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ARTICLE

On Progressive-Stress ALT under Generalized Progressive Hybrid Censoring Scheme for Quasi Xgamma Distribution

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ABSTRACT: Accelerated life tests play a vital role in reliability analysis, especially as advanced technologies lead to the production of highly reliable products to meet market demands and competition. Among these tests, progressive-stress accelerated life tests (PSALT) allow for continuous changes in applied stress. Additionally, the generalized progressive hybrid censoring (GPHC) scheme has attracted significant attention in reliability and survival analysis, particularly for handling censored data in accelerated testing. It has been applied to various failure models, including competing risks and step-stress models. However, despite its growing relevance, a notable gap remains in the literature regarding the application of GPHC in PSALT models. This paper addresses that gap by studying PSALT under a GPHC scheme with binomial removal. Specifically, it considers lifetimes following the quasi-Xgamma distribution. Model parameters are estimated using both maximum likelihood and Bayesian methods under gamma priors. Interval estimation is provided through approximate confidence intervals, bootstrap methods, and Bayesian credible intervals. Bayesian estimators are derived under squared error and entropy loss functions, using informative priors in simulation and non-informative priors in real data applications. A simulation study is conducted to evaluate various censoring schemes, with coverage probabilities and interval widths assessed via Monte Carlo simulations. Additionally, Bayesian predictive estimates and intervals are presented. The proposed methodology is illustrated through the analysis of two real-world accelerated life test datasets.

KEYWORDS: Progressive-stress; progressive hybrid censoring; maximum likelihood estimation; Bayes estimation; simulation study

1 Introduction

Advancements in science and technology have led to the development of highly durable and complex products, such as silicone seals, computers, missiles, and LEDs. Reliability is essential for maintaining product quality, prompting manufacturers to invest heavily in testing and design. However, for highly reliable products, their extended lifespans often result in few or no failures during standard testing under normal conditions. Traditional life-testing methods struggle to handle such scenarios.

In traditional life testing and reliability experiments, obtaining sufficient failure data can be challenging, especially for highly reliable products with long lifespans. Under normal conditions, tests conducted within limited timeframes often result in very few failures. To address this, accelerated life testing (ALT) is commonly used. ALT involves subjecting products to elevated stress levels, such as increased humidity, temperature, pressure, voltage, or vibration, to induce failures more quickly. The resulting failure data are then analyzed to estimate the product's life characteristics under normal operating conditions.

ALT methods are broadly categorized into three main types based on how the stress is applied over time: constant-stress, step-stress, and progressive-stress models [1]. These categories reflect different strategies to accelerate failure in highly reliable products in order to estimate their lifetime distribution within practical timeframes and costs.

- **Constant-stress ALT:** Applies a fixed, elevated stress level throughout the entire test period.
- **Step-stress ALT:** Increases the stress in discrete steps at predetermined time intervals or after specific events.
- **Progressive-stress ALT:** Increases the stress continuously (e.g., linearly) over time, which better mimics gradual degradation processes.

Fig. 1 illustrates the conceptual differences between these ALT strategies.

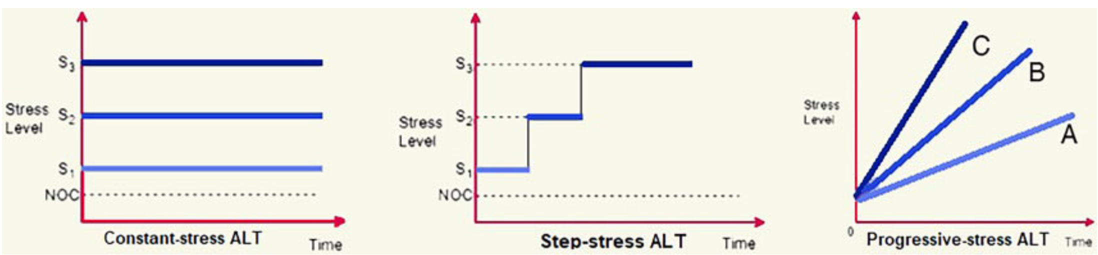

Figure 1: ALT types

Constant-stress ALT applies a fixed high stress until failure or a preset time, and has been widely used in reliability analysis. Several studies have proposed models and estimation techniques under this scheme, including works by Kumar et al. [2], Hakamipour [3], Balakrishnan et al. [4], Abd El-Raheem et al. [5], Sief et al. [6], El-Sherpieny et al. [7], and Abd El-Raheem et al. [8]. Step-stress ALT gradually increases stress at set intervals or after specific failures and is well established in reliability studies. Key contributions include works by Gouno et al. [9], Balakrishnan et al. [10], Xu et al. [11], Mohie El-Din [12], Xu et al. [13], Alotaibi et al. [14,15], and Hassan and Abdelghaffar [16], who developed various inference methods under different censoring and lifetime models.

Progressive-stress ALT elevates stress over time, unlike constant- and step-stress methods. The ramp-stress test, which increases tension linearly, is a common example. This method closely resembles real-world deterioration processes and is popular in dependability studies. For instance, Yin and Sheng [17] examined maximum likelihood estimation for exponential models under progressive stress. Bai and Cha [18] studied optimal test design and failure rate modeling under increasing load. Wang and Fei [19] studied statistical inference for Weibull distributions in modified failure rate models under increasing stress. Moreover, AL-Hussaini et al. [20] introduced Bayesian prediction ranges for the half-logistic distribution using Type-II censored data, demonstrating its versatility and effectiveness.

Censored data are commonly used in studies on reliability and life testing. Due to factors like preserving working experimental units for future use, reducing the overall time for the test, and financial limits, investigators have to gather data using censored samples. Time-censoring (Type-I) and failure-censoring (Type-II) strategies are the two widely used censoring strategies in life-testing and reliability studies (see, for additional details, the work by Bain and Engelhardt [21]). These methods are not flexible enough to allow units to be removed from the experiment at any point other than the terminal point.

The progressive Type-II censoring scheme (PTII) is a widely utilized method in reliability and survival analysis, often preferred over the standard Type-II censoring scheme due to its adaptability in practical applications such as industrial and medical studies. Unlike the conventional approach, PTII permits the removal of still-operational units during the experiment, rather than at its conclusion [22]. In PTII, suppose m out of n units are intended to fail during a life test. A predetermined sequence $(R_1, R_2, \ldots, R_m)$ dictates the number of units removed after each failure. When the first failure occurs at time $X_{1:m:n}$, R_1 remaining units are arbitrarily removed. Similarly, after the second failure at $X_{2:m:n}$, R_2 additional units are removed. This process continues until the m-th failure at $X_{m:m:n}$, where all R_m remaining units are discarded, concluding the test. While this method has been extensively studied, it has a notable drawback: if the units are highly reliable, the experiment duration may become excessively long.

To address this limitation, Kundu and Joarder [23] introduced the progressive hybrid censoring scheme (PHCS), where the test ends at $\min(T, X_{m:m:n})$, with T being a predetermined maximum time. However, PHCS struggles when only a few failures occur before T, leading to limited or imprecise parameter estimation. To mitigate these challenges, Cho et al. [24] proposed the generalized progressive hybrid censoring scheme (GPHCS). This scheme ensures a minimum number of failures are observed, improving parameter estimation accuracy. Under GPHCS, both the number of failures and their corresponding lifetimes are predetermined, balancing test duration and cost. The experiment concludes at $T^* = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$, where k, m, and $(R_1, \ldots, R_m)$ are predefined, see [23]. The observations fall into one of three categories:

- **Case I:** $X_{1:m:n}, X_{2:m:n}, \ldots, X_{k:m:n}$, if $T < X_{k:m:n} < X_{m:m:n}$,
- **Case II:** $X_{1:m:n}, \ldots, X_{k:m:n}, \ldots, X_{D:m:n}$, if $X_{k:m:n} < T < X_{m:m:n}$,
- **Case III:** $X_{1:m:n}, \ldots, X_{k:m:n}, \ldots, X_{m:m:n}$, if $X_{k:m:n} < X_{m:m:n} < T$,

a schematic representation of the generalized progressive hybrid censoring scheme is presented in Fig. 2.

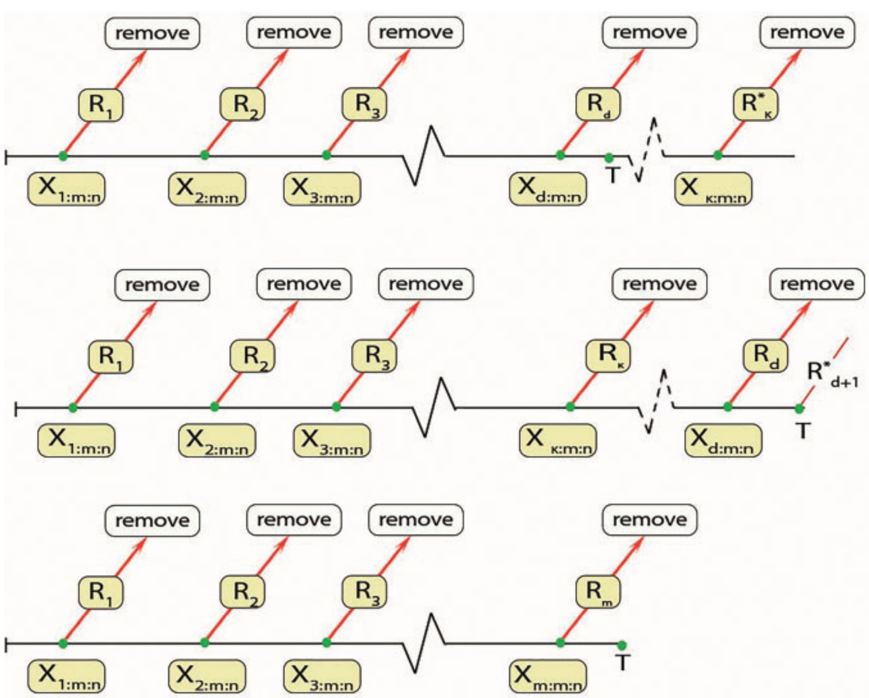

Figure 2: Diagram of GPHCS strategy

More paper used GPHCS for model a new dialog of censored sample as: Koley and Kundu [25] introduced GPHCS in the presence of competing risks models. Maswadah [26] improved maximum likelihood estimation of the shape-scale family based on GPHCS. Salem et al. [27] introduced joint Type-II of GPHCS. Abdelwahab et al. [28] discussed classical and Bayesian inference for lifetime distribution based on GPHCS. Mohie El-Din et al. [29] obtained predictions of lifetimes under GPHCS. Mahto et al. [30] developed a partially observed competing risk model under GPHSC. Zhu obtained reliability inference for the multi-component stress–strength model under GPHCS. Wang et al. [31] discussed the dependence of competing risks with partially observed failure causes from bivariate distribution under GPHCS. Çetinkaya [32] discussed inference of P (X > Y) under GPHCS. Shi et al. [33] introduced reliability analysis of the J-out-of-N system under GPHCS.

