left(\boldsymbol{\beta} \cdot \mathbf{B}\right) - \beta^2\mathbf{B}
\end{aligned} \tag{A5}$$

Then substitute E into E* of Eq. (A1) and it results

$$\begin{aligned}
\mathbf{E} &= \gamma(\mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^*) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}^*)\\
&= \gamma\left\{\left[\gamma\left(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})\right]\right.\\
&\quad\left. - \boldsymbol{\beta} \times \left[\gamma\left(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B})\right]\right\}\\
&\quad - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta} \cdot \left[\gamma\left(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})\right]\right\}\\
&= \gamma\left\{\gamma\mathbf{E} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) + \gamma\left[\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) - \beta^2\mathbf{E}\right]\right\}\\
&\quad - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta} \cdot \left[\gamma\left(\mathbf{E} - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E})\right)\right]\right\}
\end{aligned}$$

$$\begin{aligned}
&+\gamma^2\boldsymbol{\beta}\times\mathbf{B}-\gamma\boldsymbol{\beta}\times\left[\gamma\mathbf{B}-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})\right]\\
&=-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta}\cdot\left[-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})\right]\right\}\\
&=\mathbf{E}\left(\gamma^2-\gamma^2\beta^2\right)+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})\left\{-\frac{\gamma^3}{\gamma+1}+\gamma^2-\frac{\gamma^3}{\gamma+1}+\frac{\gamma^4}{(\gamma+1)^2}\beta^2\right\}\\
&=\mathbf{E}
\end{aligned}\tag{A6}$$

Now substitute B into B* of Eq. (A2) and it results

$$\begin{aligned}
\mathbf{B}=\;&\gamma(\mathbf{B}^*+\boldsymbol{\beta}\times\mathbf{E}^*)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B}^*)\\
&=\gamma\left\{[\gamma(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]+\boldsymbol{\beta}\times[\gamma(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})]\right\}\\
&-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta}\cdot[\gamma(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]\right\}\\
&=\gamma\left\{-\gamma\boldsymbol{\beta}\times\mathbf{E}+\boldsymbol{\beta}\times[\gamma(\mathbf{E})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})]\right\}+\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta}\cdot[\gamma(\boldsymbol{\beta}\times\mathbf{E})]\right\}\\
&+\gamma\left\{[\gamma\mathbf{B}-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]+\boldsymbol{\beta}\times[\gamma(\boldsymbol{\beta}\times\mathbf{B})]\right\}\\
&-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta}\cdot[\gamma(\mathbf{B})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]\right\}\\
&=\gamma\left\{[\gamma\mathbf{B}-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]+\gamma[\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})-\beta^2\mathbf{B}]\right\}\\
&-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left\{\boldsymbol{\beta}\cdot[\gamma(\mathbf{B})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})]\right\}\\
&=\mathbf{B}\left\{\gamma^2-\gamma^2\beta^2\right\}+\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})\left\{-\frac{\gamma^3}{\gamma+1}+\gamma^2-\frac{\gamma^3}{\gamma+1}+\frac{\gamma^4\beta^2}{(\gamma+1)^2}\right\}\\
&=\mathbf{B}
\end{aligned}\tag{A7}$$

This proves that Eqs. (A1) and (A2) are valid formula in Lorentz Transformation.

Following the same pattern of Eqs. (A1) and (A2), we write

$$\mathbf{D}^*=\gamma\left(\mathbf{D}+\boldsymbol{\beta}\times\mathbf{H}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{D}\right)\tag{A8}$$

$$\mathbf{H}^*=\gamma\left(\mathbf{H}-\boldsymbol{\beta}\times\mathbf{D}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{H}\right)\tag{A9}$$

$$\mathbf{M}^*=\gamma\left(\mathbf{M}+\boldsymbol{\beta}\times\mathbf{P}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{M}\right)\tag{A10}$$

$$\mathbf{P}^*=\gamma\left(\mathbf{P}-\boldsymbol{\beta}\times\mathbf{M}\right)-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{P}\right)\tag{A11}$$

Similar to the derivation of Eqs. (A6) and (A7), one may prove that

$$\mathbf{D} = \gamma\left(\mathbf{D}^* - \boldsymbol{\beta} \times \mathbf{H}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{D}^*\right) \tag{A12}$$

$$\mathbf{H} = \gamma\left(\mathbf{H}^* + \boldsymbol{\beta} \times \mathbf{D}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{H}^*\right) \tag{A13}$$

$$\mathbf{M} = \gamma\left(\mathbf{M}^* - \boldsymbol{\beta} \times \mathbf{P}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{M}^*\right) \tag{A14}$$

$$\mathbf{P} = \gamma\left(\mathbf{P}^* + \boldsymbol{\beta} \times \mathbf{M}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{P}^*\right) \tag{A15}$$

Since polarization and magnetization are defined as

$$\begin{aligned}&\mathbf{P} = \mathbf{D} - \mathbf{E}, \quad \mathbf{P}^* = \mathbf{D}^* - \mathbf{E}^* \\ &\mathbf{M} = \mathbf{B} - \mathbf{H}, \quad \mathbf{M}^* = \mathbf{B}^* - \mathbf{H}^*\end{aligned} \tag{A16}$$

One may verify that

$$\begin{aligned}\mathbf{P} &= \mathbf{D} - \mathbf{E} \\ &= \gamma\left(\mathbf{D}^* - \boldsymbol{\beta} \times \mathbf{H}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{D}^*\right) - \gamma\left(\mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^*\right) + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{E}^*\right) \\ &= \gamma\left(\mathbf{P}^* + \boldsymbol{\beta} \times \mathbf{M}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{P}^*\right) \\ &= \mathbf{P}\end{aligned} \tag{A17}$$

$$\begin{aligned}\mathbf{M} &= \mathbf{B} - \mathbf{H} \\ &= \gamma\left(\mathbf{B}^* + \boldsymbol{\beta} \times \mathbf{E}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{B}^*\right) - \gamma\left(\mathbf{H}^* + \boldsymbol{\beta} \times \mathbf{D}^*\right) + \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\boldsymbol{H}^*\right) \\ &= \gamma\left(\mathbf{M}^* - \boldsymbol{\beta} \times \mathbf{P}^*\right) - \frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\mathbf{M}^*\right) \\ &= \mathbf{M}\end{aligned} \tag{A18}$$

