



REVIEW

Advances in the Improved Element-Free Galerkin Methods: A Comprehensive Review

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ABSTRACT: The element-free Galerkin (EFG) method, which constructs shape functions via moving least squares (MLS) approximation, represents a fundamental and widely studied meshless method in numerical computation. Although it achieves high computational accuracy, the shape functions are more complex than those in the conventional finite element method (FEM), resulting in great computational requirements. Therefore, improving the computational efficiency of the EFG method represents an important research direction. This paper systematically reviews significant contributions from domestic and international scholars in advancing the EFG method. Including the improved element-free Galerkin (IEFG) method, various interpolating EFG methods, four distinct complex variable EFG methods, and a series of dimension splitting meshless methods. In the numerical examples, the effectiveness and efficiency of the three methods are validated by analyzing the solutions of the IEFG method for 3D steady-state anisotropic heat conduction, 3D elastoplasticity, and large deformation problems, as well as the performance of two-dimensional splitting meshless methods in solving the 3D Helmholtz equation.

KEYWORDS: Meshless method; improved element-free Galerkin method; singular weight function; nonsingular weight function; interpolating element-free Galerkin method; complex variable element-free Galerkin method; dimension splitting method; dimension splitting meshless method

1 Introduction

As an important numerical computation method, the meshless (or meshfree) method operates solely on discrete points rather than predefined meshes. Consequently, it eliminates the need for remeshing when solving large deformation problems where mesh distortion typically occurs. This inherent advantage over the conventional finite element method (FEM) [1] has led to its widespread adoption across multiple scientific and engineering disciplines.

The development of the meshfree method originated in 1977 when Lucy pioneered the smoothed particle hydrodynamics (SPH) method [2], representing the first meshless computational method. Later in 1981, Lancaster and Salkauskas first proposed the moving least squares (MLS) approximation [3] for surface fitting applications. Building on this foundation, Nayroles et al. developed the diffuse element method (DEM) [4] in 1992, implementing MLS approximation for nodal approximation. Subsequently, in 1994, Belytschko et al. developed the element-free Galerkin (EFG) method [5], which also utilized MLS approximation with Lagrange multipliers for boundary condition enforcement. Since then, numerous other



meshless methods have been proposed, including the reproducing kernel particle method (RKPM) [6], the finite point method [7], the partition of unity method [8], the radial basis function method [9], the boundary node method [10], the natural element method [11], the local Petrov-Galerkin method [12], the local boundary integral equation [13], the point interpolation method [14], the collocation meshless method [15], the hybrid boundary node method [16], the moving Kriging interpolation method [17], the meshless finite element method [18].

Among the various meshless methods, the EFG method proposed by Belytschko et al. [5] represents a fundamental and widely studied method in numerical computation. After the strong form of the governing differential equation is transformed into its weak integral form, and the shape function is introduced to derive the discrete equations, which are solved by using the computational tool of background integration mesh, ultimately yielding the numerical solution for the corresponding problem. The EFG method constructs shape functions using the MLS approximation [4], with error estimates analyzed by Li [19]. Noted for its foundation in least squares mathematical theory, the MLS approximation achieves high computational accuracy. A key advantage of the MLS method over the least squares method is its mobility and compact support. This allows for the use of a linear basis function to obtain stable and highly accurate numerical solutions. Currently, the EFG method has been applied to solve diverse problems, including transient heat transfer problem [20], hyperbolic problem [21], fracture mechanics [22,23], optimization design [24], dynamic boundary flow [25], magneto-electro-elastic plates [26], shrinkage analysis of round billets [27], free vibration of rotating nanobeams [28], wave propagation [29], Signorini-Tresca contact problem [30], topology optimization [31–33], free vibration [34], buckling [35] and nonlinear analyses [36,37]. Additionally, the coupled FEM-EFG method [38,39], the variational multiscale EFG method [40,41] and the enrichment of EFG method [42] have also been developed. However, the computation of the shape function in the EFG method demands significant computational resources, resulting in reduced efficiency. Additionally, it may encounter singular matrices during numerical implementation. These limitations necessitate further methodological improvements.

In 2003, Cheng and Chen [43] proposed the improved moving least squares (IMLS) approximation by replacing the basis function in MLS with orthogonal ones, resulting in higher computational efficiency. Later, in 2008, Zhang et al. studied the improved element-free Galerkin (IEFG) method based on the IMLS approximation and applied it to analyze 2D fracture problems. Subsequently, the IEFG method is successfully extended to solve various problems, including elasticity, potential problems [44,45], wave equation, heat conduction [46] and elastic dynamics problems. Cheng and Liew further applied the IEFG method to study the modified equal width (MEW) wave equation [47] and generalized Camassa and Holm equation [48]. Cheng et al. made significant contributions to advancing the IEFG method, applying it to problems such as the Schrödinger equation, viscoelasticity [49], fracture mechanics, topology optimization, elastoplasticity, and diffusional drug release [50] problems. Cheng et al. further integrated deep learning with the IEFG method for solving inverse potential problems [51]. Zhang et al. investigated the biological population problem using the IEFG method [52], while Li et al. conducted error analysis for the IEFG method [53] and later applied it to the elliptic problem [54], p-Laplacian problem [55], and Boussinesq equation [56]. Debbabi and BelhadjSalah selected the IEFG method for the thermo-elastic problem [57]. Mousavi et al. coupled it with the finite strip method to analyze plate vibration [58]. Guo et al. further extended its application to study vibration [59], nonlinear behavior [60], and flutter [61] in composite quadrilateral plates, in addition to the dynamics of cantilever plates [62]. The IEFG method offers the advantage of fast computational speed, effectively addressing the limitations of the EFG method, such as slow computation and the tendency to form singular matrices.

The shape functions in both EFG and IIEFG methods lack interpolation properties, traditionally requiring the penalty method to impose the essential boundary conditions, which inevitably reduces computational efficiency. To address this limitation, Lancaster and Salkauskas proposed the interpolating MLS approximation [3], enabling direct imposition of Dirichlet boundary conditions without auxiliary methods. Ren et al. further simplified the derivation of the shape function and proposed the improved interpolating MLS approximation [63]. Wang and Sun analyzed its error estimates [64], while Li and Wang investigated its inherent instability [65]. By constructing shape function with this method, the improved boundary EFG method was proposed. Later, Ren et al. presented the interpolating EFG method for 2D elasticity [66], potential and heat conduction [67,68] problems. Liu and Cheng extended their application to study two-point boundary value, potential [69], heat conduction, elastoplasticity and large deformation [70] problems. Sun and Wang adopted the method for the regularized long wave equation [71]. Zhang and Peng applied it to the viscoelasticity problem [72]. Dehghan and Abbaszadeh solved the magnetohydrodynamics equation using the interpolating EFG method [73]. Shen et al. employed it for evolutionary variational inequalities [74] and the mixed complementarity problem [75]. Finally, Zhang and Li developed the variational multiscale interpolating EFG method [76,77] based on the improved interpolating MLS approximation [63]. Since the interpolating EFG method can directly impose boundary conditions like FEM and features a simpler formulation than the standard EFG method, it offers improved computational efficiency for solving scientific and engineering problems.

A limitation of the improved interpolating MLS method is that its weight function becomes singular at nodes, which inevitably introduces truncation errors. To overcome this issue, Wang et al. studied a novel interpolating MLS approximation [78] by replacing the singular weight function with the nonsingular alternative, subsequently developing the new interpolating EFG method [78]. Later, Sun et al. extended this method to solve elasticity [79] and elastoplasticity [80] problems. Dehghan et al. applied this method to diverse problems, including solutions for fractional diffusion-wave [81], inverse tempered fractional diffusion equations [82], and Oldroyd models [83], with additional contributions to decomposition variational multiscale interpolating EFG method [84]. Liu and Cheng further advanced the method for 2D large deformation analysis [85,86] and 3D elastoplasticity simulations [87]. Additionally, the interpolating boundary EFG method [88,89], variational multiscale interpolating EFG method [90], interpolating EFG scaled boundary method [91–93], coupled interpolating EFG method [94], regularized and adaptive orthogonal interpolating MLS approximation [95,96] were also developed by researchers. The interpolating meshless method based on non-singular weight functions offers a wide range of weight function options and overcomes the limitations associated with singular weight functions.

In 2022, Ma et al. combined the Hermite interpolation with MLS approximation to approximate the field variables, and then proposed the Hermite interpolation EFG method for solving the elasticity problem [97], functionally graded structures [98], piezoelectric materials [99], and functionally graded piezoelectric structures [100]. Its distinct advantage lies in providing superior accuracy for calculating function derivatives at any point within the solution domain. This stems from explicitly incorporating the normal derivatives of field functions during shape function construction, which is the distinctive feature absent in previous EFG improvements.

The EFG, IIEFG, and three interpolation EFG methods all use scalar basis functions to establish shape functions. To reduce the number of undetermined coefficients in the shape function, Cheng and Li integrated the complex variable theory combined with the MLS approximation, developing the complex variable moving least squares (CVMLS) approximation. This innovation led to the complex variable element-free Galerkin (CVEFG) method [101] for solving elasticity problems. Typically, a linear basis function is selected for shape function construction. When a scalar basis is selected, the number of undetermined coefficients

is three, which reduces to two for a vector basis function, thereby lowering the matrix order in the shape function computation.

The CVMLS approximation exhibits a minor limitation in that the mathematical and physical interpretation of its functional construction remains unclear. To address this issue, Bai et al. studied the improved complex variable moving least squares (ICVMLS) approximation [102] by replacing the basis function in the CVMLS approximation with a conjugate basis function. Subsequently, the improved complex variable element-free Galerkin (ICVEFG) method was successfully applied to solve a wide range of 2D problems including: elasticity [102], potential, advection-diffusion, Schrödinger, wave propagation, viscoelasticity [103], elastoplasticity, topology optimization [104], large deformation [105,106], as well as the bending problems of Kirchhoff plate and thin plate on elastic foundations [107]. Li and Tian further extended the ICVEFG method to large deformation [108–110] and fracture [111–113] problems. Li and Li conducted error estimation analyses for both the ICVMLS approximation and the ICVEFG method [114]. Additionally, Niu et al. integrated the conjugate basis function into the RKPM for solving the 2D elasticity problem [115]. However, the studies on the ICVEFG method in the aforementioned literature are confined to two-dimensional problems. To extend its applicability, Li and Li derived the 3D shape function calculation formula for CVMLS approximation [116], and Li subsequently developed the CVEFG method [117] for analyzing 3D partial differential equations. More recently, Yang et al. applied this method to solve 3D elastoplastic mechanics problems [118], thereby significantly expanding its scope of applicability.

The CVEFG and ICVEFG methods lack interpolation properties. Ren et al. addressed this limitation by developing the interpolating CVMLS approximation [119] through the incorporation of singular weight functions into the CVMLS framework. Building on this advancement, Deng and He successfully implemented the interpolating CVEFG method for temperature field analysis, and subsequently enhanced the approach by proposing an improved interpolating CVEFG method [120–124] by integrating the singular weight functions with ICVEFG approximations. This method addresses the inability of the ICVEFG method to directly apply boundary conditions and achieves improved computational efficiency.

Regardless of which of the aforementioned improved methods is employed, the influence domain of a point in 3D problems still constitutes a three-dimensional region. The number of points included within this domain is significantly larger than that in a 2D problem. Furthermore, shape functions must be computed for every point within the influence domain. As a result, the improvement in computational speed of the EFG method remains relatively constrained, particularly for 3D problems.

In 2017, Cheng et al. pioneered the integration of the dimension split method [125,126] with the ICVEFG method, leading to the development of the dimension splitting complex variable element-free Galerkin (DS-CVEFG) method [127]. This innovative approach demonstrated significant computational time savings compared to the IEFG method for solving 3D partial differential equations. The method operates by reducing the 3D problem to a series of 2D problems, which are then solved using the ICVEFG method with specially constructed 2D ICVMLS approximations. The resulting 2D discrete equations are further processed in the splitting direction through the simple finite difference method (FDM) discretization. Subsequent applications successfully addressed heat conduction [128], convection diffusion and elasticity problems with improved efficiency. Drawing on the successful research of DS-CVEFG method, researchers have progressively developed several related methods, including: the dimension splitting IEFG (DSIEFG) [129], the dimension splitting interpolating EFG methods employing singular weight functions [130], and their counterparts utilizing nonsingular weight functions [131–135], and the hybrid reproducing kernel particle method [136]. These studies collectively demonstrate that the dimension splitting meshless methods can subsequently enhance the calculation speed of traditional meshless methods for solving 3D problems.

A common feature of these dimension-splitting meshless methods is their use of the simple FDM along the splitting directions to discretize the equations derived from the meshless discretization of 2D problems. Considering that the interpolating meshless method and the FEM have broader applicability than the FDM and offer greater convenience in handling natural boundary conditions, Cheng et al. replaced the FDM-based splitting-direction discretization with the FEM while maintaining the IIEFG method for 2D domain treatment, thereby developing the dimension coupling method [137,138]. Wang et al. innovatively applied the interpolating MLS method [78] for shape function construction in the splitting direction, leading to the development of dimension splitting interpolating MLS approximation [139–142]. This approach enabled the subsequent development of both the interpolating dimension splitting variational multiscale EFG [143,144] and generalized EFG [145,146] methods. These improved dimension splitting meshless methods can also improve the efficiency of traditional meshfree methods subsequently.

The dimension splitting meshless method can significantly enhance the computational efficiency of the IIEFG method for solving 3D problems. This advantage is validated in the numerical examples section by solving the 3D Helmholtz equation. However, challenges arise when addressing nonlinear mechanical problems. Such complex problems currently require the traditional meshless methods for effective solutions. Therefore, the IIEFG method is employed in the numerical examples to solve material and geometric nonlinear problems, demonstrating its superior computational speed over the standard EFG method.

This paper contributes a comprehensive review of various improved EFG methods, establishing learning foundations for new researchers while offering guidance for future advancements and research directions in EFG methods. The first section is dedicated to a literature review on enhanced EFG methods, analyzing their strengths and weaknesses. Since these advancements primarily focus on the shape function optimization, Section 2 provides detailed mathematical derivations of key shape function formulations. Section 3 introduces the IIEFG method for solving 3D heat conduction, elastoplasticity and large deformation problems, as well as two types of dimension splitting meshless methods applied to the 3D Helmholtz equation. The final section presents conclusions and proposes four specific directions for future research.

2 Shape Functions

The main distinction between meshless methods and the FEM lies in their shape function construction. This section systematically presents the shape function formulation for the MLS, IMLS, interpolating MLS, and ICVMLS approximations.