Research on progressive-stress ALT continues to expand, incorporating diverse statistical models and censoring schemes. Rong-hua and Heliang [19] applied it to Weibull distributions using tampered failure rate models. Abdel-Hamid and Al-Hussaini [34] combined progressive stress with finite mixture distributions, Bayesian analysis, and progressive censoring for various lifetime distributions. Abdel-Hamid and Al-Hussaini [35] also studied inference for Weibull distributions under PTII censoring. Recent works utilizing progressive-stress ALT with censoring schemes include studies by Mohie et al. [36], Zhuang et al. [37], Mahto et al. [38], Abushal and Abdel-Hamid [39], Ismail [40], Hussam et al. [41], and Alotaibi et al. [42].

The GPHC scheme has proven to be a valuable tool for analyzing censored data in ALTs, particularly in models like competing risks and step-stress models. However, a significant gap exists in the application of GPHC to Progressive Stress ALT models, which involve progressively increasing stress levels during the test. Most of the existing studies, such as those by Wang et al. [43] and Pandey et al. [44], focus on constant or step-stress models, without addressing the complexities introduced by progressive stress scenarios.

This paper extends the progressive-stress ALT framework by incorporating the QXG distribution—a flexible lifetime model suitable for diverse failure behaviors—under the GPHCS with binomial removal. To the best of our knowledge, this integration has not been addressed in the literature. The study contributes by:

- Developing classical and Bayesian estimation procedures for the QXG parameters under PSALT with GPHCS.
- Providing predictive survival probabilities under different censoring schemes.
- Conducting a thorough simulation to evaluate the performance of the estimators.
- Addressing a critical gap in the literature and paves the way for more realistic modeling of accelerated life tests.
- Demonstrating the methodology with real-world datasets.

This combination of modeling flexibility, advanced censoring, and estimation robustness aims to provide a more realistic and practically applicable framework for analyzing accelerated life test data.

The remainder of this paper is organized as follows: Section 2 presents the fundamental assumptions and details of the Quasi Xgamma distribution under the progressive-stress ALT model. Sections 3 and 4 describe the estimation methods, including both maximum likelihood and Bayesian approaches, respectively. Section 5 provides a comprehensive simulation study to assess the performance of the proposed estimators. Section 6 demonstrates the methodology using real-world accelerated life test datasets. Section 7 concludes the paper with final remarks and potential directions for future research.

2 Test Assumptions

2.1 Quasi Xgamma Distribution

The quasi xgamma distribution, introduced as a generalization of the xgamma distribution, adds flexibility in modeling lifetime data. Developed by Sen and Chandra [45], this distribution includes an additional parameter, α, which, when combined with the primary scale parameter θ, allows for more accurate modeling of real-world data, especially where classical models fall short. The probability density function (PDF) of the quasi xgamma (QXG) distribution is given by:

$$f(x;\alpha,\theta) = \frac{\theta}{(1+\alpha)}\left(\alpha + \frac{(\theta x)^2}{2}\right)e^{-\theta x}; \quad \alpha > 0, \theta > 0, x \geq 0. \tag{1}$$

The cumulative distribution function (CDF) of the QXG distribution is given by:

$$F(x;\alpha,\theta) = 1 - \frac{\left(1+\alpha+\theta x+\frac{(\theta x)^2}{2}\right)}{(1+\alpha)}e^{-\theta x}; \quad x \geq 0. \tag{2}$$

The Quasi Xgamma (QXG) distribution was chosen for its flexibility in modeling lifetime data, especially under progressive-stress ALT. It includes a shape parameter α that adjusts the distribution's form, allowing it to generalize gamma and xgamma distributions as special cases [45]. Structurally, QXG is a mixture of exponential and gamma distributions, enhancing its ability to capture varied failure behaviors. Prior applications in survival analysis, such as bladder cancer data, have demonstrated its superior fit and predictive accuracy compared to classical models [45]. These features make QXG particularly suitable for reliability studies involving complex censoring and stress structures.

This model is quite versatile in nature and resembles probabilistic behavior in the gamma distribution and exponential distribution; see Sen and Chandra [45]. Recently, papers used QXG distribution as a baseline to obtain a new good distribution as: Sen et al. [46] discussed the QXG-geometric distribution with applications in medicine. Sen et al. [47] introduced QXG-Poisson distribution. Ahsan-ul-Haq et al. [48] discussed analysis, estimation, and practical implementations of the discrete power QXG distribution. Hassan et al. [49] obtained a new generalized QXG distribution applicable to survival times. Wani and Shafi [50] introduced the generalized Lindley-QXG distribution.

Assumption: Progressive-stress ALT

1. Lifetimes of units follow $QXG(\alpha,\theta)$.
2. We have stress function $S(t) = vt, v > 0$.
3. The $S(t)$ depends upon t. Also, λ is a function of time t.
4. We also have $\lambda(t) = \frac{1}{\theta_1[S(t)]^{\theta_2}}$, where $\theta_1, \theta_2 > 0$ and are to be estimated.
5. Linear cumulative exposure model (LCEM) is considered for modeling the effect of stress change, for more details, see Nelson [1].

Under consideration of LCEM, the CDF under progressive-stress $S_i(t)$ is expressed as

$$G_i(t) = F_i(\Delta t), \quad i = 1, 2, \ldots, s,$$

where

$$\Delta t = \int_0^t \frac{dw}{\lambda_i(w)} = \frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2+1}.$$

Then by setting the value of the scale parameter $\theta = 1$, we have the CDF for the i-th progressive stress given as:

$$G_i(t;\Omega) = 1 - \frac{\left(1 + \alpha + \frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2 + 1} + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)} e^{-\frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2+1}}; \quad t > 0,\ \alpha, \theta_1, \theta_2 > 0, \tag{3}$$

where Ω is vector of parameter as $(\alpha, \theta_1, \theta_2)$, and the corresponding PDF is given by:

$$g_i(t;\Omega) = \frac{\theta_1 v_i^{\theta_2} t^{\theta_2}}{1+\alpha}\left(\alpha + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2+1}\right)^2\right) e^{-\frac{\theta_1 v_i^{\theta_2} t^{\theta_2+1}}{\theta_2+1}}. \tag{4}$$

Algorithm 1 has been used To generate sample from generalized progressive hybrid censoring scheme.

Algorithm 1: Generalized progressive hybrid censoring scheme (GPHCS)

1. **Input:**

- Select distribution; In this study, we select QXG distribution.
- Number of units: n
- Pre-determined time: $T > 0$
- Number of failures: k and m $(k < m \leq n)$
- Censoring plan: $(R_1, R_2, \ldots, R_m)$ such that $m + \sum_{i=1}^{m} R_i = n$

2. **Procedure:**

(a) Place n units under life testing.

(b) Generate lifetimes, which follow QXG distribution with CDF $F(x;\sigma)$ based on PTII [51].

 i. Set the actual values of σ.

 ii. For specific values of n, m, and R_i $(i = 1, 2, \ldots, m)$, simulate an ordinary PTII sample as follows:

  A. Generate $v_1, v_2, \ldots, v_m$, independent observations of size m, from the uniform $U(0,1)$ distribution.

  B. Set

$$\tau_i = v_i^{\left(i + \sum_{l=m-i+1}^{m} R_l\right)^{-1}}, \quad i = 1, 2, \ldots, m.$$

  C. Compute

$$U_i = 1 - \tau_m \tau_{m-1} \cdots \tau_{m-i+1}, \quad i = 1, 2, \ldots, m.$$

  D. Invert of QXG distribution to obtain

$$X_{i:m:n} = F^{-1}(U_i;\sigma), \quad i = 1, 2, \ldots, m.$$

  E. Obtain $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, which represent the PTII order statistic from the QXG distribution.

(c) Initialize failure time $T^* = \max\{X_{k:m:n}, \min\{X_{m:m:n}, T\}\}$.

(Continued)

Algorithm 1 (continued)

$$T^* = \begin{cases} T & \text{if } X_{k:m:n} < T < X_{m:m:n}, \\ X_{k:m:n} & \text{if } T < X_{k:m:n} < X_{m:m:n}, \\ X_{m:m:n} & \text{if } X_{k:m:n} < X_{m:m:n} < T. \end{cases}$$

(d) The number of failures due to case j for j = I, II. as follows:

$$D = \begin{cases} k,\ \delta = 0 & \text{if } T \le X_{k:m:n} < X_{m:m:n}, \\ d,\ \delta = 1 & \text{if } X_{k:m:n} < T \le X_{m:m:n}, \\ m,\ \delta = 0 & \text{if } X_{k:m:n} < X_{m:m:n} < T. \end{cases}$$

(e) **Output:** Observed failure times $X_{1:m:n}, X_{2:m:n}, \ldots, X_{d:m:n}$, where:

i. If $T < X_{k:m:n} < X_{m:m:n}$, then $d = k$.

ii. If $X_{k:m:n} < T < X_{m:m:n}$, then d is the number of failures before T.

iii. If $X_{k:m:n} < X_{m:m:n} < T$, then $d = m$.

3 Estimation Methods

In this section, we utilize the maximum likelihood estimation method based on PSALT to estimate the model parameters α, θ_1, and θ_2. A concise explanation of GPHCS under PSALT is provided. Consider s stress levels, where n_i items are subjected to testing at each level, with $i = 1, 2, \ldots, s$. Additionally, the observed items, denoted as m_i, k_i, and T_i, for $i = 1, 2, \ldots, s$, follow predefined censoring schemes $\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{im_i}$ at failure times $t_{i1:m_i:n_i}, t_{i2:m_i:n_i}, \ldots, t_{im_i:m_i:n_i}$, for $i = 1, 2, \ldots, s$.

During the first failure at the ith stress level, $\mathcal{R}_{i1}$ surviving units are removed from the experiment. Similarly, at the second failure in the ith stress level, $\mathcal{R}_{i2}$ surviving units are withdrawn from the remaining $(n_i - 2 - \mathcal{R}_{i1})$ units. Finally, at the m_ith failure, all the remaining units, calculated as $\mathcal{R}_{im_i} = n_i - m_i - \sum_{j=1}^{m_i-1} \mathcal{R}_{ij}$, are withdrawn, bringing the experiment to an end. Refer to Fig. 2 for illustration.