Recall that $\gamma \equiv \left(1 - \beta^2\right)^{-\frac{1}{2}}$, the Taylor series expansion of γ about $\beta = 0$ is obtained as

$$\gamma \approx 1 + \frac{1}{2}\beta^2 + \frac{3}{4}\beta^4 + \ldots. \tag{A19}$$

For the time being, let's keep the Taylor series only to β^2. Then we have

$$\frac{\gamma^2}{\gamma+1} \approx \frac{1+\beta^2}{2+\frac{1}{2}\beta^2} \approx \frac{1}{2} + \frac{3}{8}\beta^2 \tag{A20}$$

Therefore, for non-relativistic electromagnetic theory we have

$$\mathbf{E}^* = \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}, \quad \mathbf{E} = \mathbf{E}^* - \boldsymbol{\beta} \times \mathbf{B}^* \tag{A21}$$

$$\mathbf{B}^* = \mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}, \quad \mathbf{B} = \mathbf{B}^* + \boldsymbol{\beta} \times \mathbf{E}^* \tag{A22}$$

$$\mathbf{D}^* = \mathbf{D} + \boldsymbol{\beta} \times \mathbf{H}, \quad \mathbf{D} = \mathbf{D}^* - \boldsymbol{\beta} \times \mathbf{H}^* \tag{A23}$$

$$\mathbf{H}^* = \mathbf{H} - \boldsymbol{\beta} \times \mathbf{D}, \quad \mathbf{H} = \mathbf{H}^* + \boldsymbol{\beta} \times \mathbf{D}^* \tag{A24}$$

$$\mathbf{P}^* = \mathbf{P} - \boldsymbol{\beta} \times \mathbf{M}, \quad \mathbf{P} = \mathbf{P}^* + \boldsymbol{\beta} \times \mathbf{M}^* \tag{A25}$$

$$\mathbf{M}^* = \mathbf{M} + \boldsymbol{\beta} \times \mathbf{P}, \quad \mathbf{M} = \mathbf{M}^* - \boldsymbol{\beta} \times \mathbf{P}^* \tag{A26}$$

For illustrative purpose, let's substitute E into E* and substitute B* into B of Eqs. (A21) and (A22) and it results

$$\begin{aligned}\mathbf{E} &= \mathbf{E}' - \boldsymbol{\beta} \times \mathbf{B}' \\ &= \mathbf{E} + \boldsymbol{\beta} \times \mathbf{B} - \boldsymbol{\beta} \times \{\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\} \\ &= \mathbf{E} + \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{E}) \\ &= \mathbf{E} + \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}) - \beta^2 \mathbf{E} \\ &\approx \mathbf{E}\end{aligned} \tag{A27}$$

$$\begin{aligned}\mathbf{B} &= \mathbf{B}' + \boldsymbol{\beta} \times \mathbf{E}' \\ &= \mathbf{B} - \boldsymbol{\beta} \times \mathbf{E} + \boldsymbol{\beta} \times (\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \\ &= \mathbf{B} + \boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{B}) \\ &= \mathbf{B} + \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) - \beta^2 \mathbf{B} \\ &\approx \mathbf{B}\end{aligned} \tag{A28}$$

Notice that the approximations in Eqs. (A27) and (A28) are made due to the assumption that non-relativistic electromagnetic theory is valid up to small $\boldsymbol{\beta} \equiv \frac{\mathbf{v}}{c} \approx 0$, β^2 is not included because it is even smaller.

Appendix B

Recall the following equations

$$\rho \dot{e} = m_{klm} v_{lm,k} + t_{kl} \left(v_{l,k} - v_{lk}\right) + s_{kl} v_{kl} - q_{k,k} + \rho h + W \tag{70}$$

$$\rho \dot{\eta} + \nabla \cdot (\mathbf{q}/\theta) - \rho h/\theta \geq 0 \tag{71}$$

$$W = \mathbf{E}^* \cdot \left(\dot{\mathbf{P}} + \mathbf{P} \nabla \cdot \mathbf{v}\right) - \mathbf{M}^* \cdot \dot{\mathbf{B}} + \mathbf{J}^* \cdot \mathbf{E}^* \tag{75}$$

$$\psi \equiv e - \theta \eta - \rho^{-1} E_k^* P_k \tag{79}$$

One may verify that Eq. (33) can be re-written as

$$\rho^o \dot{e} = M_{KLM} \dot{\gamma}_{KLM} + T_{KL} \dot{\alpha}_{KL} + S_{KL} \dot{\beta}_{KL} - Q_{K,K} + \rho^o h + JW \tag{A29}$$

where the description of the Lagrangian stresses and heat flux, $M_{KLM}, T_{KL}, S_{KL}, Q_K$, and the description of the Eulerian stresses and heat flux, $m_{klm}, t_{kl}, s_{kl}, q_k$. are related as [5]

$$\begin{aligned} &T_{KL} = J t_{kl} X_{K,k} \chi_{lL}, && t_{kl} = J^{-1} T_{KL} x_{k,K} \overline{\chi}_{Ll} \\ &S_{KL} = \tfrac{1}{2} J s_{kl} \overline{\chi}_{Kk} \overline{\chi}_{Ll}, && s_{kl} = 2J^{-1} S_{KL} \chi_{kK} \chi_{lL} \\ &M_{KLM} = J m_{mkl} X_{M,m} \chi_{kK} \overline{\chi}_{Ll}, && m_{mkl} = J^{-1} M_{KLM} x_{m,M} \overline{\chi}_{Kk} \chi_{lL} \\ &Q_K = J q_k X_{K,k}, && q_k = J^{-1} Q_K x_{k,K} \end{aligned} \tag{A30}$$

and the corresponding strain rates are defined as

$$\begin{aligned}
&\dot{\alpha}_{KL} \equiv a_{kl} x_{k,K} \overline{\chi}_{Ll} \equiv \left(\nu_{l,k} - \upsilon_{lk}\right) x_{k,K} \overline{\chi}_{Ll} \\
&\dot{\beta}_{KL} \equiv 2 b_{kl} \chi_{kK} \chi_{lL} \equiv \left(\upsilon_{kl} + \upsilon_{lk}\right) \chi_{kK} \chi_{lL} \\
&\dot{\gamma}_{KLM} \equiv c_{klm} \overline{\chi}_{Kk} \chi_{lL} x_{m,M} \equiv \upsilon_{kl,m} \overline{\chi}_{Kk} \chi_{lL} x_{m,M}
\end{aligned} \tag{A31}$$