2.1 The MLS Approximation

In 1981, Lancaster and Salkauskas proposed the MLS approximation [3], originally developed for surface fitting. After 1994, this technique became the foundation for constructing shape functions in the renowned EFG method.

The approximation $u^h(\mathbf{x})$ for the function $u(\mathbf{x})$ is expressed as a polynomial, though the coefficients and degree of the polynomial are initially unknown. According to the research of Dolbow and Belytschko on the EFG method, even when using lower-order polynomial basis (such as linear basis functions), sufficiently accurate numerical solutions can be obtained by selecting appropriate weight functions [147].

When applying a polynomial to approximate an arbitrary function $u(\mathbf{x})$, the local approximation function yields the following expression:

$$u^h(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{i=1}^m q_i(\hat{\mathbf{x}}) a_i(\mathbf{x}) = \mathbf{q}^T(\hat{\mathbf{x}}) \mathbf{a}(\mathbf{x}), \mathbf{x} \in \Omega, \quad (1)$$

where $\hat{x}$ is a point within the local neighborhood of x ; m is the number of basis functions used in approximation, and

$$\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), \dots, a_m(\mathbf{x}))^T \quad (2)$$

is unknown coefficient vector corresponding to the basis function. $\mathbf{q}^T(\mathbf{x})$ denotes the basis function vector.

The coefficient vector $\mathbf{a}(\mathbf{x})$ can be obtained by minimizing the following functional

$$J = \sum_{I=1}^n w(\mathbf{x} - \mathbf{x}_I) [\mathbf{q}^T(\mathbf{x}_I) \mathbf{a}(\mathbf{x}) - u_I]^2 = \sum_{I=1}^n w(\mathbf{x} - \mathbf{x}_I) \left[\sum_{i=1}^m q_i(\mathbf{x}_I) a_i(\mathbf{x}) - u_I \right]^2, \quad (3)$$

where $w(\mathbf{x} - \mathbf{x}_I)$ ($I = 1, 2, \dots, n$) is weight function, $\mathbf{x}_I$ refer to an arbitrary node in problem domain and the corresponding influence domain cover $\mathbf{x}$.

Eq. (3) can be written as

$$J = (\mathbf{P}(\mathbf{x}) \mathbf{a}(\mathbf{x}) - \mathbf{u}(\mathbf{x}))^T \mathbf{W}(\mathbf{x}) (\mathbf{P}(\mathbf{x}) \mathbf{a}(\mathbf{x}) - \mathbf{u}(\mathbf{x})) \quad (4)$$

By minimizing the functional J , the expression for the unknown coefficient vector $\mathbf{a}(\mathbf{x})$ can be obtained. Let

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{0}, \quad (5)$$

thus the matrix $\mathbf{a}(\mathbf{x})$ can be obtained as

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \mathbf{u}(\mathbf{x}), \quad (6)$$

where

$$\mathbf{A}(\mathbf{x}) = \begin{bmatrix} (q_1, q_1) & (q_1, q_2) & \cdots & (q_1, q_m) \\ (q_2, q_1) & (q_2, q_2) & \cdots & (q_2, q_m) \\ \vdots & \vdots & \ddots & \vdots \\ (q_m, q_1) & (q_m, q_2) & \cdots & (q_m, q_m) \end{bmatrix} = \mathbf{P}^T(\mathbf{x}) \mathbf{W}(\mathbf{x}) \mathbf{P}(\mathbf{x}), \quad (7)$$

$$\mathbf{B}(\mathbf{x}) = [w(\mathbf{x} - \mathbf{x}_1) \mathbf{q}(\mathbf{x}_1), w(\mathbf{x} - \mathbf{x}_2) \mathbf{q}(\mathbf{x}_2), \dots, w(\mathbf{x} - \mathbf{x}_n) \mathbf{q}(\mathbf{x}_n)] = \mathbf{P}^T(\mathbf{x}) \mathbf{W}(\mathbf{x}), \quad (8)$$

$$\mathbf{P}(\mathbf{x}) = \begin{bmatrix} 1 & q_2(\mathbf{x}_1) & \cdots & q_m(\mathbf{x}_1) \\ 1 & q_2(\mathbf{x}_2) & \cdots & q_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & q_2(\mathbf{x}_n) & \cdots & q_m(\mathbf{x}_n) \end{bmatrix}, \quad (9)$$

$$\mathbf{u}(\mathbf{x}) = (u_1, u_2, \dots, u_n)^T, \quad (10)$$

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w(\mathbf{x} - \mathbf{x}_1) & 0 & \cdots & 0 \\ 0 & w(\mathbf{x} - \mathbf{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\mathbf{x} - \mathbf{x}_n) \end{bmatrix}. \quad (11)$$

Typically, the cubic and quartic spline functions serve as the primary weight functions. The cubic spline function, defined as

$$w(d) = \begin{cases} \frac{2}{3} - 4d^2 + 4d^3 & d \leq 0.5 \\ \frac{4}{3} - 4d + 4d^2 - \frac{4}{3}d^3 & 0.5 < d \leq 1 \\ 0 & d > 1 \end{cases}, \quad (12)$$

while the quartic spline function is

$$w(d) = \begin{cases} 1 - 6d^2 + 8d^3 - 3d^4 & d \leq 1 \\ 0 & d > 1 \end{cases}, \quad (13)$$

where d is a function of $\mathbf{x} - \mathbf{x}_I$, and

$$d = \frac{\|\mathbf{x} - \mathbf{x}_I\|}{\rho_I}, \quad (14)$$

$$\rho_I = d_{\max} \cdot c_I, \quad (15)$$

the scales parameter $d_{\max}$ defines the size of the influence domain for a node, and c_I denotes the distance between the point $\mathbf{x}_I$ and its closest neighbor node.

Therefore, the approximate function can be written as

$$u^h(\mathbf{x}) = \Phi(\mathbf{x}) \mathbf{u} = \sum_{I=1}^n \Phi_I u_I, \quad (16)$$

where

$$\Phi(\mathbf{x}) = (\Phi_1, \Phi_2, \dots, \Phi_n) = \mathbf{q}(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}(\mathbf{x}) \quad (17)$$

is the shape function.

We arrange six nodes uniformly in $[0, 1]$ and discretize the domain with five background integration cells, each having one Gauss point. Under this configuration, a cubic spline weight function is selected, and $d_{\max}$ is set to 2.0. The corresponding shape function and its derivative are shown in Figs. 1 and 2. It is shown that the shape function and its derivative are smooth at every node.

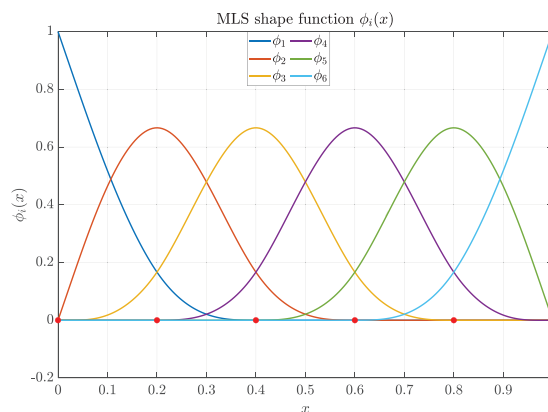


Figure 1: Schematic diagram of the MLS shape functions

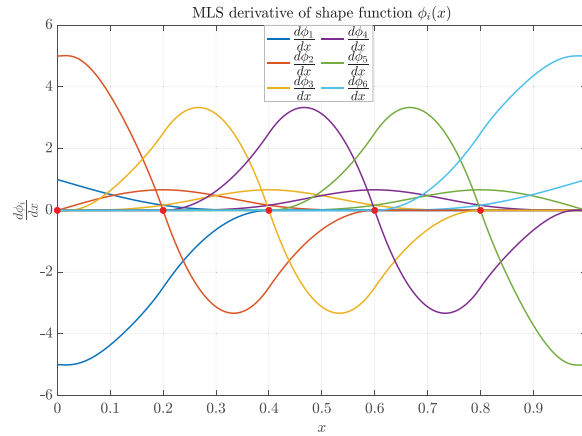


Figure 2: Schematic diagram of the MLS derivative of shape functions

This shape function can achieve high computational accuracy due to its foundation in least squares mathematical theory. However, the inversion of matrix $\mathbf{A}(\mathbf{x})$ requires significant computational effort, leading to reduced efficiency. Additionally, it may encounter singular matrices during numerical implementation.

2.2 The IMLS Approximation

To simplify the high-computational cost are primarily focus of matrix $\mathbf{A}(\mathbf{x})$ inversion in the MLS approximation, Cheng and Chen [43] proposed the IMLS approximation. The formulation derivation proceeds as follows:

Applying Gram-Schmidt orthogonalization to the basis function $\mathbf{q}^T(\mathbf{x})$ of MLS approximation yields:

$$p_i = q_i - \sum_{k=1}^{i-1} \frac{(q_i, p_k)}{(p_k, p_k)} p_k, i = 1, 2, 3, \dots, \quad (18)$$

where

$$\mathbf{q} = (q_i) = (1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_3x_1, \dots), \quad (19)$$

and

$$(p_i, p_j) = 0, i \neq j \quad (20)$$

Therefore, the matrix $\mathbf{A}$ in Eq. (7) can be simplified as the following new form

$$\tilde{\mathbf{A}}(\mathbf{x}) = \begin{bmatrix} (p_1, p_1) & 0 & \cdots & 0 \\ 0 & (p_2, p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p_m, p_m) \end{bmatrix}. \quad (21)$$

The resulting IMLS approximation shape function is therefore given by:

$$\Phi^*(\mathbf{x}) = (\Phi_1^*, \Phi_2^*, \dots, \Phi_n^*) = \mathbf{p}^T(\mathbf{x}) \tilde{\mathbf{A}}^{-1}(\mathbf{x}) \tilde{\mathbf{B}}(\mathbf{x}), \quad (22)$$

where

$$\mathbf{p}(\mathbf{x}) = (1, p_2, p_3, \dots, p_m)^T, \quad (23)$$

$$\tilde{\mathbf{B}}(\mathbf{x}) = \tilde{\mathbf{P}}^T(\mathbf{x}) \mathbf{W}(\mathbf{x}), \quad (24)$$

$$\tilde{\mathbf{P}}(\mathbf{x}) = \begin{bmatrix} 1 & p_2(\mathbf{x}_1) & \cdots & p_m(\mathbf{x}_1) \\ 1 & p_2(\mathbf{x}_2) & \cdots & p_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_2(\mathbf{x}_n) & \cdots & p_m(\mathbf{x}_n) \end{bmatrix}. \quad (25)$$

Evidently, the inverse of the modified matrix $\tilde{\mathbf{A}}(\mathbf{x})$ in the IMLS approximation has a simpler form, which consequently accelerates the computation of shape function compared to the traditional MLS method.

2.3 The Interpolating MLS Approximation

Both the MLS and IMLS approximations lack interpolation properties, the penalty method is often used to impose the essential boundary condition in the corresponding meshless methods, which inevitably affects computational efficiency. To address this limitation, Ren et al. [63] proposed the interpolating MLS approximation based on the singular weight function.

When the zeroth-order basis function $q = 1$ is selected in the MLS method, thus the approximation function is

$$u^h(\mathbf{x}) = \mathbf{v}^T(\mathbf{x} - \mathbf{x}_I) \mathbf{u}(\mathbf{x}) = \frac{\mathbf{b}(\mathbf{x}) \mathbf{u}(\mathbf{x})}{A(\mathbf{x})}, \quad (26)$$

where

$$\mathbf{v}^T(\mathbf{x} - \mathbf{x}_I) = \frac{\mathbf{b}(\mathbf{x})}{A(\mathbf{x})} = (v(\mathbf{x} - \mathbf{x}_1), v(\mathbf{x} - \mathbf{x}_2), \dots, v(\mathbf{x} - \mathbf{x}_n)), \quad (27)$$

$$A(\mathbf{x}) = \sum_{I=1}^n \tilde{w}(\mathbf{x} - \mathbf{x}_I), \quad (28)$$

$$\mathbf{b}(\mathbf{x}) = (\tilde{w}(\mathbf{x} - \mathbf{x}_1), \tilde{w}(\mathbf{x} - \mathbf{x}_2), \dots, \tilde{w}(\mathbf{x} - \mathbf{x}_n)), \quad (29)$$

In the interpolating MLS approximation, the singular weight function can be selected as

$$\tilde{w}(\mathbf{x} - \mathbf{x}_I) = \begin{cases} \frac{1}{|\mathbf{x} - \mathbf{x}_I|^\beta}, & |\mathbf{x} - \mathbf{x}_I| \leq \rho_I \\ 0, & |\mathbf{x} - \mathbf{x}_I| > \rho_I \end{cases}, \quad (30)$$

or

$$\tilde{w}(d) = \begin{cases} \frac{1}{d^\beta} \left(1 - \frac{1}{d}\right)^2, & d \leq 1 \\ 0, & d > 1 \end{cases}, \quad (31)$$

and β is a positive integer.