Let the observed GPHCS data under the progressive-stress level $S_i(t)$, for $i = 1, 2, \ldots, s$ be represented by t_{ij}. Here, $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m_i$. The corresponding likelihood function with vector parameter Ω is then expressed as follows:

$$L(\Omega) = \prod_{i=1}^{s} \left[C_i \prod_{j=1}^{D_i} g_i(t_{ij}; \Omega) \left(1 - G_i(t_{ij}; \Omega)\right)^{R_{ij}} (1 - G_i(T_i; \Omega))^{\delta R_T^*} \right], \tag{5}$$

where C_i is constant and does not depend on parameters Ω, the R_T^* can be define as $n_i - m_i - \sum_{j=1}^{D_i} R_{ij}$.

Using Eqs. (3) and (4) in (5), the likelihood function of the QXG distribution under GPHCS within the framework of the PSALT model is derived as:

$$L(\Omega) = \prod_{i=1}^{s} \left[C_i \prod_{j=1}^{D_i} \frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2}}{(1+\alpha)^{R_{ij}+1}} \left(\alpha + \frac{1}{2} \left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2 + 1} \right)^2 \right) \left(1 + \alpha + \frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2 + 1} + \frac{1}{2} \left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2 + 1} \right)^2 \right)^{R_{ij}} \times \right.$$

$$e^{-R_{ij}\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}}\left(\frac{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)}e^{-\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}}\right)^{\delta R_T^*}\Bigg], \tag{6}$$

hence, the log-likelihood function, excluding the constant term C_i, is expressed as:

$$\begin{aligned}\ell(\Omega) ={}& \sum_{i=1}^{s}\sum_{j=1}^{D_i}\left(\ln\theta_1+\theta_2\ln v_i-(R_{ij}+1)\ln(\alpha+1)\right)+\theta_2\sum_{i=1}^{s}\sum_{j=1}^{D_i}\ln t_{ij}+\sum_{i=1}^{s}\sum_{j=1}^{D_i}\ln\left(\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)\\ &+\sum_{i=1}^{s}\sum_{j=1}^{D_i}R_{ij}\ln\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)-\sum_{i=1}^{s}\sum_{j=1}^{D_i}R_{ij}\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\\ &+\delta R_T^*\left[\ln\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2\right)-\ln(\alpha+1)-\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right].\end{aligned} \tag{7}$$

3.1 Newton–Raphson Algorithm

The likelihood equations are obtained by taking the first partial derivatives of the log-likelihood function in (7) with respect to α, θ_1, and θ_2, expressed as:

$$\begin{aligned}\frac{\partial\ell(\Omega)}{\partial\alpha} ={}& \sum_{i=1}^{s}\sum_{j=1}^{D_i}-\frac{(R_{ij}+1)}{\alpha+1}+\sum_{i=1}^{s}\sum_{j=1}^{D_i}\frac{1}{\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}+\sum_{i=1}^{s}\sum_{j=1}^{D_i}\frac{R_{ij}}{1+\alpha+\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}\\ &+\frac{\delta R_T^*}{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}-\frac{\delta R_T^*}{\alpha+1},\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\partial\ell(\Omega)}{\partial\theta_1} ={}& \sum_{i=1}^{s}\frac{D_i}{\theta_1}+\sum_{i=1}^{s}\sum_{j=1}^{D_i}\frac{\theta_1\left(\frac{v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}{\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}+\sum_{i=1}^{s}\sum_{j=1}^{D_i}R_{ij}\frac{\frac{v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}+\theta_1\left(\frac{v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}{1+\alpha+\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}\\ &-\sum_{i=1}^{s}\sum_{j=1}^{D_i}R_{ij}\frac{v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}+\delta R_T^*\left[\frac{\frac{v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\theta_1\left(\frac{v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2}{1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2}-\frac{v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right],\end{aligned} \tag{9}$$

and

$$\frac{\partial \ell(\Omega)}{\partial \theta_2} = \sum_{i=1}^{s} D_i \ln v_i + \sum_{i=1}^{s}\sum_{j=1}^{D_i} \ln t_{ij} + \sum_{i=1}^{s}\sum_{j=1}^{D_i} \frac{\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2 A(\theta_2, t_{ij})}{\alpha + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2}$$
$$+ \sum_{i=1}^{s}\sum_{j=1}^{D_i} R_{ij} \frac{\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1} A(\theta_2, t_{ij}) + \left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2 A(\theta_2, t_{ij})}{1+\alpha+\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1} + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2} - \sum_{i=1}^{s}\sum_{j=1}^{D_i} R_{ij} \frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1} A(\theta_2, t_{ij})$$
$$+ \delta R_T^* \frac{\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1} A(\theta_2, T) + \left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2 A(\theta_2, T)}{1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1} + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2} - \delta R_T^* \frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1} A(\theta_2, T), \tag{10}$$

where $A(\theta_2, t) = \ln v_i + \ln t_{ij} - \frac{1}{\theta_2+1}$.

The maximum likelihood estimates of the unknown parameters α, θ_1, and θ_2 are determined by solving the three Eqs. (8)–(10) by setting them to zero. Since these equations are nonlinear, a numerical method is employed to compute the estimates such as the Newton–Raphson (NR) algorithm are employed. The NR algorithm begins with an initial estimate $(\alpha^{(0)}, \theta_1^{(0)}, \theta_2^{(0)})$ and iteratively improves the estimates through the following update formula:

$$\Omega^{(k)} = \Omega^{(k-1)} - \left[f'(\Omega^{(k-1)})\right]^{-1} f(\Omega^{(k-1)}).$$

Here, Ω denotes the vector of unknown parameters, $f(\Omega)$ corresponds to the vector of first-order derivatives, and $f'(\Omega)$ represents the Hessian matrix of second-order derivatives:

$$\Omega = \begin{bmatrix} \alpha \\ \theta_1 \\ \theta_2 \end{bmatrix}, \qquad f(\Omega) = \begin{bmatrix} \frac{\partial \ell(\Omega)}{\partial \alpha} \\ \frac{\partial \ell(\Omega)}{\partial \theta_1} \\ \frac{\partial \ell(\Omega)}{\partial \theta_2} \end{bmatrix}, \qquad f'(\Omega) = \begin{bmatrix} \frac{\partial^2 \ell(\Omega)}{\partial \alpha^2} & \frac{\partial^2 \ell(\Omega)}{\partial \alpha \partial \theta_1} & \frac{\partial^2 \ell(\Omega)}{\partial \alpha \partial \theta_2} \\ \frac{\partial^2 \ell(\Omega)}{\partial \theta_1 \partial \alpha} & \frac{\partial^2 \ell(\Omega)}{\partial \theta_1^2} & \frac{\partial^2 \ell(\Omega)}{\partial \theta_1 \partial \theta_2} \\ \frac{\partial^2 \ell(\Omega)}{\partial \theta_2 \partial \alpha} & \frac{\partial^2 \ell(\Omega)}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell(\Omega)}{\partial \theta_2^2} \end{bmatrix}.$$

3.2 Asymptotic Confidence Interval

In this subsection, approximate confidence intervals for the parameters are derived using the asymptotic distribution of the MLEs of $\Omega = (\alpha, \theta_1, \theta_2)$. The asymptotic distribution of the MLEs for $(\alpha, \theta_1, \theta_2)$ is expressed as:

$$((\hat{\alpha} - \alpha), (\hat{\theta}_1 - \theta_1), (\hat{\theta}_2 - \theta_2)) \sim N(0, f'(\Omega)^{-1}),$$

where $f'(\Omega)_{ij}^{-1}$ represents the variance–covariance matrix of the parameters Ω. The elements of the 3×3 matrix $f'(\Omega)^{-1}$ denoted as $f'(\Omega)_{ij}$ for $i, j = 1, 2, 3$ for $(\alpha, \theta_1, \theta_2)$, can be approximated by $f'(\hat{\Omega})_{ij}$.

Accordingly, an approximate $100(1-p)\%$ confidence interval for the parameter Ω is given by:

$$\hat{\Omega}_i \pm Z_{1-p/2}\sqrt{f'(\Omega)^{-1}_{ii}}, \quad i=1,2,3,$$

where $\hat{\Omega}_1 \equiv \hat{\alpha}$, $\hat{\Omega}_2 \equiv \hat{\theta}_1$, and $\hat{\Omega}_3 \equiv \hat{\theta}_2$.

The following section explores the incorporation of prior knowledge into the analysis to achieve improved results.

4 Bayesian Estimation

In this section, Bayes estimates for the model parameters α, θ_1, and θ_2 are derived under symmetric and asymmetric loss functions, specifically the squared error loss function and the entropy loss function, respectively. When all the model parameters are unknown, a joint conjugate prior may not exist. Therefore, piecewise independent priors are considered. It is assumed that the parameters α, θ_1, and θ_2 follow informative gamma priors with hyperparameters (p_i, q_i). Consequently, the priors for α, θ_1, and θ_2 are defined as follows:

$$\pi_1(\alpha) \propto \alpha^{p_1-1}e^{-\alpha q_1}, \quad \pi_2(\theta_1) \propto \theta_1^{p_2-1}e^{-\theta_1 q_2}, \quad \pi_3(\theta_2) \propto \theta_2^{p_3-1}e^{-\theta_2 q_3},$$

where $\alpha, \theta_1\theta_2 > 0, \quad p_i, q_i > 0; i=1,2,3.$

The joint prior density function of α, θ_1, and θ_2, under the assumption that these three parameters are independent, is expressed as:

$$\pi(\Omega) = \alpha^{p_1-1}e^{-\alpha q_1}\theta_1^{p_2-1}e^{-\theta_1 q_2}\theta_2^{p_3-1}e^{-\theta_2 q_3}. \tag{11}$$

The mean and variance of the MLE for each of the jth samples are computed and equated to the mean and variance of the gamma prior distribution to determine the hyperparameters. The estimators $\hat{\alpha}$, $\hat{\theta}_1$, and $\hat{\theta}_2$ for α, θ_1, and θ_2, respectively. For more information about this crtirea see [52]. Solving the mean and variance of the MLE and mean and variance of the gamma prior distribution yields the estimators for the hyper-parameters:

$$\hat{p}_i = \left(\frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{\,j}\right) \Bigg/ \left(\frac{1}{I-1}\sum_{j=1}^{I}\left(\hat{\Omega}_i^{\,j} - \frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{\,j}\right)^2\right)$$

and

$$\hat{q}_i = \left(\frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{\,j}\right)^2 \Bigg/ \left(\frac{1}{I-1}\sum_{j=1}^{I}\left(\hat{\Omega}_i^{\,j} - \frac{1}{I}\sum_{j=1}^{I}\hat{\Omega}_i^{\,j}\right)^2\right).$$

For more information about gamma prior, hyperparameters, and posterior distribution, see [53–56].