One may readily verify that Eqs. (34) and (39) can be re-written as

$$\rho^o \theta \dot{\eta} - \frac{1}{\theta} Q_K \theta_{,K} + Q_{K,K} - \rho^o h \geq 0 \tag{A32}$$

$$\rho^o \psi \equiv \rho^o \left(e - \theta \eta\right) - J E_k^* P_k \tag{A33}$$

Define $E_K^*, B_K, P_K, M_K^*,$ and J_K^* as

$$\begin{aligned}
&E_K^* \equiv E_k^* x_{k,K} \\
&B_K \equiv B_k x_{k,K} \\
&P_K \equiv J P_k X_{K,k} \\
&M_K^* \equiv J M_k^* X_{K,k} \\
&J_K^* \equiv J J_k^* X_{K,k}
\end{aligned} \tag{A34}$$

One may readily verify that

$$\begin{aligned}
&E_k^* \equiv E_K^* X_{K,k} \\
&B_k \equiv B_K X_{K,k} \\
&P_k \equiv J^{-1} P_K x_{k,K} \\
&M_k^* \equiv J^{-1} M_K^* x_{k,K} \\
&J_k^* \equiv J^{-1} J_K^* x_{k,K}
\end{aligned} \tag{A35}$$

$$\begin{aligned}
&\dot{E}_K^* = \left(\dot{E}_k^* + E_l^* \nu_{l,k}\right) x_{k,K} \\
&\dot{B}_K = \left(\dot{B}_k + B_l \nu_{l,k}\right) x_{k,K} \\
&E_K^* P_K = J E_k^* P_k
\end{aligned} \tag{A36}$$

Theorem 1: *The entropy principle can be expressed as*

$$-\rho^o \left(\dot{\psi} + \eta \dot{\theta}\right) - \frac{1}{\theta} Q_K \theta_{,K} - E_K^* \dot{P}_K - \dot{E}_K^* P_K + M_{KLM} \dot{\gamma}_{KLM} + T_{KL} \dot{\alpha}_{KL} + S_{KL} \dot{\beta}_{KL} + J W \geq 0 \tag{A37}$$

Proof:

Rewrite Eq. (A33) as

$$\rho^o \dot{\psi} = \rho^o \dot{e} - \rho^o \dot{\theta} \eta - \rho^o \theta \dot{\eta} - \dot{E}_K^* P_K - E_K^* \dot{P}_K \tag{A38}$$

Following Eq. (A33), Eq. (81) can be re-written as

$$\begin{aligned}
&\rho^o \theta \dot{\eta} - \frac{1}{\theta} Q_K \theta_{,K} + Q_{K,K} - \rho^o h \\
&\quad = -\rho^o \left(\dot{\psi} + \eta \dot{\theta}\right) + \rho^o \dot{e} - E_K^* \dot{P}_K - \dot{E}_K^* P_K - \frac{1}{\theta} Q_K \theta_{,K} + Q_{K,K} - \rho^o h
\end{aligned} \tag{A39}$$

Substituting Eq. (A29) into Eq. (A39) results

$$\begin{aligned} &-\rho^o\left(\dot{\psi}+\eta\dot{\theta}\right)-E_K^*\dot{P}_K-\dot{E}_K^*P_K \\ &+M_{KLM}\dot{\gamma}_{KLM}+T_{KL}\dot{\alpha}_{KL}+S_{KL}\dot{\beta}_{KL}-\frac{1}{\theta}Q_K\theta_{,K}+JW\geq 0 \end{aligned} \tag{A40}$$

Thus, Theorem 1 is proved. □

Theorem 2: *The entropy principle can also be expressed as*

$$\begin{aligned} &-\rho^o\left(\dot{\psi}+\eta\dot{\theta}\right)+M_{KLM}\dot{\gamma}_{KLM}+T_{KL}\dot{\alpha}_{KL}+S_{KL}\dot{\beta}_{KL}-\frac{1}{\theta}Q_K\theta_{,K} \\ &-P_K\dot{E}_K^*-M_K^*\dot{B}_K+J_K^*E_K^*+Jv_{l,k}\left(P_kE_l^*+M_k^*B_l\right)\geq 0 \end{aligned} \tag{A41}$$

Proof:

Now we rewrite Eq. (38) as

$$\begin{aligned} JW &= J\left\{E_k^*\left(\dot{P}_k+P_k\nabla\cdot\mathbf{v}\right)-M_k^*\dot{B}_k+J_k^*E_k^*\right\} \\ &\equiv \alpha^1+\alpha^2+\alpha^3 \end{aligned} \tag{A42}$$

and derive the followings

$$\begin{aligned} \alpha^1 &\equiv J\left\{E_k^*P_k\nabla\cdot\mathbf{v}+E_k^*\frac{d}{dt}\left[J^{-1}P_Kx_{k,K}\right]\right\} \\ &= J\left\{E_k^*P_k\nabla\cdot\mathbf{v}+E_k^*\left[-J^{-1}\nabla\cdot\mathbf{v}P_Kx_{k,K}+J^{-1}\dot{P}_Kx_{k,K}+J^{-1}P_Kv_{k,l}x_{l,K}\right]\right\} \\ &= J\left\{E_k^*\left[J^{-1}\dot{P}_Kx_{k,K}+J^{-1}P_Kv_{k,l}x_{l,K}\right]\right\} \\ &= E_K^*\dot{P}_K+JE_l^*P_kv_{l,k} \end{aligned} \tag{A43}$$

$$\begin{aligned} \alpha^2 &\equiv -JM_k^*\dot{B}_k \\ &= -J\left(J^{-1}M_K^*x_{k,K}\right)\frac{d}{dt}\left[B_LX_{L,k}\right] \\ &= -M_K^*x_{k,K}\left[\dot{B}_LX_{L,k}-B_Lv_{l,k}X_{L,l}\right] \\ &= -M_K^*\dot{B}_K+Jv_{l,k}M_k^*B_l \end{aligned} \tag{A44}$$

$$\begin{aligned} \alpha^3 &\equiv JJ_k^*E_k^* \\ &= J\left(E_K^*X_{K,k}\right)\left(J^{-1}J_L^*x_{k,L}\right) \\ &= J_K^*E_K^* \end{aligned} \tag{A45}$$