Because q_1 is equal to 1 in the MLS approximation, after normalization processing of q_1 at point $\mathbf{x}$, we obtain:

$$\hat{q}_1 = \frac{q_1}{\|q_1\|_{\mathbf{x}}} = \left[\sum_{I=1}^n \tilde{w}(\mathbf{x} - \mathbf{x}_I) \right]^{-1/2}, \quad (32)$$

by orthogonalizing $q_2, q_3, \dots, q_m$ against $\hat{q}_1$ at point $\mathbf{x}$, we obtain

$$\hat{q}_i = q_i - \frac{(q_i, \hat{q}_1)}{(\hat{q}_1, \hat{q}_1)} \hat{q}_1 = q_i(\mathbf{x}) - \sum_{I=1}^n v(\mathbf{x} - \mathbf{x}_I) q_i(\mathbf{x}_I), i = 2, 3, 4, \dots, \quad (33)$$

The resulting interpolating MLS approximation shape function is therefore given by:

$$\hat{\Phi}(\mathbf{x}) = (\hat{\Phi}_1, \hat{\Phi}_2, \dots, \hat{\Phi}_n) = \mathbf{v}^T(\mathbf{x}) + \hat{\mathbf{q}}^T(\mathbf{x}) \hat{\mathbf{A}}^{-1}(\mathbf{x}) \hat{\mathbf{B}}(\mathbf{x}), \quad (34)$$

where

$$\hat{\mathbf{q}}(\mathbf{x}) = (\hat{q}_2, \hat{q}_3, \dots, \hat{q}_m)^T, \quad (35)$$

$$\hat{\mathbf{A}}(\mathbf{x}) = \hat{\mathbf{P}}^T(\mathbf{x}) \tilde{\mathbf{W}}(\mathbf{x}) \hat{\mathbf{P}}(\mathbf{x}), \quad (36)$$

$$\hat{\mathbf{B}}(\mathbf{x}) = \hat{\mathbf{P}}^T(\mathbf{x}) \tilde{\mathbf{W}}(\mathbf{x}), \quad (37)$$

$$\hat{\mathbf{P}}(\mathbf{x}) = \begin{bmatrix} \hat{q}_2(\mathbf{x}_1) & \hat{q}_3(\mathbf{x}_1) & \cdots & \hat{q}_m(\mathbf{x}_1) \\ \hat{q}_2(\mathbf{x}_2) & \hat{q}_3(\mathbf{x}_2) & \cdots & \hat{q}_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{q}_2(\mathbf{x}_n) & \hat{q}_3(\mathbf{x}_n) & \cdots & \hat{q}_m(\mathbf{x}_n) \end{bmatrix}, \quad (38)$$

$$\tilde{\mathbf{W}}(\mathbf{x}) = \begin{bmatrix} \tilde{w}(\mathbf{x} - \mathbf{x}_1) & 0 & \cdots & 0 \\ 0 & \tilde{w}(\mathbf{x} - \mathbf{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{w}(\mathbf{x} - \mathbf{x}_n) \end{bmatrix}. \quad (39)$$

Therefore, the approximation function yields the following expression:

$$u^h(\mathbf{x}) = \hat{\Phi}(\mathbf{x}) \mathbf{u}(\mathbf{x}) = \mathbf{v}^T(\mathbf{x}) \mathbf{u}(\mathbf{x}) + \hat{\mathbf{q}}^T(\mathbf{x}) \hat{\mathbf{a}}(\mathbf{x}), \quad (40)$$

where

$$\hat{\mathbf{a}}(\mathbf{x}) = \hat{\mathbf{A}}^{-1}(\mathbf{x}) \hat{\mathbf{B}}(\mathbf{x}) \mathbf{u}(\mathbf{x}). \quad (41)$$

The interpolating MLS approximation satisfies the Kronecker delta property, endowing its shape functions with interpolation characteristics that permit direct imposition of essential boundary conditions. Moreover, the unknown coefficient vector $\hat{\mathbf{a}}(\mathbf{x})$ in the shape functions contains one fewer element than in conventional MLS approximation, thereby reducing computational cost associated with shape function matrix operation.

2.4 The Interpolating MLS Approximation with Nonsingular Weight Function

A drawback of the interpolating MLS approximation proposed by Ren et al. [63] is weight function singularity in constructing shape function, leading to the inevitable truncation errors. To overcome this limitation, Wang et al. studied a new interpolating MLS approximation [78] by replacing the singular weight function with the nonsingular alternative.

Inspired from Eq. (34), we can derive a new set of basis function such that the approximation function satisfies the interpolation property. Let

$$\tilde{q}_i(\hat{\mathbf{x}}) = q_i(\hat{\mathbf{x}}) - \sum_{I=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_I) q_i(\mathbf{x}_I), \quad i = 1, 2, 3, \dots, m, \quad (42)$$

where $\tilde{v}(\mathbf{x} - \mathbf{x}_I)$ must be satisfied as

$$\tilde{v}(\mathbf{x}_I, \mathbf{x}_J) = \delta_{IJ}, \quad (43)$$

and

$$\sum_{I=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_I) = 1. \quad (44)$$

Therefore, the form of $\tilde{v}(\mathbf{x}, \mathbf{x}_I)$ can be selected as

$$\tilde{v}(\mathbf{x}, \mathbf{x}_I) = \frac{\zeta(\mathbf{x}, \mathbf{x}_I)}{\sum_{J=1}^n \zeta(\mathbf{x}, \mathbf{x}_J)}, \quad (45)$$

where

$$\zeta(\mathbf{x}, \mathbf{x}_I) = \prod_{I \neq J} \frac{\|\mathbf{x} - \mathbf{x}_J\|^2}{\|\mathbf{x}_I - \mathbf{x}_J\|^2}. \quad (46)$$

From Eq. (42), we derive the same transformation for $u(\hat{\mathbf{x}})$

$$\tilde{u}(\hat{\mathbf{x}}) = u(\hat{\mathbf{x}}) - \sum_{I=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_I) u(\mathbf{x}_I), \quad (47)$$

the corresponding local approximation function is

$$\tilde{u}^h(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{i=1}^m \tilde{q}_i(\hat{\mathbf{x}}) \tilde{a}_i(\mathbf{x}). \quad (48)$$

From Eqs. (42) and (44), we have

$$\tilde{q}_1(\hat{\mathbf{x}}) = 1 - \sum_{I=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_I) q_1(\mathbf{x}_I) = 0. \quad (49)$$

Thus Eq. (48) can be simplified as

$$\tilde{u}^h(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{i=2}^m \tilde{q}_i(\hat{\mathbf{x}}) \cdot \tilde{a}_i(\mathbf{x}) = \tilde{\mathbf{q}}^T(\hat{\mathbf{x}}) \tilde{\mathbf{a}}(\mathbf{x}), \quad (50)$$

where

$$\tilde{\mathbf{q}}(\mathbf{x}) = \left(\tilde{q}_2(\mathbf{x}), \tilde{q}_3(\mathbf{x}), \dots, \tilde{q}_m(\mathbf{x}) \right)^T, \quad (51)$$

$$\tilde{q}_i(\mathbf{x}) = q_i(\mathbf{x}) - \sum_{I=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_I) q_i(\mathbf{x}_I), \quad (52)$$

$$\tilde{\mathbf{a}}(\mathbf{x}) = \left(\tilde{a}_2(\mathbf{x}), \tilde{a}_3(\mathbf{x}), \dots, \tilde{a}_m(\mathbf{x}) \right)^T. \quad (53)$$

From Eqs. (47), (48) and (50), we have

$$u^h(\mathbf{x}, \hat{\mathbf{x}}) = \sum_{l=1}^n \tilde{v}(\mathbf{x}, \mathbf{x}_l) u(\mathbf{x}_l) + \sum_{i=2}^m \tilde{q}_i(\hat{\mathbf{x}}) \cdot \tilde{a}_i(\mathbf{x}). \quad (54)$$

Thus the resulting shape function is therefore given by:

$$u^h(\mathbf{x}) = \tilde{\Phi}(\mathbf{x}) \mathbf{u}(\mathbf{x}) = \sum_{l=1}^n \tilde{\Phi}_l u_l, \quad (55)$$

where

$$\tilde{\Phi}(\mathbf{x}) = (\tilde{\Phi}_1, \tilde{\Phi}_2, \dots, \tilde{\Phi}_n) = \tilde{\mathbf{v}}(\mathbf{x}) + \tilde{\mathbf{q}}^T(\mathbf{x}) \tilde{\mathbf{A}}^{-1}(\mathbf{x}) \tilde{\mathbf{B}}(\mathbf{x}), \quad (56)$$

$$\tilde{\mathbf{v}}(\mathbf{x}) = (\tilde{v}(\mathbf{x}, \mathbf{x}_1), \tilde{v}(\mathbf{x}, \mathbf{x}_2), \dots, \tilde{v}(\mathbf{x}, \mathbf{x}_n)). \quad (57)$$

$$\tilde{\mathbf{A}}(\mathbf{x}) = \tilde{\mathbf{P}}^T(\mathbf{x}) \mathbf{W}(\mathbf{x}) \tilde{\mathbf{P}}(\mathbf{x}), \quad (58)$$

$$\tilde{\mathbf{B}}(\mathbf{x}) = \tilde{\mathbf{P}}^T(\mathbf{x}) \mathbf{W}(\mathbf{x}) (\mathbf{E} - \mathbf{V}(\mathbf{x})), \quad (59)$$

$\mathbf{E}$ is identity matrix, and

$$\tilde{\mathbf{P}}(\mathbf{x}) = \begin{bmatrix} \tilde{q}_2(\mathbf{x}_1) & \tilde{q}_3(\mathbf{x}_1) & \cdots & \tilde{q}_m(\mathbf{x}_1) \\ \tilde{q}_2(\mathbf{x}_2) & \tilde{q}_3(\mathbf{x}_2) & \cdots & \tilde{q}_m(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{q}_2(\mathbf{x}_n) & \tilde{q}_3(\mathbf{x}_n) & \cdots & \tilde{q}_m(\mathbf{x}_n) \end{bmatrix}, \quad (60)$$

$$\mathbf{V}(\mathbf{x}) = \begin{bmatrix} \tilde{v}(\mathbf{x}, \mathbf{x}_1) & \tilde{v}(\mathbf{x}, \mathbf{x}_2) & \cdots & \tilde{v}(\mathbf{x}, \mathbf{x}_n) \\ \tilde{v}(\mathbf{x}, \mathbf{x}_1) & \tilde{v}(\mathbf{x}, \mathbf{x}_2) & \cdots & \tilde{v}(\mathbf{x}, \mathbf{x}_n) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{v}(\mathbf{x}, \mathbf{x}_1) & \tilde{v}(\mathbf{x}, \mathbf{x}_2) & \cdots & \tilde{v}(\mathbf{x}, \mathbf{x}_n) \end{bmatrix}. \quad (61)$$

In contrast to the interpolating MLS approximation developed by Ren et al. [63], the shape function formulation proposed by Wang et al. [78] allows arbitrary weight function selection from standard MLS approximation. This method not only eliminates computational challenges caused by weight functions singularities but also avoids the associated truncation errors.

2.5 2D ICVMLS Approximation

The MLS, IMLS, and two type of interpolating MLS approximations are formulated for scalar function approximation. Bai et al. [102] developed the ICVMLS approximation to handle vector-valued function approximation.

For any function $u(z)$, its approximation is

$$u^h(z) = u_1^h(z) + iu_2^h(z) = \bar{\mathbf{q}}^T(z) \mathbf{a}(z) = \sum_{i=1}^m \bar{q}_i(z) a_i(z), z = x_1 + ix_2 \in \Omega, \quad (62)$$

$\bar{\mathbf{q}}^T(z)$ denotes the basis function vector, which is conjugate to $\mathbf{q}^T(z)$,

$$\mathbf{q}^T(z) = (1, q_2(z), \dots, q_m(z)), \quad (63)$$

and

$$\mathbf{a}^T(z) = (a_1, a_2, \dots, a_m). \quad (64)$$

In general, the simple linear basis function is adopted for the ICVMLS approximation, which can be written as

$$\bar{\mathbf{q}}^T(z) = (\bar{q}_1, \bar{q}_2) = (1, \bar{z}) = (1, x_1 - ix_2), \quad (65)$$

Define the following functional

$$J = \left(\bar{\mathbf{Q}}(z) \mathbf{a}(z) - \check{\mathbf{u}}(z) \right)^T \mathbf{W}(z) \overline{\left(\bar{\mathbf{Q}}(z) \mathbf{a}(z) - \check{\mathbf{u}}(z) \right)}, \quad (66)$$

where

$$\bar{\mathbf{Q}}(z) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \bar{q}_2(z_1) & \bar{q}_2(z_2) & \cdots & \bar{q}_2(z_n) \end{bmatrix}^T, \quad (67)$$

$$\check{\mathbf{u}}(z) = (u_1(z_1) + i u_2(z_1), u_1(z_2) + i u_2(z_2), \dots, u_1(z_n) + i u_2(z_n))^T, \quad (68)$$

$$\mathbf{W}(z) = \begin{bmatrix} w(z-z_1) & 0 & \cdots & 0 \\ 0 & w(z-z_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(z-z_n) \end{bmatrix}. \quad (69)$$

Let

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{0}, \quad (70)$$

we get

$$\mathbf{a}(z) = \mathbf{A}^{-1}(z) \mathbf{B}(z) \check{\mathbf{u}}(z), \quad (71)$$

where

$$\mathbf{A}(z) = \mathbf{Q}^T(z) \mathbf{W}(z) \bar{\mathbf{Q}}(z), \quad (72)$$

$$\mathbf{B}(z) = \mathbf{Q}^T(z) \mathbf{W}(z). \quad (73)$$

Thus the approximation function can be obtained as

$$u^h(z) = \Phi(z) \check{\mathbf{u}} = \sum_{I=1}^n \Phi_I(z) u(z_I), \quad (74)$$

where the shape function is

$$\Phi(z) = [\Phi_1, \Phi_2, \dots, \Phi_n] = \bar{\mathbf{q}}^T(z) \mathbf{A}^{-1}(z) \mathbf{B}(z). \quad (75)$$

Then Eq. (74) can be discretized as

$$u_1^h(z) = \text{Re} \left[\Phi(z) \tilde{\mathbf{u}} \right], \quad (76)$$

and

$$u_2^h(\mathbf{x}) = \text{Im} \left[\Phi(z) \tilde{\mathbf{u}} \right]. \quad (77)$$

If $u(z)$ is a real valued function, $u^h(z) = u_1^h(z)$ in Eqs. (62) and (68) can be simplified as

$$\tilde{\mathbf{u}}(z) = (u_1(z_1), u_1(z_2), \dots, u_1(z_n))^T. \quad (78)$$

Since the ICVMLS method yields the approximation function, the expression in Eq. (77) tends to zero. This demonstrates that only the real component of shape function needs to be retained to approximate the real-valued functions.

The use of the ICVMLS approximation with linear basis function for 2D shape function result in only 2 undetermined coefficients, as opposed to the 3 required by the MLS approximation, thereby directly lowering the matrix order in the computation.

3 The Improved EFG Methods for Partial Differential Equations and Nonlinear Mechanics Problems

3.1 The IIEFG Method for 3D Steady Heat Conduction Problem in Anisotropic Media

This section employs a steady heat conduction problem to validate both the effectiveness and computational efficiency of the IIEFG method.