By combining Eqs. (11) and (6) and applying Bayes' theorem, the joint posterior distribution is obtained as:

$$\Pi(\Omega|t) \propto \prod_{i=1}^{s}\left[\prod_{j=1}^{D_i}\frac{v_i^{\theta_2}t_{ij}^{\theta_2}}{(1+\alpha)^{R_{ij}+1}}\left(\alpha + \frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)^{R_{ij}}\right.$$

$$\times\ \alpha^{p_1-1}e^{-\alpha q_1}\theta_1^{p_2+\sum_{i=1}^{s}D_i-1}e^{-\theta_1 q_2}\theta_2^{p_3-1}e^{-\theta_2 q_3}e^{-R_{ij}\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}}$$

$$\times\left(\frac{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)}e^{-\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}}\right)^{\delta R_T^*}\Bigg]. \quad (12)$$

Deriving analytical Bayes estimators for the three parameters α, θ_1, and θ_2 are challenging, necessitating the use of numerical techniques. The next section explores the MCMC method for obtaining the required estimates.

4.1 MCMC Method

The marginal posterior distribution of a parameter is derived by integrating the joint posterior distribution over the other parameter. Consequently, the marginal posterior probability density functions for α, θ_1, and θ_2 are expressed as follows:

$$\Pi(\alpha|\theta_1,\theta_2,t)\propto\prod_{i=1}^{s}\left[\prod_{j=1}^{D_i}\frac{1}{(1+\alpha)^{R_{ij}+1}}\left(\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)^{R_{ij}}\right.$$
$$\left.\times\ \alpha^{p_1-1}e^{-\alpha q_1}\left(\frac{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)}e^{-\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}}\right)^{\delta R_T^*}\right], \quad (13)$$

$$\Pi(\theta_1|\alpha,\theta_2,t)\propto\prod_{i=1}^{s}\theta_1^{p_2+\sum_{i=1}^{s}D_i-1}\left[\prod_{j=1}^{D_i}\left(\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)^{R_{ij}}\right.$$
$$\left.\times\ e^{-\theta_1\left(q_2+R_{ij}\frac{v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)}\left(\frac{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)}e^{-\frac{\theta_1 v_i^{\theta_2}T^{\theta_2+1}}{\theta_2+1}}\right)^{\delta R_T^*}\right], \quad (14)$$

and

$$\Pi(\theta_2|\alpha,\theta_1,t)\propto\theta_2^{p_3-1}\prod_{i=1}^{s}\left[\prod_{j=1}^{D_i}v_i^{\theta_2}t_{ij}^{\theta_2}\left(\alpha+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2}t_{ij}^{\theta_2+1}}{\theta_2+1}\right)^2\right)^{R_{ij}}\right.$$

$$\times e^{-\theta_2 q_3} e^{-R_{ij} \frac{\theta_1 v_i^{\theta_2} t_{ij}^{\theta_2+1}}{\theta_2+1}} \left(\frac{\left(1+\alpha+\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}+\frac{1}{2}\left(\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}\right)^2\right)}{(1+\alpha)} e^{-\frac{\theta_1 v_i^{\theta_2} T^{\theta_2+1}}{\theta_2+1}} \right)^{\delta R_T^*} \Bigg]. \tag{15}$$

The Metropolis-Hastings (MH) algorithm is the preferred method for generating posterior samples to compute the desired Bayes estimates. This is because obtaining the conditional posterior distributions of the parameters α, θ_1, and θ_2 in the form of well-known distributions is challenging, making direct sampling from these distributions difficult. To address this, the conditional posterior distributions are approximated using well-known distributions. The following steps of the MH algorithm are then used to generate random samples from these approximated conditional distributions:

Steps of the Metropolis-Hastings algorithm:

1. Initialize the parameter values $(\alpha, \theta_1, \theta_2)$ as $(\alpha^0, \theta_1^0, \theta_2^0)$.
2. Set the iteration index $j = 1$.
3. Generate proposed values for the parameters as:

$$\alpha \sim N(\alpha^j, \sqrt{[f'(\Omega)]_{11}^{-1}}), \quad \theta_1 \sim N(\theta_1^j, \sqrt{[f'(\Omega)]_{22}^{-1}}), \quad \theta_2 \sim N(\theta_2^j, \sqrt{[f'(\Omega)]_{33}^{-1}}),$$

where $[f'(\Omega)]^{-1}$ represents the variance-covariance matrix.

4. Calculate the acceptance ratio:

$$P = \frac{\pi(\alpha^j, \theta_1^j, \theta_2^j \mid \mathbf{x})}{\pi(\alpha^{j-1}, \theta_1^{j-1}, \theta_2^{j-1} \mid \mathbf{x})}.$$

5. Accept the proposed values $(\alpha^{<j>}, \theta_1^{<j>}, \theta_2^{<j>})$ with probability $\min(1, P)$.
6. Repeat steps (3) to (5) B times to generate B samples for the parameters $(\alpha, \theta_1, \theta_2)$.

As described in [57], the Bayes estimators for the parameters α, θ_1, and θ_2 under the general entropy loss function can be defined as:

$$\tilde{\alpha} = \left(\frac{1}{B'}\sum_{j=1}^{B'}(\alpha^{<j>})^{-c}\right)^{-1/c}, \quad \tilde{\theta}_1 = \left(\frac{1}{B'}\sum_{j=1}^{B'}(\theta_1^{<j>})^{-c}\right)^{-1/c}, \quad \tilde{\theta}_2 = \left(\frac{1}{B'}\sum_{j=1}^{B'}(\theta_2^{<j>})^{-c}\right)^{-1/c}.$$

We remove first B_0 samples to discard the possible dependency on the initial guess, where $B' = B - B_0$ and B_0 is also known as burn-in sample.

The Bayes estimators for the parameters α, θ_1, and θ_2 under the squared error loss function can be defined as:

$$\tilde{\alpha} = \frac{1}{B'}\sum_{j=1}^{B'}(\alpha^{<j>}), \quad \tilde{\theta}_1 = \frac{1}{B'}\sum_{j=1}^{B'}(\theta_1^{<j>}), \quad \tilde{\theta}_2 = \frac{1}{B'}\sum_{j=1}^{B'}(\theta_2^{<j>}).$$

4.2 Highest Posterior Density (HPD) Credible Interval

In this subsection, the HPD credible interval (L, U) is determined for a random variable $\tilde{\Omega}$ by ensuring that it satisfies the following expression:

$$P(L \le \tilde{\Omega} \le U) = \int_L^U \pi^*(\tilde{\Omega} \mid \mathbf{t})\, d\tilde{\Omega} = 1 - \alpha.$$

Given the difficulty of determining the interval (L, U) analytically, we utilize the posterior samples derived to calculate the desired HPD credible intervals using the approach outlined by Chen and Shao [58].

5 Simulation

To evaluate the performance of the methods presented in this paper, a Monte Carlo simulation is carried out. The MLEs and Bayes estimates are compared based on their relative absolute bias (RAB), and mean squared errors (MSEs). Additionally, the asymptotic confidence intervals (ACI) and HPD intervals are assessed in terms of their average interval lengths and coverage probabilities. The simulation study involves three different progressive-stress accelerated life tests (ALTs): the first is a simple ramp-stress ALT with two stress levels ($s = 2$), and the second is a multiple ramp-stress ALT with four stress levels ($s = 4$). Furthermore, two censoring schemes are considered by binomial random removal. Assume that the removal of an individual unit from the life test is independent of the other units, with each unit having the same probability p of being removed. Consequently, the number of units removed at each failure time follows a binomial distribution, expressed as:

$$P(R_{i1} = r_{i1}) = \binom{n_i - m_i}{r_{i1}} p^{r_{i1}} (1-p)^{n_i - m_i - r_{i1}}, \quad 0 \le r_{i1} \le n_i \tag{16}$$

and

$$P(R_{ij} = r_{ij} \mid R_{i(j-1)} = r_{i(j-1)}, \ldots, R_{i1} = r_{i1}) = \binom{n_i - m_i - \sum_{k=1}^{(j-1)} r_{ik}}{r_{ij}} p^{r_{ij}} (1-p)^{n_i - m_i - \sum_{k=1}^{j} r_{ik}}, \tag{17}$$

where $0 \le r_{ij} \le n_i - m_i - \sum_{k=1}^{j-1} r_{ik}$, $j = 2, 3, \ldots, m_i - 1$, and $i = 1, 2, ..., s$ is the stress levels. Also supposing further that R_i is independent of t_j for all stress level. Therefore, the joint likelihood function with binomial removal can be expressed as:

$$L(\Omega, p) = L(\Omega) P(R = r). \tag{18}$$

Substituting (16) and (17) into (18), we get

$$P(R = r) = \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i! (n - m - \sum_{i=1}^{m-1} r_i)!} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{m-1} \prod_{i=1}^{m-1} \binom{n-m}{r_i}.$$

In all scenarios considered, we have assumed $p = 0.2$, 0.9 without any loss of generality. Samples were generated for specified values of each stress level n_i and m_i using a binomial removal technique in conjunction with a predefined sampling scheme.

The true values of the parameters α, θ_1, and θ_2 are assumed as follows: Case 1: $\alpha = 0.5$, $\theta_1 = 0.3$, and $\theta_2 = 0.4$; Case 2: $\alpha = 1.5$, $\theta_1 = 0.3$, and $\theta_2 = 0.8$; Case 3: $\alpha = 1.5$, $\theta_1 = 2$, and $\theta_2 = 0.8$; Case 4:$\alpha = 1.5$, $\theta_1 = 2$, and $\theta_2 = 2$. For Bayesian estimation, it is typical to exclude a portion of the initial samples as a burn-in period to ensure that the Markov chain has converged to the correct stationary distribution. In our approach, we generate a total of M-H $B = 12{,}000$ samples, with the first $B_0 = 2000$ samples discarded as burn-in. The remaining samples can then be used to construct point estimates and interval estimates for α and λ. In the first ramp-stress ALT, the ramp stresses are set to $\nu_1 = 170$ and $\nu_2 = 200$. For the second ramp-stress ALT, the ramp stresses are $\nu_1 = 170$, $\nu_2 = 200$, $\nu_3 = 240$ and $\nu_4 = 280$. The simulation results, based on 5000 samples, are summarized in Tables 1–6.