Therefore, we arrive at

$$JW=E_K^*\dot{P}_K-M_K^*\dot{B}_K+J_K^*E_K^*+Jv_{l,k}\left(P_kE_l^*+M_k^*B_l\right) \tag{A46}$$

Since $S_{KL}=S_{LK}$ and $s_{kl}=s_{lk}$, one may readily verify that

$$T_{KL}\dot{\alpha}_{KL}=Jt_{kl}\left(v_{l,k}-\upsilon_{lk}\right),\quad S_{KL}\dot{\beta}_{KL}=\frac{1}{2}Js_{kl}\left(\upsilon_{kl}+\upsilon_{lk}\right) \tag{A47}$$

Now one may further derive $T_{KL}\dot{\alpha}_{KL} + S_{KL}\dot{\beta}_{KL} + J\nu_{l,k}\left(P_k E_l^* + M_k^* B_l\right)$ as

$$\begin{aligned}
&T_{KL}\dot{\alpha}_{KL} + S_{KL}\dot{\beta}_{KL} + J\nu_{l,k}\left(P_k E_l^* + M_k^* B_l\right)\\
&= J\left\{t_{kl}\left(\nu_{l,k} - \upsilon_{lk}\right) + s_{kl}\upsilon_{lk} + \nu_{l,k}\left(P_k E_l^* + M_k^* B_l\right)\right\}\\
&= J\left\{\nu_{l,k}\left(t_{kl} + P_k E_l^* + M_k^* B_l\right) + \left(s_{kl} - t_{kl}\right)\upsilon_{lk}\right\}
\end{aligned} \tag{A48}$$

Define the mechanical part of the Cauchy stresses and microstresses, t_{kl}^m and s_{kl}^m, and the electromagnetic part of the Cauchy stresses and microstresses, t_{kl}^{em} and s_{kl}^{em}, as

$$t_{kl} = t_{kl}^m + t_{kl}^{em}, \quad s_{kl} = s_{kl}^m + s_{kl}^{em} \tag{A49}$$

$$\begin{aligned}
&t_{kl}^{em} \equiv -P_k E_l^* - M_k^* B_l, \quad t_{kl}^m \equiv t_{kl} + P_k E_l^* + M_k^* B_l\\
&s_{kl}^{em} \equiv -P_k E_l^* - M_k^* B_l, \quad s_{kl}^m \equiv s_{kl} + P_k E_l^* + M_k^* B_l
\end{aligned} \tag{A50}$$

Now one may verify that Eq. (A48) as

$$T_{KL}\dot{\alpha}_{KL} + S_{KL}\dot{\beta}_{KL} + J\nu_{l,k}\left(P_k E_l^* + M_k^* B_l\right) = J\left\{t_{kl}^m\left(\nu_{l,k} - \upsilon_{lk}\right) + s_{kl}^m \upsilon_{lk}\right\} \tag{A51}$$

Define

$$S_{KL}^m \equiv \frac{1}{2} J s_{kl}^m \overline{\chi}_{Kk}\overline{\chi}_{Ll} \tag{A52}$$

Then verify that

$$s_{kl}^m = 2J^{-1} S_{MN}^m \chi_{kM}\chi_{lN} \tag{A53}$$

Notice that $S_{KL}^m \neq S_{LK}^m$, $s_{kl}^m \neq s_{lk}^m$, and $\dot{\beta}_{KL} = \dot{\beta}_{LK}$. Now we further derive

$$\begin{aligned}
S_{KL}^m \dot{\beta}_{KL} &= \overline{S}_{KL}^m \dot{\beta}_{KL}\\
&= \frac{1}{2} J \overline{\chi}_{Kk}\overline{\chi}_{Ll}\overline{s}_{kl}^m \chi_{mK}\chi_{nL}\left(\upsilon_{mn} + \upsilon_{nm}\right)\\
&= \frac{1}{2} J\left(\upsilon_{kl} + \upsilon_{lk}\right)\overline{s}_{kl}^m\\
&= J\upsilon_{lk}\overline{s}_{kl}^m
\end{aligned} \tag{A54}$$

Notice that $\overline{S}_{KL}^m$ is the average of S_{KL}^m and S_{LK}^m, i.e., $\overline{S}_{KL}^m \equiv \frac{1}{2}\left(S_{KL}^m + S_{LK}^m\right) = \overline{S}_{LK}^m$. Now we follow the pattern of Eqs. (A52) and (A53) and obtain the following

$$\overline{S}_{KL}^m \dot{\beta}_{KL} = J\upsilon_{lk}\overline{s}_{kl}^m \tag{A55}$$

where $\overline{s}_{kl}^m$ is the average of s_{kl}^m and s_{lk}^m, i.e., $\overline{s}_{kl}^m \equiv \frac{1}{2}\left(\overline{s}_{kl}^m + \overline{s}_{lk}^m\right) = \overline{s}_{lk}^m$. From Eq. (A50), we have

$$\overline{s}_{kl}^m = s_{kl} + \frac{1}{2}\left(P_k E_l^* + P_l E_k^* + M_k^* B_l + M_l^* B_k\right) \tag{A56}$$