The governing equation is

$$k_{11} \frac{\partial^2 T}{\partial x_1^2} + k_{22} \frac{\partial^2 T}{\partial x_2^2} + k_{33} \frac{\partial^2 T}{\partial x_3^2} + 2k_{12} \frac{\partial^2 T}{\partial x_1 \partial x_2} + 2k_{23} \frac{\partial^2 T}{\partial x_2 \partial x_3} + 2k_{31} \frac{\partial^2 T}{\partial x_3 \partial x_1} = f(\mathbf{x}), \mathbf{x} = (x_1, x_2, x_3) \in \Omega, \quad (79)$$

the Dirichlet and Neumann boundary conditions are

$$T(\mathbf{x}) = \bar{T}(\mathbf{x}), \mathbf{x} \in \Gamma_u, \quad (80)$$

$$\begin{aligned} q(\mathbf{x}) = & k_{11} \frac{\partial T}{\partial x_1} n_1 + k_{12} \frac{\partial T}{\partial x_2} n_1 + k_{13} \frac{\partial T}{\partial x_3} n_1 + k_{21} \frac{\partial T}{\partial x_1} n_2 + k_{22} \frac{\partial T}{\partial x_2} n_2 + k_{23} \frac{\partial T}{\partial x_3} n_2 \\ & + k_{31} \frac{\partial T}{\partial x_1} n_3 + k_{32} \frac{\partial T}{\partial x_2} n_3 + k_{33} \frac{\partial T}{\partial x_3} n_3 = \bar{q}, \mathbf{x} \in \Gamma_q \end{aligned} \quad (81)$$

where $k_{11} > 0, k_{22} > 0$ refer to thermal conductivity coefficients, and $k_{12} = k_{21}, k_{23} = k_{32}, k_{31} = k_{13}, k_{11}k_{22} > k_{12}^2, k_{22}k_{33} > k_{23}^2, k_{33}k_{11} > k_{31}^2$. $f(\mathbf{x})$ denotes the heat source density. $\Gamma = \Gamma_u \cup \Gamma_q, \Gamma_u \cap \Gamma_q = \emptyset, \bar{T}$ is known temperature and $\bar{q}$ is given density of heat source. n_i is component of the unit normal to boundary along i .

The functional corresponding to the governing heat conduction equation is constructed as follows:

$$\begin{aligned} \Pi = & \int_{\Omega} \frac{1}{2} \left[k_{11} \left(\frac{\partial T}{\partial x_1} \right)^2 + k_{22} \left(\frac{\partial T}{\partial x_2} \right)^2 + k_{33} \left(\frac{\partial T}{\partial x_3} \right)^2 \right] d\Omega \\ & + \int_{\Omega} \left(k_{12} \frac{\partial T}{\partial x_1} \frac{\partial T}{\partial x_2} + k_{23} \frac{\partial T}{\partial x_2} \frac{\partial T}{\partial x_3} + k_{31} \frac{\partial T}{\partial x_3} \frac{\partial T}{\partial x_1} \right) d\Omega + \int_{\Omega} T f d\Omega - \int_{\Gamma_q} T \bar{q} d\Gamma \\ & + \frac{\alpha}{2} \int_{\Gamma_u} (T - \bar{T})(T - \bar{T}) d\Gamma \end{aligned} \quad (82)$$

where α is penalty factor.

The corresponding Galerkin weak form can be derived as

$$\int_{\Omega} \delta (L_1 T)^T (L_2 T) d\Omega - \int_{\Omega} \delta T f d\Omega - \int_{\Gamma_q} \delta T \bar{q} d\Gamma + \alpha \int_{\Gamma_u} \delta T T d\Gamma - \alpha \int_{\Gamma_u} \delta T \bar{T} d\Gamma = 0, \quad (83)$$

where

$$L_1 (\cdot) = \left(\sqrt{k_{11}} \frac{\partial}{\partial x_1}, \sqrt{k_{22}} \frac{\partial}{\partial x_2}, \sqrt{k_{33}} \frac{\partial}{\partial x_3}, \sqrt{2k_{12}} \frac{\partial}{\partial x_1}, \sqrt{2k_{23}} \frac{\partial}{\partial x_2}, \sqrt{2k_{31}} \frac{\partial}{\partial x_3} \right)^T (\cdot), \quad (84)$$

$$L_2 (\cdot) = \left(\sqrt{k_{11}} \frac{\partial}{\partial x_1}, \sqrt{k_{22}} \frac{\partial}{\partial x_2}, \sqrt{k_{33}} \frac{\partial}{\partial x_3}, \sqrt{2k_{12}} \frac{\partial}{\partial x_2}, \sqrt{2k_{23}} \frac{\partial}{\partial x_3}, \sqrt{2k_{31}} \frac{\partial}{\partial x_1} \right)^T (\cdot). \quad (85)$$

Assume that M discretized nodes $\mathbf{x}_I$ ($I = 1, 2, \dots, M$) are distributed within the problem domain. We introduce the IMLS shape function to approximate $T(\mathbf{x})$

$$T(\mathbf{x}) = \Phi^*(\mathbf{x}) \mathbf{T} = \sum_{I=1}^n \Phi_I^* T_I, \quad (86)$$

$$\mathbf{T}(\mathbf{x}) = (T_1, T_2, \dots, T_n)^T. \quad (87)$$

Thus

$$L_1 T(\mathbf{x}) = L_1 [\Phi^*(\mathbf{x}) \mathbf{T}] = \mathbf{B}_1(\mathbf{x}) \mathbf{T} = \sum_{I=1}^n \mathbf{B}_{1I}(\mathbf{x}) T_I, \quad (88)$$

$$L_2 T(\mathbf{x}) = L_2 [\Phi^*(\mathbf{x}) \mathbf{T}] = \mathbf{B}_2(\mathbf{x}) \mathbf{T} = \sum_{I=1}^n \mathbf{B}_{2I}(\mathbf{x}) T_I, \quad (89)$$

where

$$\mathbf{B}_1(\mathbf{x}) = (\mathbf{B}_{11}(\mathbf{x}), \mathbf{B}_{12}(\mathbf{x}), \dots, \mathbf{B}_{1n}(\mathbf{x})), \quad (90)$$

$$\mathbf{B}_2(\mathbf{x}) = (\mathbf{B}_{21}(\mathbf{x}), \mathbf{B}_{22}(\mathbf{x}), \dots, \mathbf{B}_{2n}(\mathbf{x})), \quad (91)$$

$$\mathbf{B}_{1I}(\mathbf{x}) = \begin{bmatrix} \sqrt{k_{11}} \Phi_{I,1}^*(\mathbf{x}) \\ \sqrt{k_{22}} \Phi_{I,2}^*(\mathbf{x}) \\ \sqrt{k_{33}} \Phi_{I,3}^*(\mathbf{x}) \\ \sqrt{2k_{12}} \Phi_{I,1}^*(\mathbf{x}) \\ \sqrt{2k_{23}} \Phi_{I,2}^*(\mathbf{x}) \\ \sqrt{2k_{31}} \Phi_{I,3}^*(\mathbf{x}) \end{bmatrix}, \quad (92)$$

$$\mathbf{B}_{2I}(\mathbf{x}) = \begin{bmatrix} \sqrt{k_{11}} \Phi_{I,1}^*(\mathbf{x}) \\ \sqrt{k_{22}} \Phi_{I,2}^*(\mathbf{x}) \\ \sqrt{k_{33}} \Phi_{I,3}^*(\mathbf{x}) \\ \sqrt{2k_{12}} \Phi_{I,2}^*(\mathbf{x}) \\ \sqrt{2k_{23}} \Phi_{I,3}^*(\mathbf{x}) \\ \sqrt{2k_{31}} \Phi_{I,1}^*(\mathbf{x}) \end{bmatrix}. \quad (93)$$

By introducing the shape function, we obtain

$$\begin{aligned} & \int_{\Omega} \delta[\mathbf{B}_1(\mathbf{x})\mathbf{T}]^T[\mathbf{B}_2(\mathbf{x})\mathbf{T}]d\Omega + \int_{\Omega} \delta[\Phi^*(\mathbf{x})\mathbf{T}]^T f d\Omega - \int_{\Gamma_q} \delta[\Phi^*(\mathbf{x})\mathbf{T}]^T \bar{q} d\Gamma \\ & + \alpha \int_{\Gamma_u} \delta[\Phi^*(\mathbf{x})\mathbf{T}]^T [\Phi^*(\mathbf{x})\mathbf{T}] d\Gamma - \alpha \int_{\Gamma_u} \delta[\Phi^*(\mathbf{x})\mathbf{T}]^T \bar{T} d\Gamma = 0 \end{aligned} \quad (94)$$

The final equation can be derived

$$\mathbf{KT} = \mathbf{F}, \quad (95)$$

where

$$\mathbf{K} = \int_{\Omega} \mathbf{B}_1^T(\mathbf{x}) \mathbf{B}_2(\mathbf{x}) d\Omega + \alpha \int_{\Gamma_u} \Phi^{*T}(\mathbf{x}) \Phi^*(\mathbf{x}) d\Gamma, \quad (96)$$

$$\mathbf{F} = - \int_{\Omega} \Phi^{*T}(\mathbf{x}) f d\Omega + \alpha \int_{\Gamma_u} \Phi^{*T}(\mathbf{x}) \bar{T} d\Gamma + \int_{\Gamma_q} \Phi^{*T}(\mathbf{x}) \bar{q} d\Gamma. \quad (97)$$

To verify the accuracy of the derived numerical solution formulation, we present the following numerical example, the governing equation is

$$\frac{\partial^2 T}{\partial x_1^2} + \frac{\partial^2 T}{\partial x_2^2} + \frac{\partial^2 T}{\partial x_3^2} + \frac{\partial^2 T}{\partial x_1 \partial x_2} + \frac{\partial^2 T}{\partial x_2 \partial x_3} + \frac{\partial^2 T}{\partial x_3 \partial x_1} = 0, (x_1, x_2, x_3) \in [0, 1] \times [0, 1] \times [0, 1], \quad (98)$$

the boundary conditions are

$$T(0, x_2, x_3) = 0.5x_2^2 - 1.5x_3^2 + 0.5x_2x_3 + 2, \quad (99)$$

$$T(1, x_2, x_3) = 0.5x_2^2 - 1.5x_3^2 - 0.5x_2 + x_3 + 0.5x_2x_3 + 2.5, \quad (100)$$

$$T(x_1, 0, x_3) = 0.5x_1^2 - 1.5x_3^2 + x_1x_3 + 2, \quad (101)$$

$$T(x_1, 1, x_3) = 0.5x_1^2 - 1.5x_3^2 - 0.5x_1 + x_3 + 0.5x_2x_3 + 2.5, \quad (102)$$

$$T(x_1, x_2, 0) = 0.5x_1^2 + 0.5x_2^2 - 0.5x_1x_2 + 2, \quad (103)$$

$$T(x_1, x_2, 1) = 0.5x_1^2 + 0.5x_2^2 - 0.5x_1x_2 + x_1 + 0.5x_2 + 0.5. \quad (104)$$

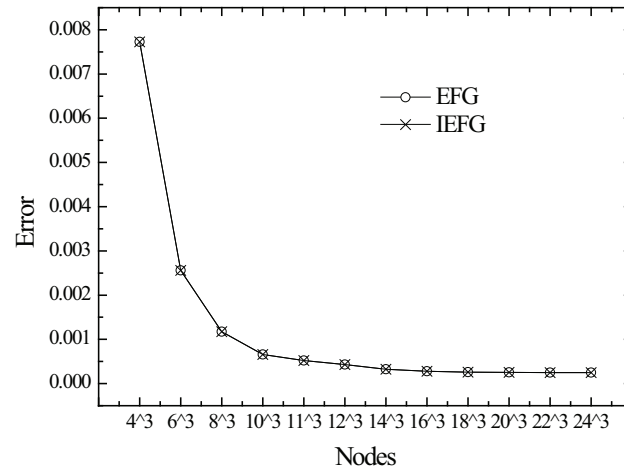
The exact solution [148] is

$$T = 0.5x_1^2 + 0.5x_2^2 - 1.5x_3^2 - 0.5x_1x_2 + x_1x_3 + 0.5x_2x_3 + 2. \quad (105)$$

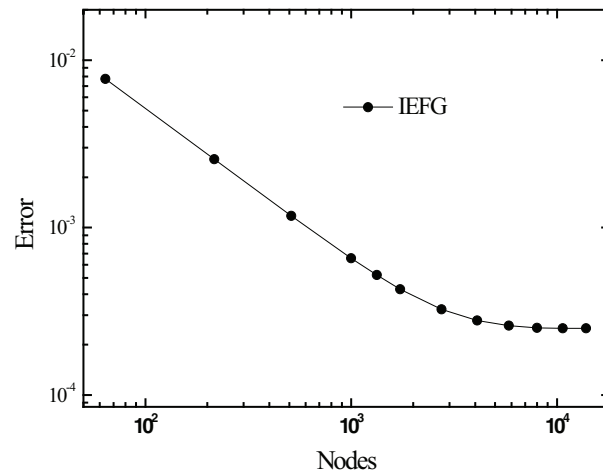
In order to calculate the accuracy of IIEFG method, the relative error of the numerical solution is computed using the following formula:

$$\|u - u^h\|_{L^2(\Omega)}^{rel} = \frac{\left(\int_{\Omega} (u - u^h)^2 d\Omega \right)^{1/2}}{\|u\|_{L^2(\Omega)}}. \quad (106)$$

In this example, the scaling parameter $d_{\max}$ is set to 1.3 and the penalty factor α is adjusted to 1.4×10^3 in both EFG and IIEFG methods. Fig. 3a shows the error variation of the EFG and IIEFG methods for solving the heat conduction problem with different nodes in the problem domain. As shown in Fig. 3b, which presents the convergence rate of the IIEFG method. It can be observed that the relative error decreases and tends to stabilize as the nodes increases, indicating that the IIEFG method is convergent.



(a) The case where the two axes are scaled proportionally.



(b) The case where logarithmic scales are applied to both axes.

Figure 3: The relative error for different number of nodes

The IEFG and EFG methods are implemented with a node distribution of $11 \times 11 \times 11$ and a corresponding background integral grid is $10 \times 10 \times 10$, $d_{\max}$ is set to 1.3, and penalty factor α is set to 1.4×10^3 . Table 1 presents the computational errors and CPU time of two methods under appropriate parameters. Figs. 4–6 compare the exact and numerical solutions. Although both methods achieve a close approximation to the exact one, the IEFG method can save computational time.