Table 1: Simple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 0.5, \theta_1 = 0.3, \theta_2 = 0.4$

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	p	Q	r_m, r_k		RAB	MSE	RAB	MSE	RAB	MSE	RAB	MSE	LACI	CP	LBP	LBT	LCCU	CP	LCCU	LCCU
20,15	0.2	0.6	0.5,0.7	α	0.2155	0.0620	0.0311	0.0411	0.0536	0.0441	0.0312	0.0385	0.8801	96.5%	0.0277	0.0280	0.8142	99%	0.8180	0.7687
				θ_1	0.1485	0.0516	0.2226	0.0389	0.2512	0.0431	0.1946	0.0350	0.8732	96.5%	0.0275	0.0270	0.6947	96.5%	0.7589	0.6966
				θ_2	0.1979	0.0258	0.0317	0.0101	0.0502	0.0127	0.0399	0.0130	0.5487	96%	0.0173	0.0174	0.3965	99%	0.3958	0.3873
			0.7,0.9	α	0.0392	0.0474	0.0095	0.0212	0.0163	0.0214	0.0285	0.0210	0.8504	96.5%	0.0274	0.0276	0.5505	99%	0.5733	0.5682
				θ_1	0.0475	0.0346	0.0604	0.0131	0.0671	0.0135	0.0538	0.0127	0.7275	96.5%	0.0234	0.0234	0.4318	99%	0.4483	0.4380
				θ_2	0.1158	0.0166	0.0126	0.0058	0.0152	0.0059	0.0100	0.0058	0.4708	96%	0.0144	0.0144	0.3053	99%	0.3007	0.2975
		0.85	0.5,0.7	α	0.1499	0.0609	0.0303	0.0360	0.0529	0.0389	0.0203	0.0334	0.6986	96.5%	0.0316	0.0316	0.7521	99%	0.7666	0.7161
				θ_1	0.0898	0.0504	0.2042	0.0376	0.2318	0.0413	0.1771	0.0341	0.7907	96.5%	0.0305	0.0301	0.6810	99%	0.7488	0.6936
				θ_2	0.1229	0.0224	0.0305	0.0101	0.0466	0.0111	0.0306	0.0110	0.5355	97%	0.0179	0.0179	0.4021	96%	0.4074	0.3995
			0.7,0.9	α	0.0348	0.0466	0.0092	0.0202	0.0151	0.0203	0.0126	0.0202	0.8413	99%	0.0256	0.0258	0.5496	97%	0.5581	0.5550
				θ_1	0.0418	0.0325	0.0591	0.0122	0.0657	0.0126	0.0526	0.0119	0.7044	96.5%	0.0226	0.0225	0.4107	99%	0.4328	0.4240
				θ_2	0.1035	0.0150	0.0044	0.0055	0.0070	0.0056	0.0018	0.0055	0.4508	99%	0.0164	0.0164	0.2975	99%	0.2935	0.2904
	0.9	0.6	0.5,0.7	α	0.1985	0.0689	0.0342	0.0406	0.0546	0.0440	0.0141	0.0376	0.9534	96.5%	0.0320	0.0317	0.7659	96.5%	0.8155	0.7603
				θ_1	0.0429	0.0612	0.2724	0.0456	0.3026	0.0505	0.2427	0.0410	0.9686	96.5%	0.0319	0.0318	0.7143	96.5%	0.8061	0.7407
				θ_2	0.1685	0.0240	0.0214	0.0097	0.0133	0.0098	0.0294	0.0096	0.5473	97%	0.0182	0.0183	0.3923	96%	0.3877	0.3810
			0.7,0.9	α	0.0248	0.0474	0.0159	0.0193	0.0101	0.0194	0.0216	0.0193	0.8521	99%	0.0271	0.0269	0.5318	96%	0.5465	0.5426
				θ_1	0.0299	0.0304	0.0485	0.0126	0.0547	0.0129	0.0422	0.0124	0.6828	96.5%	0.0218	0.0221	0.4120	99%	0.4412	0.4331
				θ_2	0.0988	0.0175	0.0203	0.0062	0.0229	0.0063	0.0177	0.0061	0.4956	96%	0.0157	0.0155	0.2987	96%	0.3088	0.3054
		0.85	0.5,0.7	α	0.1906	0.0618	0.0315	0.0365	0.0516	0.0425	0.0129	0.0314	0.9043	96.5%	0.0329	0.0328	0.7998	96.5%	0.8710	0.8097
				θ_1	0.0239	0.0604	0.2517	0.0430	0.2812	0.0475	0.2227	0.0389	0.9159	96.5%	0.0606	0.0517	0.6708	99%	0.7879	0.7278
				θ_2	0.1670	0.0234	0.0123	0.0081	0.0043	0.0080	0.0203	0.0091	0.5265	97%	0.0187	0.0186	0.4104	96%	0.4076	0.4000
			0.7,0.9	α	0.0239	0.0419	0.0057	0.0182	0.0081	0.0121	0.0116	0.0121	0.7988	99%	0.0263	0.0266	0.5679	96%	0.5683	0.5627
				θ_1	0.0236	0.0304	0.0486	0.0125	0.0550	0.0128	0.0423	0.0122	0.5809	96.5%	0.0249	0.0252	0.4146	96.5%	0.4385	0.4297
				θ_2	0.0798	0.0150	0.0191	0.0060	0.0217	0.0061	0.0166	0.0059	0.4631	97%	0.0152	0.0151	0.3017	99%	0.3032	0.3003
60,40	0.2	0.6	0.5,0.7	α	0.1891	0.0415	0.0175	0.0216	0.0034	0.0228	0.0313	0.0206	0.7072	96.5%	0.0229	0.0230	0.5674	96.5%	0.5916	0.5595
				θ_1	0.0808	0.0324	0.1375	0.0206	0.1566	0.0224	0.1188	0.0190	0.6993	96.5%	0.0223	0.0226	0.5113	99%	0.5570	0.5224
				θ_2	0.1123	0.0141	0.0078	0.0056	0.0132	0.0057	0.0024	0.0055	0.4307	97%	0.0133	0.0133	0.3038	96%	0.2941	0.2900
			0.7,0.9	α	0.0416	0.0336	0.0030	0.0144	0.0016	0.0145	0.0077	0.0142	0.7141	99%	0.0233	0.0233	0.4536	99%	0.4724	0.4675
				θ_1	0.0535	0.0314	0.0430	0.0094	0.0482	0.0096	0.0378	0.0092	0.6926	96.5%	0.0226	0.0220	0.3560	96%	0.3795	0.3743
				θ_2	0.0275	0.0121	0.0065	0.0042	0.0117	0.0043	0.0021	0.0042	0.4298	99%	0.0144	0.0142	0.2529	99%	0.2545	0.2522
		0.85	0.5,0.7	α	0.1259	0.0363	0.0132	0.0209	0.0032	0.0213	0.0304	0.0180	0.5495	96.5%	0.0370	0.0375	0.6632	96.5%	0.6867	0.6521
				θ_1	0.0790	0.0273	0.0994	0.0201	0.1189	0.0203	0.0804	0.0189	0.6381	96.5%	0.0435	0.0431	0.5235	99%	0.5764	0.5363
				θ_2	0.1023	0.0117	0.0071	0.0052	0.0113	0.0047	0.0024	0.0046	0.3779	97%	0.0265	0.0259	0.3304	96%	0.3182	0.3137
			0.7,0.9	α	0.0366	0.0251	0.0029	0.0139	0.0012	0.0139	0.0063	0.0139	0.6090	96.5%	0.0464	0.0463	0.4559	97%	0.4781	0.4745
				θ_1	0.0118	0.0181	0.0396	0.0091	0.0452	0.0093	0.0341	0.0089	0.5286	96.5%	0.0433	0.0422	0.3203	96%	0.3761	0.3693
				θ_2	0.0156	0.0076	0.0061	0.0038	0.0106	0.0038	0.0019	0.0037	0.3322	96%	0.0239	0.0238	0.2278	97%	0.2410	0.2392
	0.9	0.6	0.5,0.7	α	0.0934	0.0478	0.0636	0.0394	0.0809	0.0425	0.0463	0.0365	0.8376	96.5%	0.0270	0.0270	0.7667	96.5%	0.7930	0.7442
				θ_1	0.0247	0.0335	0.1710	0.0262	0.1917	0.0284	0.1506	0.0241	0.7171	96.5%	0.0233	0.0231	0.5445	96.5%	0.6212	0.5827
				θ_2	0.0833	0.0123	0.0056	0.0067	0.0112	0.0068	0.0001	0.0066	0.4146	97%	0.0129	0.0128	0.3486	97%	0.3228	0.3182
			0.7,0.9	α	0.0266	0.0360	0.0275	0.0196	0.0347	0.0203	0.0203	0.0189	0.7427	99%	0.0232	0.0229	0.5058	96%	0.5548	0.5375
				θ_1	0.0163	0.0245	0.0718	0.0120	0.0800	0.0126	0.0637	0.0116	0.6134	96.5%	0.0192	0.0189	0.3979	99%	0.4296	0.4148
				θ_2	0.0487	0.0109	0.0051	0.0045	0.0104	0.0045	0.0001	0.0044	0.4015	96%	0.0130	0.0129	0.2526	97%	0.2628	0.2597
		0.85	0.5,0.7	α	0.0825	0.0480	0.0637	0.0384	0.0701	0.0414	0.0447	0.0356	0.8456	96.5%	0.0679	0.0716	0.7086	96.5%	0.7795	0.7325
				θ_1	0.0049	0.0297	0.1520	0.0211	0.1699	0.0227	0.1345	0.0196	0.6775	96.5%	0.0515	0.0541	0.4987	99%	0.5571	0.5262
				θ_2	0.0686	0.0090	0.0008	0.0050	0.0043	0.0051	0.0001	0.0050	0.3578	96%	0.0258	0.0261	0.3073	96%	0.2807	0.2764
			0.7,0.9	α	0.0244	0.0322	0.0145	0.0132	0.0096	0.0132	0.0194	0.0132	0.6996	99%	0.0456	0.0469	0.4297	99%	0.4513	0.4493
				θ_1	0.0042	0.0210	0.0511	0.0081	0.0566	0.0083	0.0457	0.0079	0.6071	96.5%	0.0464	0.0428	0.3439	96%	0.3515	0.3443
				θ_2	0.0289	0.0082	0.0007	0.0036	0.0039	0.0037	0.0001	0.0036	0.3529	97%	0.0256	0.0256	0.2296	96.5%	0.2372	0.2342

Table 2: Multiple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 0.5, \theta_1 = 0.3, \theta_2 = 0.4$

K = 4					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	p	Q	r_m, r_k		RAB	MSE	RAB	MSE	RAB	MSE	RAB	MSE	LACI	CP	LBP	LBT	LCCU	CP	LCCU	LCCU
20,15, 12,10	0.2	0.6	0.5,0.7	α	0.2093	0.0468	0.0545	0.0359	0.0434	0.0275	0.0314	0.0442	0.7434	96.5%	0.0335	0.0335	0.6361	99%	0.6479	0.6131
				θ_1	0.0955	0.0358	0.1217	0.0196	0.1429	0.0214	0.1010	0.0280	0.7054	96.5%	0.0312	0.0310	0.5412	99%	0.5481	0.5122
				θ_2	0.1725	0.0193	0.0179	0.0070	0.0115	0.0071	0.0242	0.0070	0.4729	96%	0.0215	0.0217	0.3230	99%	0.3307	0.3250
			0.7,0.9	α	0.0245	0.0452	0.0045	0.0163	0.0098	0.0165	0.0082	0.0261	0.8924	96.5%	0.0410	0.0399	0.4935	96%	0.5033	0.4975
				θ_1	0.0469	0.0337	0.0503	0.0112	0.0561	0.0114	0.0446	0.0109	1.0359	96.5%	0.0504	0.0493	0.4145	96%	0.4134	0.4069