Now Eq. (A51) can be re-written as

$$\begin{aligned}
&T_{KL}\dot{\alpha}_{KL} + S_{KL}\dot{\beta}_{KL} + J\nu_{l,k}\left(P_k E_l^* + M_k^* B_l\right)\\
&\quad = J\left\{t_{kl}^m\left(\nu_{l,k} - \upsilon_{lk}\right) + s_{kl}^m \upsilon_{lk}\right\}
\end{aligned}$$

$$= T^m_{KL}\dot{\alpha}_{KL} + \overline{S}^m_{KL}\dot{\beta}_{KL} \tag{A57}$$

Now the conservation of energy and the Clausius-Duhem inequality can be expressed as

$$\begin{aligned}\rho^o\dot{e} = {} & M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + \overline{S}^m_{KL}\dot{\beta}_{KL} \\ & - Q_{K,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_K E^*_K + \rho^o h\end{aligned} \tag{A58}$$

$$\begin{aligned}& -\rho^o\left(\dot{\psi} + \eta\dot{\theta}\right) + M_{KLM}\dot{\gamma}_{KLM} + T^m_{KL}\dot{\alpha}_{KL} + \overline{S}^m_{KL}\dot{\beta}_{KL} \\ & - \frac{1}{\theta}Q_K\theta_{,K} - P_K\dot{E}^*_K - M^*_K\dot{B}_K + J^*_K E^*_K \geq 0\end{aligned} \tag{A59}$$

From now on if there is no ambiguity, we drop the bar on top of S^m_{KL} and s^m_{kl}. □

Appendix C

First, we link the displacement, micromotion, temperature, scalar potential, and vector potential at a generic point in an element to the corresponding nodal values associated with this element as follows:

$$\begin{aligned}& U_K = N_\alpha U_{K\alpha} \\ & \chi_{kK} = N_\alpha \chi_{kK\alpha} \\ & T = N_\alpha T_\alpha \\ & \phi = N_\alpha \phi_\alpha \\ & A_k = N_\alpha A_{k\alpha}\end{aligned} \tag{A60}$$

The weak form of Eq. (165) can be written as

$$\int_V \left\{\left(T_{KL}\overline{\chi}_{Ll}\right)_{,K} + \rho^o\left(f_l - \dot{v}_l\right) + JF_l\right\}\delta u_l dV = 0 \tag{A61}$$

Term by term, we may obtain the followings:

$$\begin{aligned}\int_V \left(T_{KL}\overline{\chi}_{Ll}\right)_{,K}\delta u_l dV &= \oint_S T_{KL}\overline{\chi}_{Ll} n_K \delta u_l dS - \int_V T_{KL}\overline{\chi}_{Ll}\delta u_{l,K} dV \\ &= \delta U_{M\alpha}\left\{\oint_S T_{KL}\overline{\chi}_{Ll} n_K \delta_{lM} N_\alpha dS - \int_V T_{KL}\overline{\chi}_{Ll}\delta_{lM} N_{\alpha,K} dV\right\} \\ &= \delta U_{M\alpha}\left\{\int_{S_\sigma} \hat{F}_M N_\alpha dS - \int_V T_{KL}\overline{\chi}_{Ll}\delta_{lM} N_{\alpha,K} dV\right\}\end{aligned} \tag{A62}$$

where on the boundary the surface force is specified as

$$\hat{F}_M = T_{KL}\overline{\chi}_{Ll} n_K \delta_{lM} \tag{A63}$$

Also, we define

$$f^1_{M\alpha} \equiv \int_{S_F} \hat{F}_M N_\alpha dS$$

$$f^2_{M\alpha} \equiv -\int_V T_{KL}\overline{\chi}_{Ll}\delta_{lM}N_{\alpha,K}dV \tag{A64}$$

The other two terms are derived to be

$$\int_V \left(\rho^o f_l + JF_l\right)\delta u_l dV = \delta U_{M\alpha}\int_V \left(\rho^o f_M + JF_M\right)N_\alpha dV \equiv \delta U_{M\alpha} f^3_{M\alpha} \tag{A65}$$

$$\int_V \rho^o \dot{v}_l \delta u_l dV = \ddot{U}_{M\beta}\delta U_{M\alpha}\int_V \rho^o N_\alpha N_\beta dV \equiv \ddot{U}_{M\beta}\delta U_{M\alpha} M_{\alpha\beta} \tag{A66}$$

Now Eq. (A61) can be re-written as

$$\delta U_{M\alpha}\left\{f^1_{M\alpha} + f^2_{M\alpha} + f^3_{M\alpha} - M_{\alpha\beta}\ddot{U}_{M\beta}\right\} = 0 \tag{A67}$$

Because Eq. (A67) has to be valid for arbitrary virtual displacement $\delta U_{M\alpha}$, the finite element equation for the balance of linear momentum becomes

$$M_{\alpha\beta}\ddot{U}_{M\beta} = f^1_{M\alpha} + f^2_{M\alpha} + f^3_{M\alpha} \tag{A68}$$

For the finite element formulation of the Balance of Moment of Momentum, first we recall the microgyration and spin inertia as [7]

$$\nu_{kl} = \dot{\chi}_{kK}\overline{\chi}_{Kl} \tag{A69}$$

$$\sigma_{kl} \equiv i_{ml}\left(\dot{\nu}_{km} + \nu_{kn}\nu_{nm}\right) \equiv i_{ml}C_{km} \tag{A70}$$

From Eq. (A69), one may obtain

$$\dot{\chi}_{kK} = \nu_{kl}\chi_{lK} \tag{A71}$$

Now we derive the followings

$$\begin{aligned}\ddot{\chi}_{kK} &= \frac{d\dot{\chi}_{kK}}{dt} = \frac{d}{dt}\left(\nu_{kl}\chi_{lK}\right)\\ &= \dot{\nu}_{kl}\chi_{lK} + \nu_{kl}\dot{\chi}_{lK} = \dot{\nu}_{kl}\chi_{lK} + \nu_{kl}\nu_{ln}\chi_{nK}\\ &= \left(\dot{\nu}_{kn} + \nu_{kl}\nu_{ln}\right)\chi_{nK} = C_{kn}\chi_{nK}\end{aligned} \tag{A72}$$

Define

$$\delta A_{kl} \equiv \overline{\chi}_{Kl}\delta\chi_{kK} \tag{A73}$$

Then it is seen that

$$\begin{aligned}\sigma_{kl}\delta A_{kl} &= i_{ml}C_{km}\delta A_{kl} = I_{ML}\chi_{mM}\chi_{lL}C_{km}\delta A_{kl}\\ &= I_{ML}\chi_{mM}\chi_{lL}C_{km}\overline{\chi}_{Kl}\delta\chi_{kK} = I_{ML}\chi_{mM}C_{km}\delta\chi_{kL}\\ &= I_{ML}\ddot{\chi}_{kM}\delta\chi_{kL}\end{aligned} \tag{A74}$$