Table 1: The errors and CPU times of two methods under appropriate parameters

Numerical method	$d_{\max}$	α	Integral grid	Nodes	Error (%)	Time (s)
IEFG	1.3	1.4×10^3	$10 \times 10 \times 10$	$11 \times 11 \times 11$	0.0521%	7.68
EFG	1.3	1.4×10^3	$10 \times 10 \times 10$	$11 \times 11 \times 11$	0.0521%	9.14

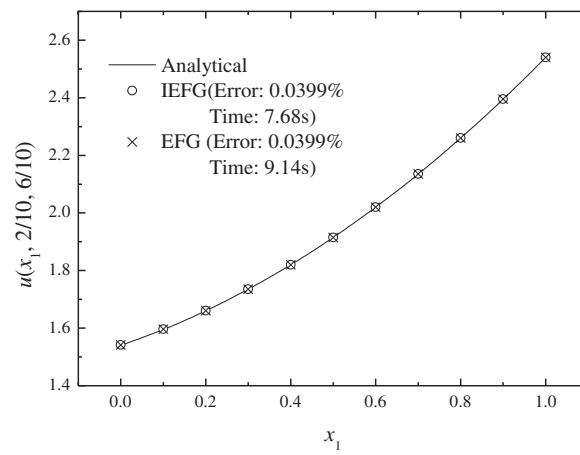


Figure 4: The temperature distribution along x_1 direction

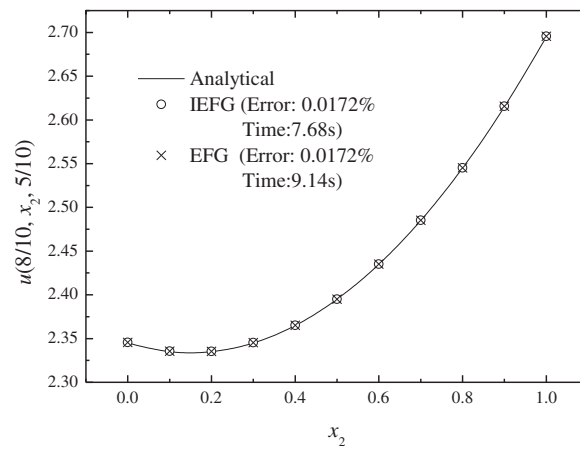


Figure 5: The temperature distribution along x_2 direction

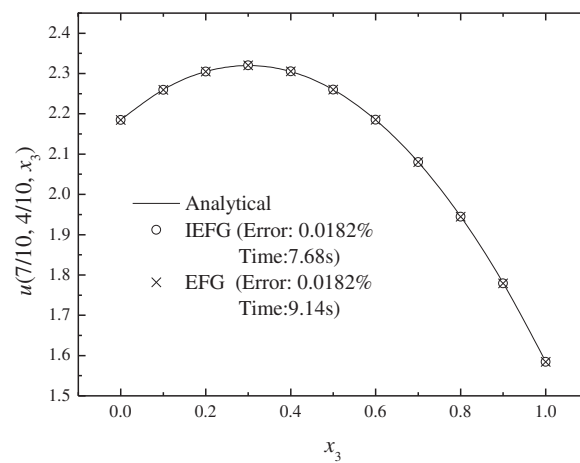


Figure 6: The temperature distribution along x_3 direction

3.2 The IIEFG Method for 3D Elastoplasticity Problem

In reference [149], Yu et al. selected the IIEFG method to solve 3D elastoplasticity problems with four numerical examples. In this paper, we analyze an example with the hollow sphere subjected to uniform internal pressure.

Due to its symmetry, 1/8 of the structure is selected as the subject of the study. Fig. 7 illustrates 1/8 of the hollow sphere. The inner radius of the sphere is $a = 10.0$ m, and the outer radius is $b = 20.0$ m. Its inner surface is subjected to a uniformly distributed load $p = 1000$ N/m², with other material parameters: Poisson's ratio $\mu = 0.25$, Young's modulus $E = 1.0 \times 10^5$ Pa, and yield limit $\sigma_s = 150$ Pa.

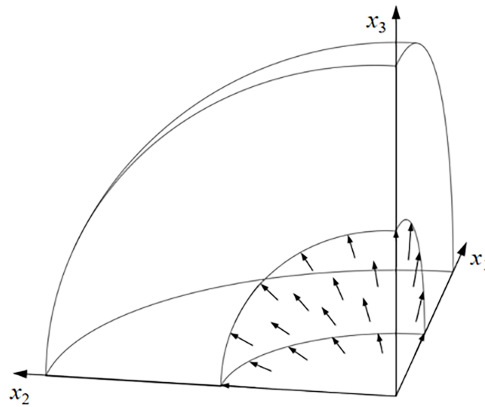


Figure 7: 1/8 hollow sphere under uniform internal pressure

We establish the nodal distribution for solving this problem using the IIEFG method. The nodes on the coordinate plane are uniformly distributed along the circumferential direction and arranged evenly in the radial direction. $7 \times 7 \times 13$ node distribution scheme is selected in Fig. 8.

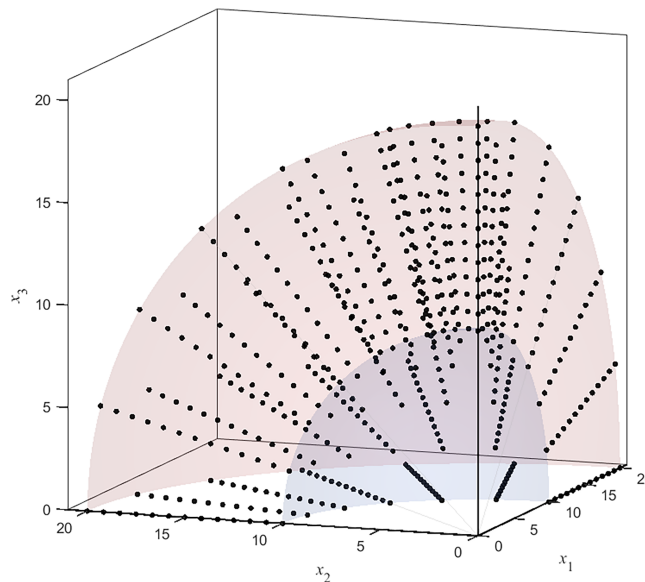


Figure 8: Node distribution in the problem domain

When the IIEFG method is employed to solve this problem, the parameters are set as follows: $d_{\max}$ is 1.6, α is 1.0×10^8 , with the number of loading steps set to 50, and the weight function is a quintic spline function. The obtained error is 2.535%, and the CPU time is 759.4 s. When the EFG method is used to solve this problem, the parameters set are the same as those of the IIEFG method, resulting in the same error. However, the computation time for the EFG method is 928.9 s.

Fig. 9 presents the numerical solutions of the IIEFG method and the EFG method at $x_1 = 0$ and $x_2 = 0$. The figure indicates that both methods yield results consistent with the numerical solution obtained from ABAQUS.

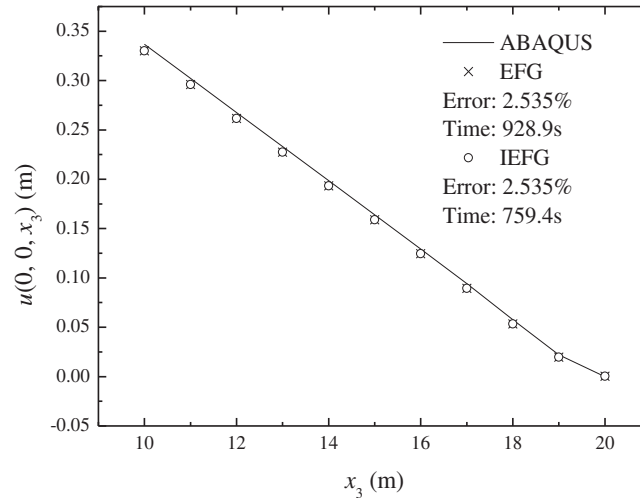


Figure 9: The numerical solution of displacement when $x_1 = 0$ and $x_2 = 0$

Fig. 10 shows 3D node displacement cloud plots of this example. It can be observed that the displacement gradually decreases along the radial direction.

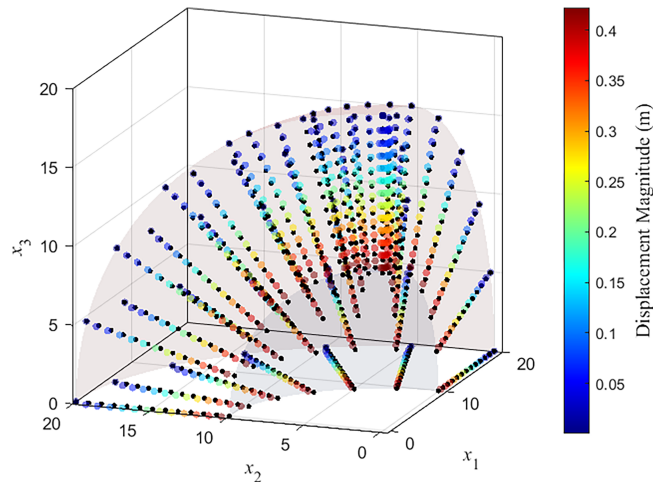


Figure 10: 3D node displacement cloud plot

3.3 The IIEFG Method for 3D Large Deformation Problems

One of the fundamental advantages of meshless methods lies in their inherent capability to handle large deformation and crack propagation problems without mesh distortion that is unavoidable in FEM [150,151]. In this section, we implement the IIEFG method for solving 3D elastic large deformation problems.

The equilibrium equation for large deformation problems at time $t + \Delta t$ can be derived using the principle of virtual displacements

$$\int_{t+\Delta t \Omega} {}^{t+\Delta t} \tau_{ij} \delta {}^{t+\Delta t} e_{ij} d\Omega + \alpha \int_{t+\Delta t \Gamma_u} \delta u_i \cdot (u_i - \bar{u}_i) d\Gamma = {}^{t+\Delta t} Q, \quad (107)$$

where ${}^{t+\Delta t} \tau_{ij}$ is the Cauchy stress, $\delta {}^{t+\Delta t} e_{ij}$ is the variation of strain,

$$\delta {}^{t+\Delta t} e_{ij} = \frac{1}{2} \delta ({}^{t+\Delta t} u_{i,j} + {}^{t+\Delta t} u_{j,i}), \quad (108)$$

and

$$u_i = {}^{t+\Delta t} u_i - {}^t u_i, \quad (109)$$

${}^t u_i$ and ${}^{t+\Delta t} u_i$ ($i = 1, 2, 3$) denote the displacements of any point in the computational domain at times t and $t + \Delta t$, respectively; $\bar{u}_i$ is the known displacement on the boundary Γ_u , ${}^{t+\Delta t} Q$ is the virtual work,

$${}^{t+\Delta t} Q = \int_{t+\Delta t \Gamma} {}^{t+\Delta t} \bar{t}_k \delta u_k d\Gamma + \int_{t+\Delta t \Omega} {}^{t+\Delta t} \rho {}^{t+\Delta t} b_k \delta u_k d\Omega, \quad (110)$$

${}^{t+\Delta t} b_k$, ${}^{t+\Delta t} \bar{t}_k$ and ${}^{t+\Delta t} \rho$ refer to the body force, surface force and mass density at the moment of $t + \Delta t$.

Following linearization of the equilibrium equations, we derive the corresponding weak integral form as:

$$\begin{aligned} & \int_{\Omega} \delta {}_0 e_{ij} \cdot {}_0 D_{ijkl} \cdot {}_0 e_{kl} d\Omega + \int_{\Omega} \delta {}_0 \eta_{ij} \cdot {}^t {}_0 S_{ij} d\Omega + \alpha \int_{\Gamma_u} \delta u_i \cdot (u_i - \bar{u}_i) d\Gamma \\ & = \int_{t+\Delta t \Gamma} \delta u_k \cdot {}^{t+\Delta t} \bar{t}_k d\Gamma + \int_{t+\Delta t \Omega} \delta u_k \cdot {}^t {}_0 \rho {}^{t+\Delta t} b_k d\Omega - \int_{\Omega} \delta {}_0 e_{ij} \cdot {}^t {}_0 S_{ij} d\Omega \end{aligned} \quad (111)$$

${}_0 e_{ij}$ and ${}_0 \eta_{ij}$ represent the linear and quadratic terms with respect to the displacement increment u_i , respectively, and

$${}_0 e_{ij} = 0.5 ({}_0 u_{i,j} + {}_0 u_{j,i}) + 0.5 ({}_0^t u_{k,i} \cdot {}_0 u_{k,j} + {}_0^t u_{k,j} \cdot {}_0 u_{k,i}), \quad (112)$$

$${}_0 \eta_{ij} = 0.5 {}_0 u_{k,i} {}_0 u_{k,j}, \quad (113)$$

$${}_0 S_{ij} = {}_0 D_{ijkl} \cdot {}_0 \varepsilon_{kl}, \quad (114)$$

where ${}_0 D_{ijkl}$ is the tangent constitutive tensor, which is a function of t .

By introducing the IMLS approximation, the displacement increment ${}_0 u(\mathbf{x})$ can be obtained as

$${}_0 u(\mathbf{x}) = \sum_{I=1}^n \Phi_I^*(\mathbf{x}) {}_0 u(\mathbf{x}_I) = \Phi^*(\mathbf{x}) {}_0 \mathbf{U}, \quad (115)$$

and

$${}_0 \mathbf{U} = ({}_0 u(\mathbf{x}_1), {}_0 u(\mathbf{x}_2), \dots, {}_0 u(\mathbf{x}_n))^T. \quad (116)$$