(Continued)

Table 2 (continued)

K = 4					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
				θ_2	0.0642	0.0132	0.0081	0.0048	0.0103	0.0049	0.0060	0.0048	0.5976	99%	0.0274	0.0267	0.2552	96%	0.2738	0.2714
		0.85	0.5,0.7	α	0.1203	0.0388	0.0302	0.0306	0.0318	0.0244	0.0208	0.0372	0.6972	99%	0.0339	0.0338	0.7303	99%	0.8121	0.7528
				θ_1	0.0694	0.0338	0.1140	0.0182	0.1335	0.0205	0.1001	0.0201	0.7006	96.5%	0.0347	0.0354	0.5041	99%	0.5853	0.5386
				θ_2	0.0643	0.0154	0.0166	0.0068	0.0106	0.0067	0.0173	0.0068	0.4707	99%	0.0228	0.0223	0.3231	97%	0.3083	0.3034
			0.7,0.9	α	0.0217	0.0340	0.0041	0.0163	0.0092	0.0165	0.0055	0.0162	0.8689	99%	0.0392	0.0388	0.5146	99%	0.5035	0.4991
				θ_1	0.0414	0.0317	0.0496	0.0101	0.0517	0.0102	0.0416	0.0101	1.0195	96.5%	0.0515	0.0507	0.3897	96%	0.4154	0.4083
				θ_2	0.0408	0.0146	0.0043	0.0045	0.0042	0.0046	0.0019	0.0045	0.4948	97%	0.0236	0.0238	0.2622	96%	0.2648	0.2623
	0.9	0.6	0.5,0.7	α	0.1925	0.0529	0.0064	0.0305	0.0240	0.0333	0.0108	0.0281	0.7580	96.5%	0.0361	0.0344	0.6103	99%	0.7144	0.6572
				θ_1	0.0376	0.0524	0.1814	0.0265	0.2059	0.0292	0.1575	0.0241	0.8934	96.5%	0.0407	0.0388	0.6001	96.5%	0.6254	0.5799
				θ_2	0.1373	0.0165	0.0072	0.0071	0.0055	0.0072	0.0137	0.0070	0.4565	96%	0.0212	0.0214	0.3363	99%	0.3318	0.3271
			0.7,0.9	α	0.0221	0.0402	0.0063	0.0181	0.0095	0.0184	0.0012	0.0179	0.8713	96.5%	0.0397	0.0393	0.5151	96%	0.5313	0.5256
				θ_1	0.0301	0.0308	0.0460	0.0122	0.0521	0.0124	0.0398	0.0119	0.8081	96.5%	0.0491	0.0488	0.3792	99%	0.4333	0.4251
				θ_2	0.0469	0.0162	0.0072	0.0054	0.0042	0.0055	0.0130	0.0053	0.5566	99%	0.0243	0.0244	0.2631	96%	0.2891	0.2847
		0.85	0.5,0.7	α	0.1925	0.0490	0.0060	0.0297	0.0238	0.0323	0.0030	0.0274	0.7238	96.5%	0.0335	0.0311	0.6932	99%	0.7014	0.6497
				θ_1	0.0210	0.0401	0.1327	0.0242	0.1542	0.0264	0.1116	0.0222	0.7761	96.5%	0.0344	0.0346	0.5350	99%	0.6115	0.5700
				θ_2	0.1253	0.0159	0.0067	0.0069	0.0041	0.0070	0.0018	0.0067	0.4283	96%	0.0211	0.0210	0.3287	96%	0.3282	0.3213
			0.7,0.9	α	0.0214	0.0407	0.0052	0.0161	0.0083	0.0114	0.0011	0.0159	0.8493	96.5%	0.0405	0.0405	0.4844	96%	0.4998	0.4947
				θ_1	0.0201	0.0298	0.0417	0.0113	0.0451	0.0113	0.0306	0.0113	0.7002	96.5%	0.0532	0.0508	0.4164	96%	0.4431	0.4344
				θ_2	0.0416	0.0119	0.0062	0.0051	0.0040	0.0052	0.0015	0.0051	0.5447	99%	0.0248	0.0240	0.2786	99%	0.2824	0.2801
	0.2	0.6	0.5,0.7	α	0.1522	0.0391	0.0078	0.0205	0.0036	0.0201	0.0215	0.0192	0.6445	96.5%	0.0297	0.0303	0.6212	99%	0.6454	0.6009
				θ_1	0.0612	0.0258	0.0879	0.0181	0.1043	0.0193	0.0717	0.0170	0.6144	96.5%	0.0300	0.0298	0.4734	99%	0.5314	0.5047
				θ_2	0.1089	0.0106	0.0081	0.0057	0.0133	0.0058	0.0021	0.0056	0.3654	97%	0.0172	0.0173	0.2894	96%	0.2978	0.2926
			0.7,0.9	α	0.0217	0.0308	0.0010	0.0143	0.0017	0.0145	0.0057	0.0142	0.6738	99%	0.0313	0.0310	0.4612	97%	0.4730	0.4670
				θ_1	0.0289	0.0255	0.0371	0.0094	0.0425	0.0095	0.0417	0.0092	0.6255	96.5%	0.0291	0.0286	0.3654	96%	0.3760	0.3719
				θ_2	0.0229	0.0083	0.0055	0.0041	0.0075	0.0041	0.0021	0.0040	0.3511	96%	0.0164	0.0172	0.2354	96%	0.2518	0.2489
		0.85	0.5,0.7	α	0.1042	0.0312	0.0062	0.0198	0.0015	0.0207	0.0205	0.0190	0.5840	96%	0.0268	0.0268	0.5391	99%	0.5647	0.5369
				θ_1	0.0472	0.0255	0.0840	0.0174	0.1017	0.0190	0.0705	0.0161	0.5915	96.5%	0.0440	0.0431	0.5004	96%	0.5226	0.4880
				θ_2	0.0320	0.0101	0.0070	0.0051	0.0058	0.0026	0.0022	0.0046	0.3406	97%	0.0183	0.0184	0.2989	96%	0.3084	0.3040
40,30, 20,10	0.9	0.6	0.7,0.9	α	0.0207	0.0205	0.0009	0.0138	0.0013	0.0139	0.0009	0.0115	0.6687	96.5%	0.0463	0.0421	0.4530	96%	0.4794	0.4739
				θ_1	0.0104	0.0161	0.0399	0.0091	0.0458	0.0091	0.0341	0.0091	0.6125	96.5%	0.0426	0.0409	0.3583	96%	0.3974	0.3893
				θ_2	0.0118	0.0071	0.0054	0.0035	0.0052	0.0024	0.0012	0.0035	0.3483	99%	0.0231	0.0229	0.2483	96%	0.2631	0.2608
			0.5,0.7	α	0.0831	0.0413	0.0104	0.0291	0.0252	0.0307	0.0041	0.0277	0.7116	99%	0.0334	0.0321	0.6359	99%	0.6863	0.6529
				θ_1	0.0202	0.0239	0.1704	0.0219	0.1917	0.0241	0.1496	0.0200	0.6041	96.5%	0.0277	0.0278	0.5318	99%	0.5654	0.5261
				θ_2	0.0375	0.0096	0.0021	0.0057	0.0035	0.0058	0.0001	0.0056	0.3615	96%	0.0175	0.0174	0.3133	96%	0.2993	0.2947
			0.7,0.9	α	0.0167	0.0299	0.0101	0.0151	0.0199	0.0152	0.0040	0.0149	0.6652	96%	0.0290	0.0291	0.4604	99%	0.4825	0.4790
				θ_1	0.0110	0.0214	0.0534	0.0096	0.0590	0.0098	0.0477	0.0093	0.6093	96.5%	0.0288	0.0288	0.3683	96.5%	0.3820	0.3746
				θ_2	0.0375	0.0079	0.0018	0.0038	0.0030	0.0039	0.0001	0.0038	0.3432	96%	0.0153	0.0152	0.2401	97%	0.2432	0.2412
		0.85	0.5,0.7	α	0.0409	0.0403	0.0101	0.0231	0.0248	0.0302	0.0044	0.0291	0.7172	96.5%	0.0321	0.0332	0.6885	99%	0.7044	0.6694
				θ_1	0.0041	0.0224	0.1560	0.0211	0.1763	0.0230	0.1361	0.0194	0.6071	96.5%	0.0274	0.0269	0.5260	99%	0.5578	0.5230
				θ_2	0.0309	0.0075	0.0007	0.0046	0.0031	0.0046	0.0001	0.0060	0.3571	96%	0.0152	0.0154	0.3201	97%	0.3104	0.3044
			0.7,0.9	α	0.0152	0.0231	0.0016	0.0138	0.0037	0.0139	0.0059	0.0137	0.6849	99%	0.0331	0.0315	0.4520	99%	0.4625	0.4586
				θ_1	0.0038	0.0183	0.0434	0.0031	0.0489	0.0030	0.0379	0.0069	0.6126	96.5%	0.0632	0.0479	0.3829	96%	0.3940	0.3873
				θ_2	0.0263	0.0069	0.0005	0.0030	0.0022	0.0024	0.0001	0.0030	0.4429	99%	0.0217	0.0214	0.2395	99%	0.2499	0.2477

Table 3: Simple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 1.5, \theta_1 = 0.3, \theta_2 = 0.8$