Now the weak form of Eq. (166) can be written as

$$\int_V \left\{\left(M_{LMK}\overline{\chi}_{Ll}\chi_{mM}\right)_{,K} + T_{ML}x_{m,M}\overline{\chi}_{Ll} - 2S_{LM}\chi_{lL}\chi_{mM} + \rho^o\left(l_{lm} - \sigma_{lm}\right) + JL_{lm}\right\}\delta A_{lm}dV = 0 \tag{A75}$$

The inertia term can be derived as

$$\begin{aligned}\int_V \rho^o\sigma_{lm}\delta A_{lm}dV &= \int_V \rho^o I_{KL}\ddot{\chi}_{kK}\delta\chi_{kL}dV\\ &= \ddot{\chi}_{kL\beta}\delta\chi_{kK\alpha}\int_V \rho^o I_{KL}N_\alpha N_\beta dV \equiv \delta\chi_{kK\alpha}M_{KL\alpha\beta}\ddot{\chi}_{kL\beta}\end{aligned} \tag{A76}$$

If we assume the material is spin isotropic, i.e., $I_{KL} = I\delta_{KL}$, then Eq. (A76) is reduced to

$$\int_V \rho^o\sigma_{lm}\delta A_{lm}dV = \delta\chi_{kK\alpha}IM_{\alpha\beta}\ddot{\chi}_{kK\beta} \tag{A77}$$

The first term in the weak form becomes [5]

$$\begin{aligned}\int_V \left(M_{LMK}\overline{\chi}_{Ll}\chi_{mM}\right)_{,K}\delta A_{lm}dV &= \int_V \left(M_{LMK}\overline{\chi}_{Lk}\chi_{mM}\right)_{,K}\delta A_{km}dV\\ &= \int_V \left(M_{LMK}\overline{\chi}_{Lk}\chi_{mM}\right)_{,K}\delta A_{km}dV = \int_V \left(M_{LMP}\overline{\chi}_{Lk}\chi_{mM}\right)_{,P}\overline{\chi}_{Km}\delta\chi_{kK}dV\\ &= \delta\chi_{kK\alpha}\left\{\int_V \left(M_{LMP}\overline{\chi}_{Lk}\chi_{mM}\right)_{,P}\overline{\chi}_{Km}N_\alpha dV - \int_V M_{LMP}\overline{\chi}_{Lk}\chi_{mM}\left(\overline{\chi}_{Km}N_\alpha\right)_{,P}dV\right\}\\ &= \delta\chi_{kK\alpha}\left\{\int_{S_M}\hat{M}_{kK}N_\alpha dS - \int_V M_{LMP}\overline{\chi}_{Lk}\chi_{mM}\overline{\chi}_{Km,P}N_\alpha dV - \int_V M_{LKM}\overline{\chi}_{Lk}N_{\alpha,M}dV\right\}\end{aligned} \tag{A78}$$

where on the boundary the surface moment is specified as

$$\hat{M}_{kK} \equiv M_{LKM}\overline{\chi}_{Lk}n_M \tag{A79}$$

The second, third, and fourth terms become

$$\begin{aligned}\int_V T_{ML}x_{m,M}\overline{\chi}_{Ll}\delta A_{lm}dV &= \int_V T_{ML}x_{m,M}\overline{\chi}_{Lk}\delta A_{km}dV\\ &= \int_V T_{ML}x_{m,M}\overline{\chi}_{Lk}\overline{\chi}_{Km}\delta\chi_{kK}dV = \delta\chi_{kK\alpha}\int_V T_{ML}x_{m,M}\overline{\chi}_{Lk}N_\alpha dV\end{aligned} \tag{A80}$$

$$\begin{aligned}\int_V -2S_{LM}\chi_{lL}\chi_{mM}\delta A_{lm}dV &= -\int_V 2S_{LM}\chi_{kL}\chi_{mM}\delta A_{km}dV\\ &= -\delta\chi_{kK\alpha}\int_V 2S_{LM}\chi_{kL}\chi_{mM}\overline{\chi}_{Km}N_\alpha dV = -\delta\chi_{kK\alpha}\int_V 2S_{LK}\chi_{kL}N_\alpha dV\end{aligned} \tag{A81}$$

Define

$$\begin{aligned}
\phi^1_{kK\alpha} &\equiv \int\limits_{S_M} \hat{M}_{kK} N_\alpha dS \\
\phi^2_{kK\alpha} &\equiv -\int\limits_{V} M_{LMP} \overline{\chi}_{Lk} \chi_{mM} \overline{\chi}_{Km,P} N_\alpha dV \\
\phi^3_{kK\alpha} &\equiv -\int\limits_{V} M_{LKM} \overline{\chi}_{Lk} N_{\alpha,M} dV \\
\phi^4_{kK\alpha} &\equiv \int\limits_{V} T_{ML} x_{m,M} \overline{\chi}_{Lk} \overline{\chi}_{Km} N_\alpha dV \\
\phi^5_{kK\alpha} &\equiv \int\limits_{V} -2S_{LK} \chi_{kL} N_\alpha dV \\
\phi^6_{kK\alpha} &\equiv \int\limits_{V} \left\{\rho^o l_{km} + JL_{km}\right\} \overline{\chi}_{Km} N_\alpha dV
\end{aligned} \tag{A82}$$

With the assumption of spin isotropy, i.e., $I_{KL} = I\delta_{KL}$, Eq. (A75) can now be re-written as

$$\delta\chi_{kK\alpha} \left\{\phi^1_{kK\alpha} + \phi^2_{kK\alpha} + \phi^3_{kK\alpha} + \phi^4_{kK\alpha} + \phi^5_{kK\alpha} + \phi^6_{kK\alpha} - IM_{\alpha\beta} \ddot{\chi}_{kK\beta}\right\} = 0 \tag{A83}$$