Substituting Eqs. (112)–(115) into (111) yields:

$$({}_0\mathbf{K}^{(L0)} + {}_0\mathbf{K}^{(L1)} + {}_0\mathbf{K}^{(NL)} + {}_0\mathbf{G})_0 \mathbf{U} = {}^{t+\Delta t}\mathbf{F} - {}^t\mathbf{F}, \quad (117)$$

where

$${}_0\mathbf{K}^{(L0)} = \int_{\Omega} {}_0\mathbf{B}^{(L0)\mathrm{T}} {}_0\mathbf{D}_0 \mathbf{B}^{(L0)} \mathrm{d}\Omega, \quad (118)$$

$${}_0\mathbf{K}^{(L1)} = \int_{\Omega} {}_0\mathbf{B}^{(L0)\mathrm{T}} {}_0\mathbf{D}_0 \mathbf{B}^{(L1)} \mathrm{d}\Omega + \int_{\Omega} {}_0\mathbf{B}^{(L1)\mathrm{T}} {}_0\mathbf{D}_0 \mathbf{B}^{(L0)} \mathrm{d}\Omega + \int_{\Omega} {}_0\mathbf{B}^{(L1)\mathrm{T}} {}_0\mathbf{D}_0 \mathbf{B}^{(L1)} \mathrm{d}\Omega, \quad (119)$$

$${}_0\mathbf{K}^{(NL)} = \int_{\Omega} {}_0\mathbf{B}^{\mathrm{T}t} {}_0\tilde{\mathbf{S}}_0 \mathbf{B} \mathrm{d}\Omega, \quad (120)$$

$${}_0\mathbf{G} = \alpha \int_{\Gamma_u} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}} \mathbf{C} \tilde{\boldsymbol{\Phi}} \mathrm{d}\Gamma, \quad (121)$$

$${}^t\mathbf{F} = \int_{\Omega} {}_0\mathbf{B}^{(L0)\mathrm{T}} {}_0^t\mathbf{S} \mathrm{d}\Omega + \int_{\Omega} {}_0\mathbf{B}^{(L1)\mathrm{T}} {}_0^t\mathbf{S} \mathrm{d}\Omega, \quad (122)$$

$${}^{t+\Delta t}\mathbf{F} = \int_{\Omega} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}} {}_0\mathbf{b} \mathrm{d}\Omega + \int_{\Gamma_t} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}} {}_0\bar{\mathbf{t}} \mathrm{d}\Gamma + \tilde{\boldsymbol{\Phi}}^{\mathrm{T}} {}_0\bar{\mathbf{T}} + \alpha \int_{\Gamma_u} \tilde{\boldsymbol{\Phi}}^{\mathrm{T}} \mathbf{C}_0 \bar{\mathbf{u}} \mathrm{d}\Gamma, \quad (123)$$

$${}_0\bar{\mathbf{t}} = ({}_0\bar{t}_1, {}_0\bar{t}_2, {}_0\bar{t}_3)^{\mathrm{T}}, \quad (124)$$

$${}_0\bar{\mathbf{u}} = ({}_0\bar{u}_1, {}_0\bar{u}_2, {}_0\bar{u}_3)^{\mathrm{T}}, \quad (125)$$

$${}_0\mathbf{b} = ({}_0b_1, {}_0b_2, {}_0b_3)^{\mathrm{T}}, \quad (126)$$

$${}_0\bar{\mathbf{T}} = ({}_0\bar{T}_1, {}_0\bar{T}_2, {}_0\bar{T}_3)^{\mathrm{T}}, \quad (127)$$

$$\mathbf{B}^{(L0)} = \begin{bmatrix} \Phi_{1,1}^* & 0 & 0 & \cdots & \Phi_{n,1}^* & 0 & 0 \\ 0 & \Phi_{1,2}^* & 0 & \cdots & 0 & \Phi_{n,2}^* & 0 \\ 0 & 0 & \Phi_{1,3}^* & \cdots & 0 & 0 & \Phi_{n,3}^* \\ 0 & \Phi_{1,3}^* & \Phi_{1,2}^* & \cdots & 0 & \Phi_{n,3}^* & \Phi_{n,2}^* \\ \Phi_{1,3}^* & 0 & \Phi_{1,1}^* & \cdots & \Phi_{n,3}^* & 0 & \Phi_{n,1}^* \\ \Phi_{1,2}^* & \Phi_{1,1}^* & 0 & \cdots & \Phi_{n,2}^* & \Phi_{n,1}^* & 0 \end{bmatrix}, \quad (128)$$

$$\mathbf{B}^{(L1)} = \begin{bmatrix} L_{11}\Phi_{1,1}^* & L_{21}\Phi_{1,1}^* & L_{31}\Phi_{1,1}^* \\ L_{12}\Phi_{1,2}^* & L_{22}\Phi_{1,2}^* & L_{32}\Phi_{1,2}^* \\ L_{13}\Phi_{1,3}^* & L_{23}\Phi_{1,3}^* & L_{33}\Phi_{1,3}^* \\ L_{12}\Phi_{1,3}^* + L_{13}\Phi_{1,2}^* & L_{22}\Phi_{1,3}^* + L_{23}\Phi_{1,2}^* & L_{32}\Phi_{1,3}^* + L_{33}\Phi_{1,2}^* \\ L_{11}\Phi_{1,3}^* + L_{13}\Phi_{1,1}^* & L_{21}\Phi_{1,3}^* + L_{23}\Phi_{1,1}^* & L_{31}\Phi_{1,3}^* + L_{33}\Phi_{1,1}^* \\ L_{11}\Phi_{1,2}^* + L_{12}\Phi_{1,1}^* & L_{21}\Phi_{1,2}^* + L_{22}\Phi_{1,1}^* & L_{31}\Phi_{1,2}^* + L_{32}\Phi_{1,1}^* \\ \cdots & L_{11}\Phi_{n,1}^* & L_{21}\Phi_{n,1}^* & L_{31}\Phi_{n,1}^* \\ \cdots & L_{12}\Phi_{n,2}^* & L_{22}\Phi_{n,2}^* & L_{32}\Phi_{n,2}^* \\ \cdots & L_{13}\Phi_{n,3}^* & L_{23}\Phi_{n,3}^* & L_{33}\Phi_{n,3}^* \\ \cdots & L_{12}\Phi_{n,3}^* + L_{13}\Phi_{n,2}^* & L_{22}\Phi_{n,3}^* + L_{23}\Phi_{n,2}^* & L_{32}\Phi_{n,3}^* + L_{33}\Phi_{n,2}^* \\ \cdots & L_{11}\Phi_{n,3}^* + L_{13}\Phi_{n,1}^* & L_{21}\Phi_{n,3}^* + L_{23}\Phi_{n,1}^* & L_{31}\Phi_{n,3}^* + L_{33}\Phi_{n,1}^* \\ \cdots & L_{11}\Phi_{n,2}^* + L_{12}\Phi_{n,1}^* & L_{21}\Phi_{n,2}^* + L_{22}\Phi_{n,1}^* & L_{31}\Phi_{n,2}^* + L_{32}\Phi_{n,1}^* \end{bmatrix}, \quad (129)$$

$$\mathbf{D}_0 = \frac{E}{(1+\mu)(1-2\mu)} \begin{bmatrix} 1-\mu & \mu & \mu & 0 & 0 & 0 \\ \mu & 1-\mu & \mu & 0 & 0 & 0 \\ \mu & \mu & 1-\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\mu}{2} \end{bmatrix}, \quad (130)$$

$$L_{11} = \sum_{k=1}^n \Phi_{k,1}^* {}^t u_1(\mathbf{x}_k), \quad (131)$$

$$L_{12} = \sum_{k=1}^n \Phi_{k,2}^* {}^t u_1(\mathbf{x}_k), \quad (132)$$

$$L_{13} = \sum_{k=1}^n \Phi_{k,3}^* {}^t u_1(\mathbf{x}_k), \quad (133)$$

$$L_{21} = \sum_{k=1}^n \Phi_{k,1}^* {}^t u_2(\mathbf{x}_k), \quad (134)$$

$$L_{22} = \sum_{k=1}^n \Phi_{k,2}^* {}^t u_2(\mathbf{x}_k), \quad (135)$$

$$L_{23} = \sum_{k=1}^n \Phi_{k,3}^* {}^t u_2(\mathbf{x}_k), \quad (136)$$

$$L_{31} = \sum_{k=1}^n \Phi_{k,1}^* {}^t u_3(\mathbf{x}_k), \quad (137)$$

$$L_{32} = \sum_{k=1}^n \Phi_{k,2}^* {}^t u_3(\mathbf{x}_k), \quad (138)$$

$$L_{33} = \sum_{k=1}^n \Phi_{k,3}^* {}^t u_3(\mathbf{x}_k), \quad (139)$$

$${}_0\mathbf{B} = \begin{bmatrix} \Phi_{1,1}^* & 0 & 0 & \cdots & \Phi_{n,1}^* & 0 & 0 \\ 0 & \Phi_{1,1}^* & 0 & \cdots & 0 & \Phi_{n,1}^* & 0 \\ 0 & 0 & \Phi_{1,1}^* & \cdots & 0 & 0 & \Phi_{n,1}^* \\ \Phi_{1,2}^* & 0 & 0 & \cdots & \Phi_{n,2}^* & 0 & 0 \\ 0 & \Phi_{1,2}^* & 0 & \cdots & 0 & \Phi_{n,2}^* & 0 \\ 0 & 0 & \Phi_{1,2}^* & \cdots & 0 & 0 & \Phi_{n,2}^* \\ \Phi_{1,3}^* & 0 & 0 & \cdots & \Phi_{n,3}^* & 0 & 0 \\ 0 & \Phi_{1,3}^* & 0 & \cdots & 0 & \Phi_{n,3}^* & 0 \\ 0 & 0 & \Phi_{1,3}^* & \cdots & 0 & 0 & \Phi_{n,3}^* \end{bmatrix}, \quad (140)$$

$${}^t_0\tilde{\mathbf{S}} = \begin{bmatrix} {}^t_0S_{11} & 0 & 0 & {}^t_0S_{12} & 0 & 0 & {}^t_0S_{13} & 0 & 0 \\ 0 & {}^t_0S_{11} & 0 & 0 & {}^t_0S_{12} & 0 & 0 & {}^t_0S_{13} & 0 \\ 0 & 0 & {}^t_0S_{11} & 0 & 0 & {}^t_0S_{12} & 0 & 0 & {}^t_0S_{13} \\ {}^t_0S_{21} & 0 & 0 & {}^t_0S_{22} & 0 & 0 & {}^t_0S_{23} & 0 & 0 \\ 0 & {}^t_0S_{21} & 0 & 0 & {}^t_0S_{22} & 0 & 0 & {}^t_0S_{23} & 0 \\ 0 & 0 & {}^t_0S_{21} & 0 & 0 & {}^t_0S_{22} & 0 & 0 & {}^t_0S_{23} \\ {}^t_0S_{31} & 0 & 0 & {}^t_0S_{32} & 0 & 0 & {}^t_0S_{33} & 0 & 0 \\ 0 & {}^t_0S_{31} & 0 & 0 & {}^t_0S_{32} & 0 & 0 & {}^t_0S_{33} & 0 \\ 0 & 0 & {}^t_0S_{31} & 0 & 0 & {}^t_0S_{32} & 0 & 0 & {}^t_0S_{33} \end{bmatrix}, \quad (141)$$

$${}^t_0\mathbf{S} = ({}^t_0S_{11}, {}^t_0S_{22}, {}^t_0S_{33}, {}^t_0S_{23}, {}^t_0S_{13}, {}^t_0S_{12})^T, \quad (142)$$

$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}, \quad (143)$$

${}^0\bar{T}$ is concentrated load, when there is a displacement constraint in x_i ($i = 1, 2, 3$) direction, the corresponding c_i is equal to one; otherwise, it is zero.

Let

$${}^0\mathbf{K} = {}^0\mathbf{K}^{(L0)} + {}^0\mathbf{K}^{(L1)} + {}^0\mathbf{K}^{(NL)} + {}^0\mathbf{G}, \quad (144)$$

$${}^0\mathbf{f} = {}^{t+\Delta t}\mathbf{F} - {}^t\mathbf{F}, \quad (145)$$

Eq. (117) can be simplified as

$${}^0\mathbf{K}_0\mathbf{U} = {}^0\mathbf{f}. \quad (146)$$

The above presents the IIEFG method for elastic large deformation problems. For Eq. (146), the Newton-Raphson iteration method is employed in this work for numerical computation.

To verify the accuracy and efficiency of the derived numerical solution formulation, we consider a cantilever beam subjected to uniform load of $q = 10$ N/m, and the left end of the beam is fixed, as shown in Fig. 11. $L = 10.0$ m, $H = D = 1.0$ m, elastic modulus $E = 1.2 \times 10^4$ Pa, Poisson's ratio $\mu = 0.2$, and self-weight of beam is neglected.

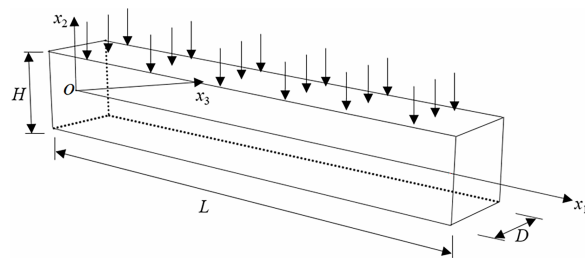


Figure 11: A cantilever beam subjected to a uniform load

The influence of node number on numerical accuracy should be considered. The total number of load steps is set to 20, with $d_{\max} = 3.5$ and $\alpha = 1.2 \times 10^9$, and the cubic spline function is selected. Fig. 12 presents

the relative errors of numerical solutions under different node distributions. It can be seen that the number of nodes along x_2 and x_3 direction is 5×5 , the error reduces with an increase in the number of nodes in x_1 direction.

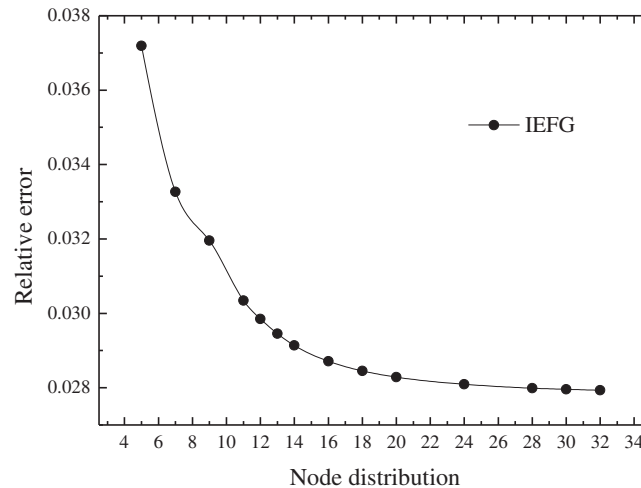


Figure 12: The error with different node distribution in x_1 direction

The IEFG method for large deformation problems is employed for computation. As shown in Fig. 13, the computational domain is discretized with $13 \times 5 \times 5$ nodes, $12 \times 4 \times 4$ background integration cells, and the loading step is 20. Fig. 14 presents the numerical solutions of node displacements along the x_1 direction with a uniform distribution of 10 nodes. Two numerical solutions agree well with the FEM software ANSYS and the proposed IEFG method achieves a 2.9451% reduction in computational time (1629.8 s) compared to the conventional EFG method (1721.5 s), confirming its enhanced efficiency while maintaining solution accuracy.

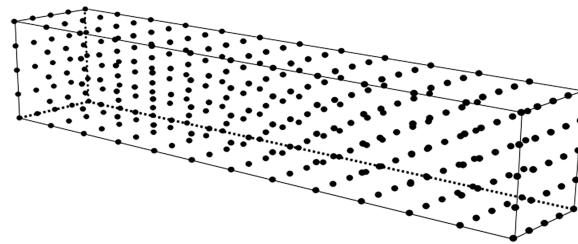


Figure 13: Nodes distribution in the beam

The next example considers a square plate with a central hole subjected to uniformly distributed loads on both sides. Fig. 15 illustrates the plate with a central circular hole under tensile loads P on both sides. The plate has geometric dimensions of $10 \text{ m} \times 10 \text{ m} \times 1 \text{ m}$ and a central hole radius of $r = 1 \text{ m}$. The material parameters are as follows:

Based on the symmetry of the model, $1/4$ of the structure is considered for analysis. Fig. 16 illustrates the coordinate system and force distribution of the quarter-open plate, while Fig. 17 shows the nodal distribution scheme: points are arranged radially in a geometric progression with a common ratio of q , and uniformly along the circumferential direction. Node distribution of $11 \times 11 \times 2$ is adopted in this numerical example.