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
20,15	0.2	0.6	0.5,0.7	α	0.0548	0.1523	0.0442	0.1888	0.0283	0.1880	0.0601	0.1906	1.2307	96%	0.0389	0.0390	1.6104	96%	1.6925	1.6753
				θ_1	0.2790	0.1070	0.1451	0.0384	0.1642	0.0205	0.1265	0.0165	0.9839	96.5%	0.0319	0.0315	0.5053	96.5%	0.5267	0.4819
				θ_2	0.2299	0.0933	0.0807	0.0094	0.0235	0.0094	0.0326	0.0095	0.9561	97%	0.0299	0.0299	0.3934	96%	0.3739	0.3681
			0.7,0.9	α	0.0021	0.1492	0.0046	0.0329	0.0019	0.0328	0.0073	0.0330	1.5148	96%	0.0488	0.0485	0.7042	96%	0.7097	0.7116
				θ_1	0.0834	0.1030	0.0463	0.0099	0.0520	0.0102	0.0406	0.0097	1.4106	96.5%	0.0448	0.0447	0.3657	96.5%	0.3918	0.3828
				θ_2	0.1008	0.0568	0.0029	0.0074	0.0017	0.0074	0.0018	0.0073	0.8797	97%	0.0281	0.0288	0.3282	96%	0.3378	0.3350
		0.85	0.5,0.7	α	0.0542	0.1064	0.0281	0.1750	0.0125	0.1763	0.0435	0.1751	1.1239	96%	0.0399	0.0399	1.5941	99%	1.6449	1.6212
				θ_1	0.1935	0.1019	0.1388	0.0257	0.1590	0.0201	0.1248	0.0152	0.9231	96.5%	0.0403	0.0404	0.5717	96.5%	0.6224	0.5703
				θ_2	0.1872	0.0868	0.0528	0.0091	0.0215	0.0091	0.0306	0.0081	0.8995	96%	0.0316	0.0315	0.4165	96%	0.4010	0.3933
			0.7,0.9	α	0.0007	0.1037	0.0033	0.0303	0.0008	0.0303	0.0058	0.0303	1.4208	96%	0.0552	0.0540	0.6731	97%	0.6829	0.6822
				θ_1	0.0798	0.1014	0.0328	0.0092	0.0386	0.0095	0.0271	0.0090	1.3446	96.5%	0.0428	0.0426	0.3771	96%	0.3792	0.3706
				θ_2	0.0953	0.0478	0.0025	0.0073	0.0017	0.0074	0.0017	0.0072	0.8694	96%	0.0293	0.0298	0.3255	96%	0.3364	0.3335
	0.9	0.6	0.5,0.7	α	0.0401	0.1010	0.0491	0.1877	0.0331	0.1889	0.0651	0.1882	1.2263	96%	0.0920	0.0912	1.6026	97%	1.6969	1.6611
				θ_1	0.2921	0.0616	0.1633	0.0249	0.1830	0.0276	0.1442	0.0224	0.9126	96.5%	0.0698	0.0668	0.4973	96.5%	0.6156	0.5634
				θ_2	0.2488	0.0929	0.0071	0.0097	0.0024	0.0100	0.0118	0.0095	0.9070	97%	0.0614	0.0634	0.4000	97%	0.3921	0.3821
			0.7,0.9	α	0.0177	0.0886	0.0069	0.0363	0.0038	0.0360	0.0100	0.0366	1.7036	96.5%	0.1180	0.1242	0.7902	96.5%	0.7452	0.7498
				θ_1	0.0177	0.0610	0.0513	0.0104	0.0572	0.0107	0.0454	0.0102	1.2539	96.5%	0.1123	0.0862	0.3976	99%	0.4014	0.3928
				θ_2	0.1151	0.0513	0.0110	0.0076	0.0128	0.0077	0.0093	0.0075	0.8136	97%	0.0626	0.0635	0.3343	96%	0.3414	0.3392
		0.85	0.5,0.7	α	0.0155	0.0912	0.0270	0.1742	0.0107	0.1742	0.0434	0.1757	1.1349	97%	0.0415	0.0415	1.5431	96%	1.6356	1.6240
				θ_1	0.2488	0.0591	0.1546	0.0208	0.1722	0.0231	0.1355	0.0204	0.9145	96.5%	0.0383	0.0352	0.5571	99%	0.6434	0.5783
				θ_2	0.2402	0.0910	0.0068	0.0094	0.0022	0.0094	0.0103	0.0094	0.9014	96%	0.0323	0.0327	0.4077	96%	0.3758	0.3700
			0.7,0.9	α	0.0076	0.0811	0.0058	0.0308	0.0036	0.0308	0.0091	0.0309	1.5579	96%	0.0499	0.0495	0.6738	99%	0.6874	0.6862

(Continued)

Table 3 (continued)

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
				θ_1	0.0174	0.0590	0.0465	0.0101	0.0471	0.0101	0.0426	0.0101	1.1383	96.5%	0.0458	0.0455	0.3778	99%	0.4067	0.3972
				θ_2	0.1002	0.0508	0.0010	0.0074	0.0007	0.0075	0.0027	0.0074	0.8009	96%	0.0280	0.0285	0.3398	97%	0.3391	0.3362
60,40	0.2	0.6	0.5,0.7	α	0.0200	0.0281	0.0296	0.1409	0.0163	0.1412	0.0428	0.1416	0.6471	96%	0.0204	0.0205	1.3888	96%	1.4704	1.4542
				θ_1	0.3006	0.0280	0.1037	0.0123	0.1166	0.0132	0.0911	0.0115	0.5526	96.5%	0.0180	0.0178	0.3907	99%	0.4295	0.4065
				θ_2	0.1597	0.0490	0.0069	0.0053	0.0040	0.0053	0.0098	0.0053	0.7090	99%	0.0216	0.0214	0.2973	97%	0.2852	0.2827
			0.7,0.9	α	0.0032	0.0275	0.0014	0.0274	0.0010	0.0274	0.0038	0.0275	1.0717	97%	0.0350	0.0343	0.6608	97%	0.6486	0.6500
				θ_1	0.0575	0.0218	0.0481	0.0074	0.0530	0.0076	0.0432	0.0073	0.5757	96.5%	0.0196	0.0196	0.3260	96%	0.3360	0.3301
				θ_2	0.0562	0.0297	0.0028	0.0047	0.0041	0.0048	0.0015	0.0047	0.6525	96.5%	0.0197	0.0194	0.2599	96%	0.2702	0.2684
		0.85	0.5,0.7	α	0.0173	0.0277	0.0256	0.1401	0.0125	0.1383	0.0386	0.1403	0.6445	97%	0.0197	0.0201	1.4293	99%	1.5084	1.4878
				θ_1	0.2316	0.0259	0.0753	0.0105	0.0881	0.0114	0.0627	0.0097	0.5100	96.5%	0.0164	0.0164	0.3789	99%	0.4061	0.3794
				θ_2	0.1463	0.0396	0.0002	0.0048	0.0028	0.0048	0.0032	0.0047	0.5884	96%	0.0182	0.0181	0.2813	99%	0.2719	0.2690
			0.7,0.9	α	0.0031	0.0261	0.0012	0.0273	0.0005	0.0272	0.0029	0.0274	0.9702	95%	0.0300	0.0298	0.6316	97%	0.6465	0.6470
				θ_1	0.0459	0.0212	0.0405	0.0068	0.0507	0.0064	0.0407	0.0068	0.5788	96.5%	0.0194	0.0191	0.3372	99%	0.3532	0.3476
				θ_2	0.0470	0.0221	0.0002	0.0046	0.0024	0.0046	0.0012	0.0046	0.5462	99%	0.0171	0.0168	0.2922	96%	0.2931	0.2910
	0.9	0.6	0.5,0.7	α	0.0052	0.0417	0.0294	0.1402	0.0158	0.1405	0.0428	0.1409	0.8001	96%	0.0253	0.0253	1.3956	96%	1.4670	1.4507
				θ_1	0.2042	0.0258	0.1367	0.0140	0.1519	0.0152	0.1218	0.0130	0.5818	96.5%	0.0185	0.0185	0.4348	96%	0.4496	0.4231
				θ_2	0.1264	0.0330	0.0019	0.0057	0.0012	0.0057	0.0050	0.0056	0.5914	96%	0.0178	0.0184	0.3074	96%	0.2965	0.2939
			0.7,0.9	α	0.0023	0.0416	0.0038	0.0270	0.0014	0.0270	0.0062	0.0271	0.9807	96%	0.0325	0.0328	0.6378	96%	0.6444	0.6448
				θ_1	0.0426	0.0229	0.0396	0.0081	0.0443	0.0083	0.0349	0.0079	0.5913	96.5%	0.0182	0.0183	0.3447	99%	0.3531	0.3468
				θ_2	0.0557	0.0318	0.0018	0.0054	0.0012	0.0055	0.0047	0.0054	0.6769	96.5%	0.0213	0.0196	0.2890	96%	0.2883	0.2861
		0.85	0.5,0.7	α	0.0049	0.0330	0.0199	0.1391	0.0056	0.1342	0.0342	0.1377	0.7121	96%	0.0489	0.0488	1.3790	96%	1.4786	1.4441
				θ_1	0.1212	0.0239	0.1359	0.0138	0.1476	0.0121	0.1204	0.0127	0.5540	96.5%	0.0385	0.0385	0.4615	99%	0.5247	0.4773
				θ_2	0.1204	0.0302	0.0016	0.0026	0.0012	0.0046	0.0041	0.0046	0.5687	96%	0.0366	0.0365	0.3304	96%	0.3133	0.3089
			0.7,0.9	α	0.0017	0.0307	0.0015	0.0269	0.0007	0.0269	0.0037	0.0269	0.7584	97%	0.0517	0.0505	0.6318	99%	0.6443	0.6440
				θ_1	0.0368	0.0229	0.0394	0.0079	0.0441	0.0081	0.0348	0.0077	0.5926	96.5%	0.0455	0.0468	0.3036	96%	0.3495	0.3434
				θ_2	0.0509	0.0213	0.0009	0.0025	0.0011	0.0045	0.0041	0.0051	0.5434	96.5%	0.0419	0.0402	0.2855	99%	0.2988	0.2968

Table 4: Simple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 1.5, \theta_1 = 2, \theta_2 = 0.8$