Because Eq. (A83) has to be valid for arbitrary virtual micromotion, $\delta\chi_{kK\alpha}$ the finite element equation for the balance law of moment of momentum becomes

$$IM_{\alpha\beta} \ddot{\chi}_{kK\beta} = \phi^1_{kK\alpha} + \phi^2_{kK\alpha} + \phi^3_{kK\alpha} + \phi^4_{kK\alpha} + \phi^5_{kK\alpha} + \phi^6_{kK\alpha} \tag{A84}$$

Based on the energy equation, the weak form of the conservation law of energy can be written as (cf. Eq. (168)) [5]

$$\begin{aligned}
\int\limits_{V} \Big\{ &\rho^o \gamma \frac{(T + T^o)}{T^o} \dot{T} + (T + T^o) \left[\overline{A}_{KL} \dot{\alpha}^e_{KL} + \overline{B}_{KL} \dot{\beta}^e_{KL}\right] - T^{me}_{KL} \dot{\alpha}^p_{KL} - S^{me}_{KL} \dot{\beta}^p_{KL} - M^e_{KLM} \dot{\gamma}^p_{KLM} \\
&- T^{md}_{KL} \dot{\alpha}_{KL} - S^{md}_{KL} \dot{\beta}_{KL} - M^d_{KLM} \dot{\gamma}_{KLM} + P_K \dot{E}^*_K + M^*_K \dot{B}_K - J^*_K E^*_K + Q_{K,K} - \rho^o h \Big\} \delta T dV = 0
\end{aligned} \tag{A85}$$

The first term in the weak form becomes

$$\int\limits_{V} \rho^o \gamma \frac{(T + T^o)}{T^o} \dot{T} \delta T dV = \dot{T}_\beta \delta T_\alpha \int\limits_{V} \rho^o \gamma \frac{\left(N_\xi T_\xi + T^o\right)}{T^o} N_\alpha N_\beta dV \equiv \dot{T}_\beta \delta T_\alpha G_{\alpha\beta} \tag{A86}$$

The second term yields

$$\begin{aligned}
&\int\limits_{V} -(T + T^o) \left[\overline{A}_{KL} \dot{\alpha}^e_{KL} + \overline{B}_{KL} \dot{\beta}^e_{KL}\right] \delta T dV \\
&\quad = \delta T_\alpha \int\limits_{V} -(T + T^o) \left[\overline{A}_{KL} \dot{\alpha}^e_{KL} + \overline{B}_{KL} \dot{\beta}^e_{KL}\right] N_\alpha dV \equiv \delta T_\alpha q^1_\alpha
\end{aligned} \tag{A87}$$

The third term results

$$\int\limits_{V} \left\{T^{me}_{KL} \dot{\alpha}^p_{KL} + S^{me}_{KL} \dot{\beta}^p_{KL} + M^e_{KLM} \dot{\gamma}^p_{KLM}\right\} \delta T dV$$

$$= \delta T_\alpha \int_V \left\{ T_{KL}^{me} \dot{\alpha}_{KL}^{p} + S_{KL}^{me} \dot{\beta}_{KL}^{p} + M_{KLM}^{e} \dot{\gamma}_{KLM}^{p} \right\} N_\alpha dV \equiv \delta T_\alpha q_\alpha^2 \tag{A88}$$

The fourth term is identified and derived as

$$\begin{aligned} &\int_V \left\{ T_{KL}^{d} \dot{\alpha}_{KL} + S_{KL}^{d} \dot{\beta}_{KL} + M_{KLM}^{d} \dot{\gamma}_{KLM} \right\} \delta T dV \\ &= \delta T_\alpha \int_V \left\{ T_{KL}^{d} \dot{\alpha}_{KL} + S_{KL}^{d} \dot{\beta}_{KL} + M_{KLM}^{d} \dot{\gamma}_{KLM} \right\} N_\alpha dV \equiv \delta T_\alpha q_\alpha^3 \end{aligned} \tag{A89}$$

The fifth term is identified and derived as

$$\begin{aligned} &\int_V \left\{ -P_K \dot{E}_K^* - M_K^* \dot{B}_K + J_K^* E_K^* \right\} \delta T dV \\ &= \delta T_\alpha \int_V \left\{ -P_K \dot{E}_K^* - M_K^* \dot{B}_K + J_K^* E_K^* \right\} N_\alpha dV \equiv \delta T_\alpha q_\alpha^4 \end{aligned} \tag{A90}$$

The sixth term can be derived as

$$\begin{aligned} &\int_V -Q_{K,K} \delta T dV = -\oint_S Q_K n_K \delta T dS + \int_V Q_K \delta T_{,K} dV \\ &= \delta T_\alpha \left\{ -\int_{S_q} \hat{Q} N_\alpha dS + \int_V Q_K N_{\alpha,K} dV \right\} \\ &\equiv \delta T_\alpha \{ q_\alpha^5 + q_\alpha^6 \} \end{aligned} \tag{A91}$$

The last term becomes

$$\int_V \rho^o h \delta T dV = \delta T_\alpha \int_V \rho^o h N_\alpha dV \equiv \delta T_\alpha q_\alpha^7 \tag{A92}$$

Now Eq. (A85) can be re-written as

$$\delta T_\alpha \left\{ G_{\alpha\beta} \dot{T}_\beta - \left[q_\alpha^1 + q_\alpha^2 + q_\alpha^4 + q_\alpha^4 + q_\alpha^5 + q_\alpha^6 + q_\alpha^7 \right] \right\} = 0 \tag{A93}$$