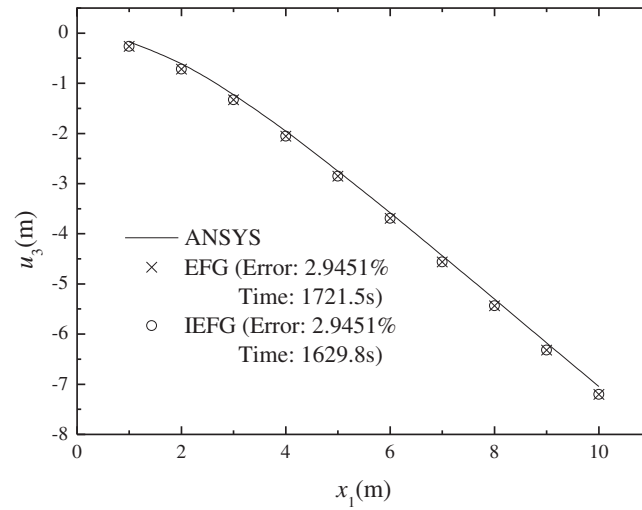


Figure 14: Numerical solutions of vertical displacements along x_1 axis

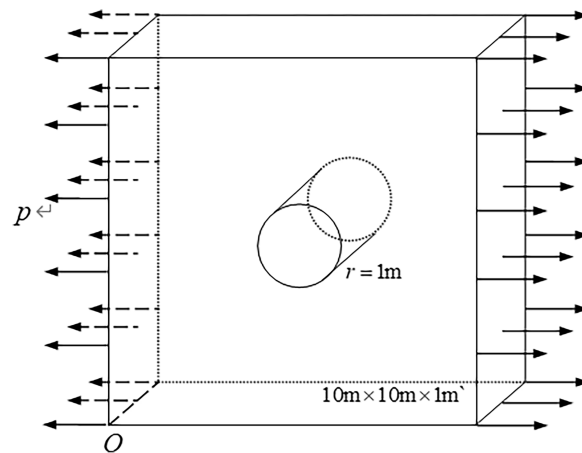


Figure 15: A centrally perforated plate under axial uniformly distributed load

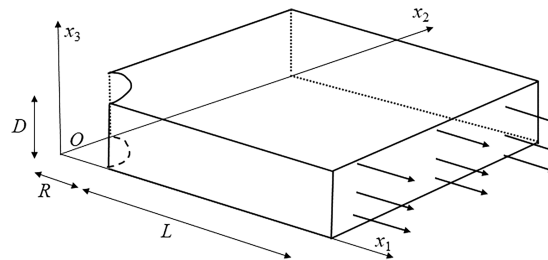


Figure 16: Force distribution diagram of a quarter-plate with a hole

When the EFG method was employed to solve this problem, a cubic spline function was adopted as the weight function with the following parameter settings: $d_{\max} = 2.8$, $\alpha = 1.0 \times 10^7$, and the number of loading steps was 10. The resulting error was 0.9999%, and the computation time was 638.63 s. Under the same parameter settings, the IEFM method achieved the same error of 0.9999% while reducing the computation

time to 563.62 s. Fig. 18 compares the results of the three numerical methods. It is shown that the IEFG method can save CPU time under similar computational accuracy.

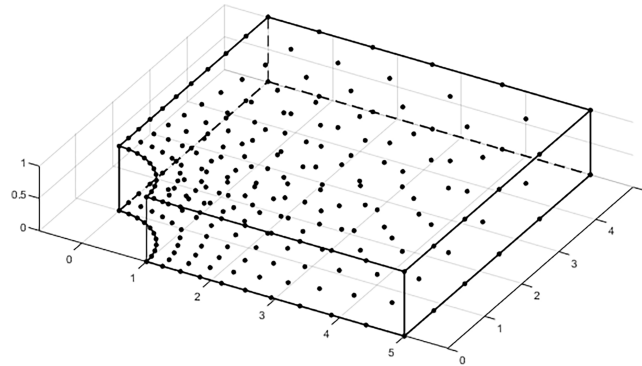


Figure 17: Node distribution

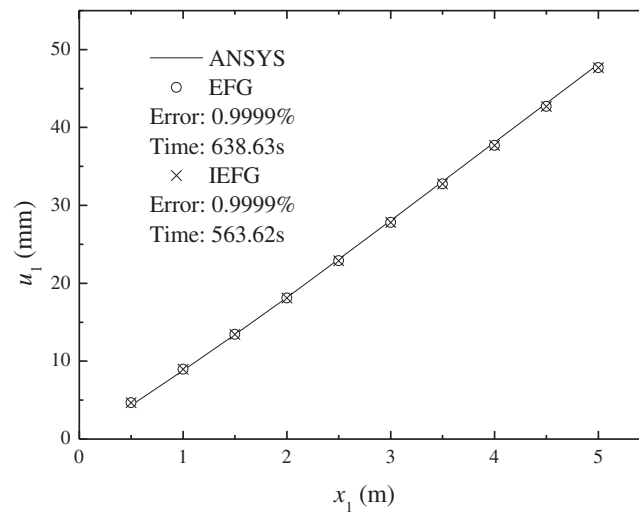


Figure 18: Comparison of results from different numerical methods

3.4 The DSIEFG Method and Dimension Coupling Method (DCM) for 3D Helmholtz Equations

As shown in Sections 3.1–3.3, the IEFG method can improve the computational speed of the EFG method, but the improvement remains relatively constrained for solving 3D problems.

To address the computational inefficiency of meshless methods in solving 3D complex mechanical problems, this section introduces the DSIEFG method and DCM for solving relatively simple 3D Helmholtz equations. A comparative analysis with the IEFG method is conducted to validate the superior computational efficiency of the proposed methods.

In this section, the 3D Helmholtz equation is given and the dimension splitting method (DSM) is introduced to decompose the 3D equation into multiple 2D forms. These 2D equations are discretized using the IEFG method, while the third direction is further discretized using the FDM [152] and FEM [137], respectively, yielding computational formulations for both approaches. A numerical example is then presented to validate the effectiveness and comparative advantages of each method.

3.4.1 The Weak Form of Helmholtz Equation Based on the DSM

The governing equation is

$$\Delta \psi(\mathbf{x}) + \kappa^2 \psi(\mathbf{x}) = f(\mathbf{x}), \mathbf{x} = (x_1, x_2, x_3) \in \Omega \quad (147)$$

The boundary conditions are

$$\psi(\mathbf{x}) = \bar{\psi}(\mathbf{x}), \mathbf{x} \in \Gamma_u, \quad (148)$$

$$q(\mathbf{x}) = u_{,1}n_1 + u_{,2}n_2 + u_{,3}n_3 = \bar{q}(\mathbf{x}), \mathbf{x} \in \Gamma_q; \quad (149)$$

$f(\mathbf{x})$ is the given function, n_i is the component of the unit normal to the boundary along i , $\Gamma = \Gamma_u \cup \Gamma_q$, $\Gamma_u \cap \Gamma_q = \emptyset$. The constant κ is wave number.

Without loss of generality, we split the 3D problem into $L + 1$ layers along the x_3 direction (See Fig. 19).

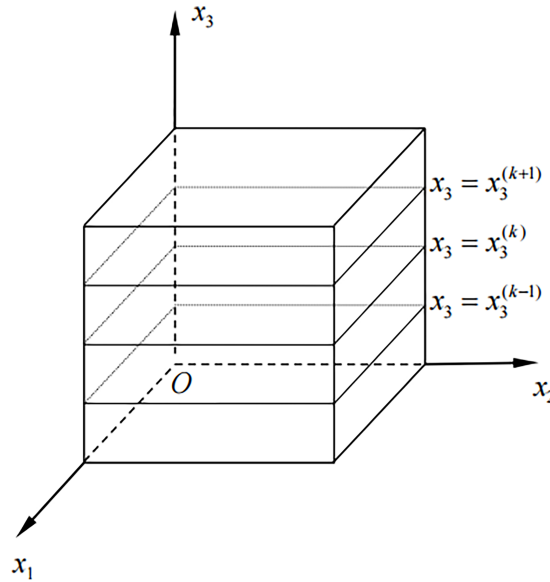


Figure 19: The problem domain is partitioned into $L + 1$ layers in x_3 -direction

The weak form obtained based on the DSM is

$$\begin{aligned} & \int_{\Omega^{(k)}} \delta u \frac{\partial^2 \psi}{\partial x_3^2} d\Omega - \int_{\Omega^{(k)}} \delta \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \right) \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \right)^T d\Omega - \int_{\Omega^{(k)}} \delta \psi f d\Omega + \int_{\Omega^{(k)}} \delta \psi \kappa^2 \psi d\Omega \\ & + \alpha \int_{\Gamma_u^{(k)}} \delta \psi \psi d\Gamma - \alpha \int_{\Gamma_u^{(k)}} \delta \psi \bar{\psi} d\Gamma - \int_{\Gamma_q^{(k)}} \delta \psi \bar{q} d\Gamma = 0 \end{aligned} \quad (150)$$

where α is penalty factor.

3.4.2 The IIEFG Method for 2D Equations in Each Layer

By introducing 2D IMLS shape function, Eq. (150) is transformed as

$$\begin{aligned} & \int_{\Omega^{(k)}} \delta [\Phi^*(x^{(k)}) \psi]^T [\Phi^*(x^{(k)}) \psi''] d\Omega + \int_{\Omega^{(k)}} \delta [\Phi^*(x^{(k)}) \psi]^T \kappa^2 [\Phi^*(x^{(k)}) \psi] d\Omega \\ & - \int_{\Omega^{(k)}} \delta [B(x^{(k)}) \psi]^T [B(x^{(k)}) \psi] d\Omega - \int_{\Omega^{(k)}} \delta [\Phi^*(x^{(k)}) \psi]^T f d\Omega - \int_{\Gamma_q^{(k)}} \delta [\tilde{\Phi}(x^{(k)}) \psi]^T \bar{q} d\Gamma \end{aligned}$$

$$+ \alpha \int_{\Gamma_u^{(k)}} \delta[\Phi^*(x^{(k)})\psi][\Phi^*(x^{(k)})\psi]d\Gamma - \alpha \int_{\Gamma_u^{(k)}} \delta[\Phi^*(x^{(k)})\psi]\bar{\psi}d\Gamma = 0 \quad (151)$$

where

$$\psi'' = \left(\frac{\partial^2 \psi(x_1^{(k)})}{\partial x_3^2}, \frac{\partial^2 \psi(x_2^{(k)})}{\partial x_3^2}, \dots, \frac{\partial^2 \psi(x_n^{(k)})}{\partial x_3^2} \right)^T, \quad (152)$$

$$\mathbf{B}(x^{(k)}) = \begin{bmatrix} \Phi_{1,1}^*(x^{(k)}) & \Phi_{2,1}^*(x^{(k)}) & \dots & \Phi_{n,1}^*(x^{(k)}) \\ \Phi_{1,2}^*(x^{(k)}) & \Phi_{2,2}^*(x^{(k)}) & \dots & \Phi_{n,2}^*(x^{(k)}) \end{bmatrix}, \quad (153)$$

thus the discretized equations are derived as

$$\mathbf{C}\psi'' + \mathbf{K}\psi = \mathbf{F}, \quad (154)$$

where

$$\mathbf{C} = \int_{\Omega^{(k)}} \tilde{\Phi}^T \tilde{\Phi} d\Omega, \quad (155)$$

$$\mathbf{K} = \kappa^2 \int_{\Omega^{(k)}} \Phi^{*T} \Phi^* d\Omega - \int_{\Omega^{(k)}} \mathbf{B}^T \mathbf{B} d\Omega + \alpha \int_{\Gamma_u^{(k)}} \Phi^{*T} \Phi^* d\Gamma, \quad (156)$$

$$\mathbf{F} = \int_{\Omega^{(k)}} \Phi^{*T} f d\Omega + \int_{\Gamma_q^{(k)}} \Phi^{*T} \bar{q} d\Gamma + \alpha \int_{\Gamma_u^{(k)}} \Phi^{*T} \bar{\psi} d\Gamma. \quad (157)$$

3.4.3 The FDM for 1D Equation in Splitting Direction

The FDM is introduced to discretize ψ'' as

$$\psi''^{(k)} \approx \frac{\psi^{(k)} - 2\psi^{(k+1)} + \psi^{(k+2)}}{(\Delta x_3)^2}, k = 1, 2, \dots, L-1. \quad (158)$$

If $\psi^{(1)}$ and $\psi^{(L+1)}$ are the given boundary value in dimension splitting direction, and we can derive that the final equation is

$$\tilde{\mathbf{K}}\tilde{\psi} = \tilde{\mathbf{F}}, \quad (159)$$

where

$$\tilde{\mathbf{K}} = \frac{1}{(\Delta x_3)^2} \begin{bmatrix} \mathbf{H} & \mathbf{C} & & & \\ \mathbf{C} & \mathbf{H} & \mathbf{C} & & \\ & \mathbf{C} & \mathbf{H} & \mathbf{C} & \\ & & \ddots & \ddots & \ddots \\ & & & \mathbf{C} & \mathbf{H} & \mathbf{C} \\ & & & & \mathbf{C} & \mathbf{H} \end{bmatrix}, \quad (160)$$

$$\mathbf{H} = (\Delta x_3)^2 \mathbf{K} - 2\mathbf{C}, \quad (161)$$

$$\tilde{\mathbf{F}} = \left(\left(\mathbf{F}^{(2)} - \frac{\mathbf{C}\psi^{(1)}}{(\Delta x_3)^2} \right)^T, \mathbf{F}^{(3)T}, \dots, \mathbf{F}^{(L-1)T}, \left(\mathbf{F}^{(L)} - \frac{\mathbf{C}\psi^{(L+1)}}{(\Delta x_3)^2} \right)^T \right)^T, \quad (162)$$

$$\tilde{\psi} = (\psi^{(2)\text{T}}, \psi^{(3)\text{T}}, \dots, \psi^{(L)\text{T}})^{\text{T}}. \quad (163)$$