Therefore, the finite element equation for conservation law of energy becomes

$$G_{\alpha\beta} \dot{T}_\beta = q_\alpha^1 + q_\alpha^2 + q_\alpha^3 + q_\alpha^4 + q_\alpha^5 + q_\alpha^6 + q_\alpha^7 \tag{A94}$$

where

$$q_\alpha^1 \equiv \int_V -(T + T^o) \left[\overline{A}_{KL} \dot{\alpha}_{KL}^{e} + \overline{B}_{KL} \dot{\beta}_{KL}^{e} \right] N_\alpha dV \tag{A95}$$

$$q_\alpha^2 \equiv \int_V \left\{ T_{KL}^{me} \dot{\alpha}_{KL}^{p} + S_{KL}^{me} \dot{\beta}_{KL}^{p} + M_{KLM}^{e} \dot{\gamma}_{KLM}^{p} \right\} N_\alpha dV \tag{A96}$$

$$q_\alpha^3 \equiv \int_V \left\{ T_{KL}^{d} \dot{\alpha}_{KL} + S_{KL}^{d} \dot{\beta}_{KL} + M_{KLM}^{d} \dot{\gamma}_{KLM} \right\} N_\alpha dV \tag{A97}$$

$$q_\alpha^4 \equiv \int_V \left\{ -P_K \dot{E}_K^* - M_K^* \dot{B}_K + J_K^* E_K^* \right\} N_\alpha dV \tag{A98}$$

$$q_\alpha^5 \equiv -\int_{S_q} \hat{Q} N_\alpha dS \tag{A99}$$

$$q_\alpha^6 \equiv \int_V Q_K N_{\alpha,K} dV \tag{A100}$$

$$q_\alpha^7 \equiv \int_V \rho^o h N_\alpha dV \tag{A101}$$

For scalar potential, the weak form of Eq. (159) is expressed as

$$\int_v \left\{ \frac{1}{\bar{c}^2}\ddot{\phi} - \left(\phi_{,i}\right)_{,i} - a \right\} \delta\phi dv = 0 \tag{A102}$$

Term by term one obtains

$$\int_v \frac{1}{\bar{c}^2}\ddot{\phi}\delta\phi dv = \delta\phi_\alpha \ddot{\phi}_\beta \int_v \frac{1}{\bar{c}^2} N_\beta N_\beta dv = \delta\phi_\alpha m_{\alpha\beta}\ddot{\phi}_\beta \tag{A103}$$

$$\begin{aligned}
\int_v -\left(\phi_{,i}\right)_{,i} \delta\phi dv &= \int_v \left\{ -\left(\phi_{,i}\delta\phi\right)_{,i} + \phi_{,i}\delta\phi_{,i} \right\} dv \\
&= \delta\phi_\alpha \oint_s -\phi_{,i} n_i N_\alpha ds + \delta\phi_\alpha \phi_\beta \int_v N_{\alpha,i} N_{\beta,i} dv \\
&= \delta\phi_\alpha \int_{s_\xi} -\phi_{,i} n_i N_\alpha ds + \delta\phi_\alpha \phi_\beta \int_v N_{\alpha,i} N_{\beta,i} dv \\
&= -\delta\phi_\alpha \hat{\xi}_\alpha + \delta\phi_\alpha k_{\alpha\beta}\phi_\beta
\end{aligned} \tag{A104}$$

$$\int_v -a\delta\phi dv = \delta\phi_\alpha \int_v -a N_\alpha dv = -\delta\phi_\alpha \tilde{a}_\alpha \tag{A105}$$

Because the weak form has to be valid for any arbitrary $\delta\phi_\alpha$, one obtains

$$m_{\alpha\beta}\ddot{\phi}_\beta + k_{\alpha\beta}\phi_\beta = \hat{\xi}_\alpha + \tilde{a}_\alpha \tag{A106}$$

where s_ξ is part of the enclosing surface of the volume v on which $\phi_{,i} n_i$ is specified as ξ; and

$$\begin{aligned}
m_{\alpha\beta} &\equiv \int_v \frac{1}{\bar{c}^2} N_\alpha N_\beta dv \\
k_{\alpha\beta} &\equiv \int_v N_{\alpha,i} N_{\beta,i} dv \\
\hat{\xi}_\alpha &\equiv \int_{s_\xi} \xi N_\alpha ds \equiv \int_{s_\xi} \phi_{,i} n_i N_\alpha ds \\
\hat{a}_\alpha &= \int_v a N_\alpha dv
\end{aligned} \tag{A107}$$

For vector potential, the weak form of Eq. (159) is expressed as

$$\int_v \left\{ \frac{1}{\bar{c}^2} \frac{\partial^2 A_k}{\partial t^2} - A_{k,ii} - C_k \right\} \delta A_k dv = 0 \tag{A108}$$

Term by term one obtains

$$\int_v \frac{1}{\bar{c}^2} \frac{\partial^2 A_k}{\partial t^2} \delta A_k dv = \delta A_{k\alpha} \ddot{A}_{k\beta} \int_v \frac{1}{\bar{c}^2} N_\alpha N_\beta dv = \delta A_{k\alpha} m_{\alpha\beta} \ddot{A}_{k\beta} \tag{A109}$$

$$\begin{aligned} \int_v -A_{k,ii} \delta A_k dv &= \int_v \left\{ -\left[A_{k,i} \delta A_k \right]_{,i} + A_{k,i} \delta A_{k,i} \right\} dv \\ &= \delta A_{k\alpha} \oint_s -A_{k,i} n_i N_\alpha ds + \delta A_{k\alpha} A_{k\beta} \int_v N_{\alpha,i} N_{\beta,i} dv \\ &= \delta A_{k\alpha} \int_{s_\varsigma} -\tilde{A}_k N_\alpha ds + \delta A_{k\alpha} A_{k\beta} \int_v N_{\alpha,i} N_{\beta,i} dv \\ &\equiv \delta A_{k\alpha} \left\{ -\hat{A}_{k\alpha} + k_{\alpha\beta} A_{k\beta} \right\} \end{aligned} \tag{A110}$$

$$\int_v -C_k \delta A_k dv = -\delta A_{k\alpha} \int_v C_k N_\alpha dv \equiv -\delta A_{k\alpha} \hat{C}_{k\alpha} \tag{A111}$$

Because the weak form has to be valid for any arbitrary $\delta A_{k\alpha}$, one obtains

$$m_{\alpha\beta} \ddot{A}_{k\beta} + k_{\alpha\beta} A_{k\beta} = \hat{A}_{k\alpha} + \hat{C}_{k\alpha} \tag{A112}$$

where s_ς is part of the enclosing surface of the volume v on which $A_{k,i} n_i$ is specified as $\tilde{A}_k$; and

$$\begin{aligned} \hat{A}_{k\alpha} &\equiv \int_{s_\varsigma} A_{k,i} n_i N_\alpha ds \equiv \int_{s_\varsigma} \tilde{A}_k N_\alpha ds \\ \hat{C}_{k\alpha} &\equiv \int_v C_k N_\alpha dv \end{aligned} \tag{A113}$$

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