If $\psi'(x_3^{(1)})$ and $\psi'(x_3^{(L+1)})$ are given, and

$$\psi'(x_3^{(1)}) = \psi'(a) = \frac{\psi(x_3^{(2)}) - \psi(x_3^{(1)})}{\Delta x_3}, \quad (164)$$

$$\psi'(x_3^{(L+1)}) = \psi'(b) = \frac{\psi(x_3^{(L+1)}) - \psi(x_3^{(L)})}{\Delta x_3}. \quad (165)$$

Thus, final equations are changed as

$$\hat{K}\tilde{\psi} = \hat{F}, \quad (166)$$

where

$$\hat{K} = \frac{1}{(\Delta x_3)^2} \begin{bmatrix} H_2 & C & & & & \\ C & H & C & & & \\ & C & H & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & C & H & C \\ & & & & C & H_L \end{bmatrix}, \quad (167)$$

$$H_2 = -C + (\Delta x_3)^2 K, \quad (168)$$

$$H_L = -C + (\Delta x_3)^2 K, \quad (169)$$

$$\hat{F} = \left(\left(F^{(2)} + \frac{C\psi'(a)}{\Delta x_3} \right)^{\text{T}}, F^{(3)\text{T}}, F^{(4)\text{T}}, \dots, F^{(L-1)\text{T}}, \left(F^{(L)} - \frac{C\psi'(b)}{\Delta x_3} \right)^{\text{T}} \right)^{\text{T}}. \quad (170)$$

3.4.4 The FEM for 1D Equation in Splitting Direction

When the FEM is introduced to discretize ψ'' in Eq. (154), essential boundary conditions are given, thus the final solved equation is

$$HU = W, \quad (171)$$

where

$$\hat{H} = \begin{bmatrix} E & & & & & \\ H_{21} & H_{22} & H_{23} & & & \\ & H_{32} & H_{33} & H_{34} & & \\ & & \ddots & \ddots & \ddots & \\ & & & H_{L,L-1} & H_{L,L} & H_{L,L+1} \\ & & & & E \end{bmatrix}, \quad (172)$$

E is identity matrix, and

$$\mathbf{H}_{ik} = \left(\sum_{k=1}^{L+1} \int_a^b \varphi_k \varphi_i dx_3 \right) \hat{\mathbf{K}} - \left(\sum_{k=1}^{L+1} \int_a^b \varphi'_k \varphi'_i dx_3 \right) \mathbf{C}, i = 2, 3, \dots, L, \quad (173)$$

φ refer to the test function of the FEM, and

$$\mathbf{U} = (\boldsymbol{\psi}^{(1)\text{T}}, \boldsymbol{\psi}^{(2)\text{T}}, \boldsymbol{\psi}^{(3)\text{T}}, \dots, \boldsymbol{\psi}^{(L)\text{T}}, \boldsymbol{\psi}^{(L+1)\text{T}})^{\text{T}}, \quad (174)$$

$$\mathbf{W} = ((\mathbf{u}(a))^{\text{T}}, \mathbf{W}_2^{\text{T}}, \mathbf{W}_3^{\text{T}}, \dots, \mathbf{W}_{(L-1)}^{\text{T}}, (\mathbf{u}(b))^{\text{T}})^{\text{T}}, \quad (175)$$

$$\mathbf{W}_i = \int_a^b \hat{\mathbf{F}}^{(k)} \varphi_i dx_3, i = 1, 2, \dots, N + 1. \quad (176)$$

If the natural boundary conditions are known, thus the $\hat{\mathbf{H}}$ and $\mathbf{W}$ can be changed as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} & & & & \\ \mathbf{H}_{21} & \mathbf{H}_{22} & \mathbf{H}_{23} & & & \\ & \mathbf{H}_{32} & \mathbf{H}_{33} & \mathbf{H}_{34} & & \\ & & & \ddots & & \\ & & & & \mathbf{H}_{L,L-1} & \mathbf{H}_{L,L} & \mathbf{H}_{L,L+1} \\ & & & & & \mathbf{H}_{L+1,L} & \mathbf{H}_{L+1,L+1} \end{bmatrix}, \quad (177)$$

and

$$\mathbf{W} = ((\mathbf{W}_1 + \mathbf{C}\boldsymbol{\psi}'(a))^{\text{T}}, \mathbf{W}_2^{\text{T}}, \mathbf{W}_3^{\text{T}}, \dots, \mathbf{W}_{(L-1)}^{\text{T}}, (\mathbf{W}_{L+1} - \mathbf{C}\boldsymbol{\psi}'(b))^{\text{T}})^{\text{T}}. \quad (178)$$

3.4.5 An Example of Helmholtz Equation with Mix Boundary Conditions

The exact solution is

$$\psi(\mathbf{x}) = \cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \mathbf{x} \in [0, 1] \times [0, 1] \times [0, 1], \quad (179)$$

and

$$f(\mathbf{x}) = (k^2 - 3\pi^2) \cos(\pi x_1) \sin(\pi x_2) \sin(\pi x_3), \quad (180)$$

κ is selected to 10, and the boundary conditions are

$$\psi_{,1}(0, x_2, x_3) = \psi_{,1}(1, x_2, x_3) = 0, \quad (181)$$

$$\psi(x_1, 0, x_3) = \psi(x_1, 1, x_3) = \psi(x_1, x_2, 0) = \psi(x_1, x_2, 1) = 0. \quad (182)$$

Table 2 shows the relative error and CPU time trends of the DSIEFG method with respect to mesh refinement in x_1 direction. The results demonstrate an inverse relationship between error and computational cost: as the mesh density increases in the splitting direction, the relative error decreases from 1.3667% to 0.7729%, while the computation time increases from 73.8 to 263.9 s. Therefore, when selecting the natural boundary condition as the splitting direction, considering both computational accuracy and time, we chose a mesh number of 360, and a relatively small error of 0.8830% was achieved, with a corresponding CPU time of 160.3 s.

Table 2: The computational accuracy and speed of DSIEFG method with the growth of grid numbers along x_1 direction

Mesheres in x_1 direction	Nodes in 2D plane	$d_{\max}$	α	Error	CPU time
180	19×19	1.24	2.0×10^5	1.3667%	73.8 s
240	19×19	1.24	2.0×10^5	1.1212%	91.2 s
300	19×19	1.24	2.0×10^5	0.9765%	126.4 s
360	19×19	1.24	2.0×10^5	0.8830%	160.3 s
420	19×19	1.24	2.0×10^5	0.8189%	209.3 s
480	19×19	1.24	2.0×10^5	0.7729%	263.9 s

A comparison was conducted on the computational accuracy and time of the IIEFG method, the DSIEFG method, and the DCM when solving this problem under different splitting directions, the parameter selections are shown in Table 3. The results indicate that when the x_1 -axis is chosen as the splitting direction, the DCM can directly and efficiently handle the natural boundary conditions along the dimensional splitting direction without requiring many layers, thereby significantly improving the computational efficiency compared to the DSIEFG method.

Table 3: Comparison of computational accuracy and time of three numerical methods

Splitting direction	Method	Nodes	$d_{\max}$	α	Error	CPU time
x_1	DCM	$19 \times 19 \times 19$	1.31	1.3×10^5	0.8481%	5.7 s
x_1	DSIEFG	$361 \times 19 \times 19$	1.24	2.0×10^5	0.8830%	160.3 s
x_2/x_3	DSIEFG	$19 \times 19 \times 19$	1.24	2.0×10^5	0.5193%	5.02 s
x_2/x_3	DCM	$19 \times 19 \times 19$	1.31	1.2×10^5	0.8386%	5.28 s
None	IEFG	$19 \times 19 \times 19$	1.10	1.9×10^7	0.8646%	103.8 s

However, when the x_2 or x_3 axis is selected as the splitting direction, the parameter selections are shown in Table 3. Both the dimension splitting meshless methods significantly improve the computational speed of the IIEFG method, and the DSIEFG method demonstrates superior computational accuracy and speed compared to the DCM. Owing to its theoretical simplicity relative to the FEM, the FDM is easier to apply to relatively simple partial differential equations (PDEs) possessing essential boundary conditions.

When the x_1 -axis is selected as the splitting direction, the numerical solutions of the three methods are compared with the analytical solution. As shown in Figs. 20–22, it can be observed that the numerical solutions of these three methods agree well with the analytical solution.

Table 4 shows the relative error with respect to the different wave numbers, demonstrating the stability of three numerical methods.

Despite the efficiency of two-dimensional splitting meshless methods in solving Helmholtz equations, their performance remains limited in handling complex 3D problems with intricate geometric domains. Therefore, the corresponding methods require further improvements to enhance their applicability.

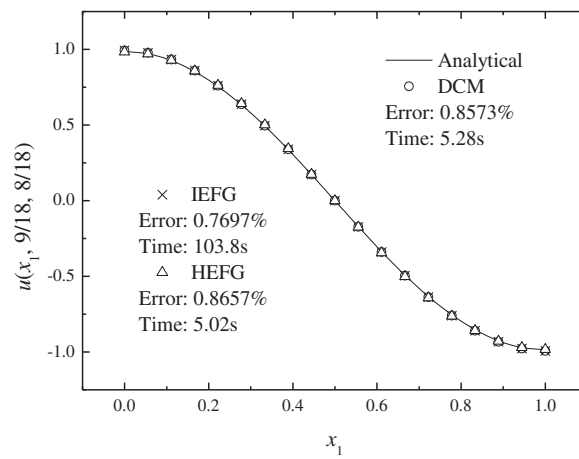


Figure 20: Numerical results of three methods in x_1 direction

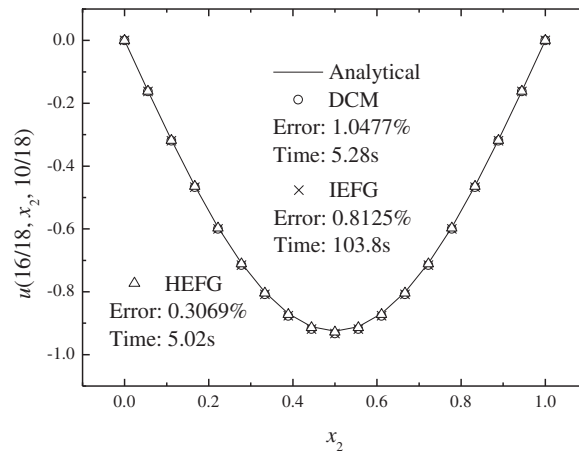


Figure 21: Numerical results of three methods in x_2 direction

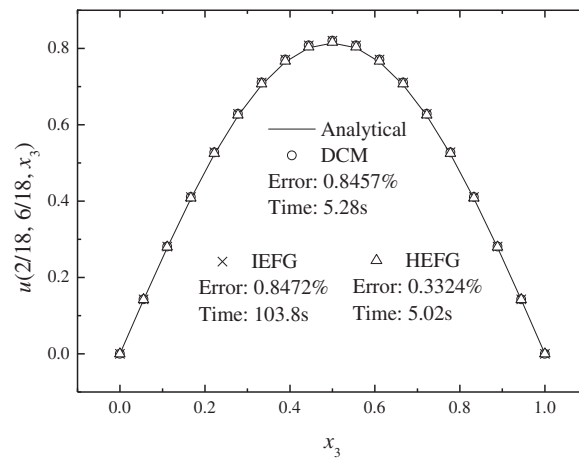


Figure 22: Numerical results of three methods in x_3 direction

Table 4: Errors of the three methods at different wave number

Wave number	Method	Nodes	$d_{\max}$	α	Error	Time (s)
20^2	DSIEFG	$19 \times 19 \times 19$	1.24	2.0×10^5	0.5128%	5.34
	DCM	$19 \times 19 \times 19$	1.31	1.2×10^5	0.7802%	5.13
	IEFG	$19 \times 19 \times 19$	1.10	1.9×10^7	0.7831%	103.11
50^2	DSIEFG	$19 \times 19 \times 19$	1.24	2.0×10^5	0.5248%	5.26
	DCM	$19 \times 19 \times 19$	1.31	1.2×10^5	0.7797%	5.06
	IEFG	$19 \times 19 \times 19$	1.10	1.9×10^7	0.7683%	99.77
80^2	DSIEFG	$19 \times 19 \times 19$	1.24	2.0×10^5	0.5202%	5.32
	DCM	$19 \times 19 \times 19$	1.31	1.2×10^5	0.7777%	5.17
	IEFG	$19 \times 19 \times 19$	1.10	1.9×10^7	0.7655%	101.29
100^2	DSIEFG	$19 \times 19 \times 19$	1.24	2.0×10^5	0.5181%	5.31
	DCM	$19 \times 19 \times 19$	1.31	1.2×10^5	0.7751%	5.04
	IEFG	$19 \times 19 \times 19$	1.10	1.9×10^7	0.7651%	97.33

4 Conclusions and Further Directions

This paper provides a comprehensive review of the EFG method and its various improved versions. As these improvements are primarily centered on the shape functions, we systematically summarize the derivations of their key formulations. To demonstrate the superiority of the improved methods, the IEFG method is applied to solve both 3D heat conduction, elastoplasticity and large deformation problems. Numerical examples verify its effectiveness and computational efficiency over the EFG method. Furthermore, the DSIEFG method and the DCM are introduced for solving 3D Helmholtz equations. Numerical experiment confirms that these methods significantly enhance computational performance compared to the IEFG in handling 3D problems, marking a substantial advancement in meshless computational mechanics.

While significant progress has been made in the improved EFG methods, several challenges remain in the research on the improved EFG methods, particularly in the following key areas requiring future investigation:

- 1) The dimension splitting meshless methods have primarily been applied to 3D partial differential equations and relatively simple 3D elasticity problems. However, some challenges persist when addressing anisotropic heat conduction problems with second-order mixed partial derivatives and more complex 3D mechanics analyses. Consequently, the theoretical framework of dimension splitting meshless methods requires further refinement to adequately address large-scale, complex 3D mechanical problems.
- 2) Although the ICVEFG method has been widely applied in 2D computational mechanics, existing applications in 3D analyses have primarily focused on solving partial differential equations and elastoplastic problems. Its extension to 3D complex mechanics applications remains an important and largely unexplored research area.
- 3) While the 2D interpolating ICVEFG method has been well established in the literature, its extension to 3D problems has not yet been investigated. The development of such 3D formulations presents a critical research frontier with significant potential applications.
- 4) The integration of the improved EFG methods with neural networks for solving inverse problems represents a promising and potentially transformative direction for future research in computational mechanics.

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Ethics Approval: Not applicable.

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