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
20,15	0.2	0.6	0.5,0.7	α	0.2104	0.8367	0.0228	0.1345	0.0074	0.1373	0.0380	0.1333	3.3672	96.5%	0.1057	0.1072	1.4178	96%	1.4526	1.4143
				θ_1	0.1333	0.6883	0.0362	0.2138	0.0223	0.2118	0.0500	0.2176	3.0811	96.5%	0.1007	0.1006	1.7993	96%	1.7963	1.7870
				θ_2	0.0893	0.0443	0.0150	0.0109	0.0145	0.0112	0.0186	0.0107	0.7768	96%	0.0230	0.0233	0.3824	96%	0.4129	0.4023
			0.7,0.9	α	0.0548	0.3757	0.0076	0.0301	0.0049	0.0300	0.0102	0.0302	2.3821	96.5%	0.0761	0.0754	0.6525	96%	0.6782	0.6793
				θ_1	0.0151	0.2471	0.0048	0.0318	0.0029	0.0316	0.0067	0.0320	1.9460	99%	0.0617	0.0616	0.7109	97%	0.6968	0.6998
				θ_2	0.0232	0.0186	0.0106	0.0042	0.0093	0.0041	0.0120	0.0042	0.5299	96%	0.0177	0.0166	0.2495	96%	0.2510	0.2510
		0.85	0.5,0.7	α	0.1308	0.8280	0.0224	0.1350	0.0062	0.1250	0.0351	0.1250	3.2248	96.5%	0.1086	0.1081	1.4724	96%	1.5135	1.4913
				θ_1	0.1265	0.6835	0.0297	0.2019	0.0161	0.1922	0.0433	0.2023	3.0119	96.5%	0.0958	0.0971	1.7431	96%	1.8209	1.8202
				θ_2	0.0665	0.0444	0.0149	0.0102	0.0135	0.0102	0.0153	0.0100	0.7700	99%	0.0263	0.0265	0.3902	96%	0.4005	0.3922
			0.7,0.9	α	0.0476	0.3441	0.0069	0.0281	0.0041	0.0281	0.0100	0.0282	2.2589	96.5%	0.0872	0.0828	0.6391	96%	0.6558	0.6548
				θ_1	0.0132	0.2396	0.0018	0.0310	0.0001	0.0309	0.0037	0.0311	1.8130	99%	0.0713	0.0703	0.6891	96%	0.6893	0.6913
				θ_2	0.0204	0.0172	0.0099	0.0037	0.0086	0.0036	0.0113	0.0037	0.5082	99%	0.0191	0.0182	0.2370	96%	0.2349	0.2350
	0.9	0.6	0.5,0.7	α	0.0794	0.8499	0.0077	0.1777	0.0086	0.1831	0.0241	0.1738	3.5854	96.5%	0.1104	0.1101	1.5803	96%	1.6773	1.6291
				θ_1	0.1582	0.7577	0.0236	0.1953	0.0098	0.1960	0.0373	0.1965	3.1804	96.5%	0.1027	0.1026	1.6999	97%	1.7345	1.7139
				θ_2	0.1280	0.0533	0.0055	0.0090	0.0089	0.0091	0.0021	0.0088	0.8120	96%	0.0270	0.0268	0.3817	99%	0.3741	0.3685
			0.7,0.9	α	0.0553	0.3548	0.0062	0.0285	0.0037	0.0285	0.0088	0.0287	2.3134	99%	0.0755	0.0745	0.6539	97%	0.6612	0.6624
				θ_1	0.0135	0.2792	0.0014	0.0303	0.0005	0.0303	0.0034	0.0303	2.0695	99%	0.0693	0.0673	0.6657	96%	0.6825	0.6818
				θ_2	0.0327	0.0197	0.0051	0.0035	0.0046	0.0035	0.0020	0.0035	0.5404	99%	0.0176	0.0176	0.2312	97%	0.2306	0.2308
		0.85	0.5,0.7	α	0.0012	0.8137	0.0061	0.1496	0.0045	0.1542	0.0205	0.1463	3.4885	96.5%	0.1216	0.1207	1.4222	99%	1.5398	1.4926
				θ_1	0.1593	0.4392	0.0181	0.1617	0.0064	0.1626	0.0297	0.1623	2.2791	96.5%	0.0724	0.0732	1.5799	97%	1.5805	1.5625
				θ_2	0.0897	0.0276	0.0045	0.0059	0.0081	0.0060	0.0017	0.0058	0.5877	99%	0.0192	0.0193	0.2972	96%	0.3020	0.2975
			0.7,0.9	α	0.0012	0.1593	0.0028	0.0269	0.0004	0.0268	0.0052	0.0270	1.5326	97%	0.0482	0.0487	0.6411	96%	0.6425	0.6435
				θ_1	0.0105	0.1721	0.0012	0.0275	0.0003	0.0274	0.0031	0.0277	1.5691	95%	0.0511	0.0510	0.6504	97%	0.6488	0.6507
				θ_2	0.0255	0.0124	0.0026	0.0017	0.0006	0.0017	0.0007	0.0017	0.5173	99%	0.0243	0.0239	0.1592	97%	0.1616	0.1614
60,40	0.2	0.6	0.5,0.7		0.0461	0.9814	0.0104	0.1496	0.0045	0.1542	0.0252	0.1463	3.8852	96.5%	0.1216	0.1207	1.4222	99%	1.5398	1.4926
				bccc1	0.1593	0.4392	0.0181	0.1617	0.0064	0.1626	0.0297	0.1623	2.2791	96.5%	0.0724	0.0732	1.5799	97%	1.5805	1.5625
				bccc.1	0.0897	0.0276	0.0084	0.0059	0.0102	0.0060	0.0067	0.0058	0.5877	99%	0.0192	0.0193	0.2972	96%	0.3020	0.2975
			0.7,0.9		0.0412	0.1593	0.0028	0.0269	0.0004	0.0268	0.0052	0.0270	1.5326	97%	0.0482	0.0487	0.6411	96%	0.6425	0.6435
					0.0549	0.1721	0.0048	0.0275	0.0030	0.0274	0.0067	0.0277	1.5691	95%	0.0511	0.0510	0.6504	97%	0.6488	0.6507
					0.0255	0.0204	0.0008	0.0017	0.0006	0.0017	0.0007	0.0017	0.7323	99%	0.0243	0.0239	0.1592	97%	0.1616	0.1614
		0.85	0.5,0.7	α	0.0419	0.7890	0.0101	0.1394	0.0042	0.1414	0.0245	0.1387	3.4719	96.5%	0.1139	0.1151	1.3668	96%	1.4717	1.4363
				θ_1	0.1251	0.3489	0.0136	0.1623	0.0052	0.1574	0.0248	0.1378	2.1092	96.5%	0.2315	0.1967	1.6106	96%	1.6338	1.6119
				θ_2	0.0865	0.0239	0.0079	0.0057	0.0101	0.0057	0.0017	0.0047	0.5723	99%	0.0243	0.0240	0.3116	96%	0.3216	0.3167
			0.7,0.9	α	0.0403	0.1467	0.0027	0.0239	0.0003	0.0259	0.0050	0.0261	1.1952	96.5%	0.1296	0.1017	0.6512	96%	0.6706	0.6720
				θ_1	0.0485	0.1622	0.0006	0.0258	0.0025	0.0265	0.0012	0.0258	1.2811	97%	0.0558	0.0561	0.6658	96%	0.6616	0.6607
				θ_2	0.0235	0.0193	0.0006	0.0017	0.0001	0.0017	0.0001	0.0017	0.7308	99%	0.0239	0.0243	0.1610	97%	0.1597	0.1596
	0.9	0.6	0.5,0.7	α	0.1163	1.1174	0.0116	0.1561	0.0263	0.1636	0.0030	0.1499	4.0890	96.5%	0.1248	0.1276	1.5191	96%	1.5787	1.5181
				θ_1	0.1177	0.5149	0.0235	0.1591	0.0125	0.1584	0.0345	0.1611	2.6586	96.5%	0.0825	0.0825	1.5010	96%	1.5578	1.5509
				θ_2	0.0686	0.0355	0.0183	0.0054	0.0199	0.0055	0.0166	0.0053	0.7066	99%	0.0245	0.0227	0.2734	99%	0.2849	0.2812

(Continued)

Table 4 (continued)

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
			0.7,0.9	α	0.0147	0.3474	0.0031	0.0285	0.0055	0.0285	0.0007	0.0284	2.3101	96.5%	0.0786	0.0717	0.6437	97%	0.6619	0.6609
				θ_1	0.0259	0.2477	0.0068	0.0290	0.0050	0.0290	0.0087	0.0291	1.9412	97%	0.0626	0.0620	0.6556	96%	0.6666	0.6657
				θ_2	0.0186	0.0248	0.0020	0.0017	0.0026	0.0017	0.0013	0.0017	0.6154	99%	0.0200	0.0193	0.1577	96%	0.1607	0.1606
		0.85	0.5,0.7	α	0.1074	1.0137	0.0102	0.1412	0.0069	0.1443	0.0024	0.1394	4.0628	96.5%	0.2790	0.2722	1.3612	96.5%	1.4923	1.4510
				θ_1	0.1148	0.3885	0.0220	0.1356	0.0121	0.1361	0.0318	0.1359	2.2774	99%	0.1831	0.1762	1.3345	96%	1.4465	1.4271
				θ_2	0.0607	0.0148	0.0120	0.0048	0.0121	0.0049	0.0158	0.0048	0.4220	96%	0.0355	0.0346	0.2624	97%	0.2679	0.2649
			0.7,0.9	α	0.0130	0.1258	0.0027	0.0231	0.0048	0.0231	0.0006	0.0203	1.3831	95%	0.1028	0.1025	0.6613	97%	0.6877	0.6886
				θ_1	0.0091	0.1365	0.0043	0.0272	0.0046	0.0273	0.0024	0.0252	1.4501	97%	0.1062	0.1004	0.6387	97%	0.6479	0.6475
				θ_2	0.0092	0.0123	0.0009	0.0012	0.0015	0.0012	0.0002	0.0016	0.5987	96.5%	0.0522	0.0396	0.1591	96%	0.1657	0.1653

Table 5: Simple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 1.5$, $\theta_1 = 2$, $\theta_2 = 2$

					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n	**p**	**Q**	**r_m, r_k**		**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**RAB**	**MSE**	**LACI**	**CP**	**LBP**	**LBT**	**LCCU**	**CP**	**LCCU**	**LCCU**
20,15	0.2	0.6	0.5,0.7	α	0.1993	0.7521	0.0099	0.1491	0.0098	0.1738	0.0232	0.1650	3.1928	96.5%	0.1017	0.1028	1.5414	99%	1.5273	1.5006
				θ_1	0.1073	1.3225	0.0272	0.2081	0.0284	0.1920	0.0557	0.1980	4.4310	96.5%	0.1422	0.1432	1.7506	97%	1.7852	1.7686
				θ_2	0.0861	0.1464	0.0098	0.0228	0.0234	0.0252	0.0294	0.0258	1.3402	99%	0.0423	0.0417	0.5788	96%	0.5925	0.5827
			0.7,0.9	α	0.0829	0.6460	0.0066	0.0309	0.0044	0.0333	0.0099	0.0332	5.0080	96.5%	0.1662	0.1641	0.6935	97%	0.6884	0.6887
				θ_1	0.0417	1.2473	0.0042	0.0305	0.0091	0.0321	0.0131	0.0327	4.7492	96.5%	0.1499	0.1494	0.6676	97%	0.6836	0.6854
				θ_2	0.0359	0.1009	0.0028	0.0084	0.0046	0.0085	0.0065	0.0086	1.2136	96%	0.0400	0.0402	0.3577	97%	0.3594	0.3594
		0.85	0.5,0.7	α	0.1672	0.7532	0.0067	0.1369	0.0055	0.1517	0.0215	0.1478	3.1259	96.5%	0.1048	0.1025	1.6203	99%	1.6342	1.5871
				θ_1	0.1005	1.2437	0.0211	0.1941	0.0136	0.1821	0.0407	0.1921	3.5451	96.5%	0.1504	0.1503	1.6627	96%	1.7039	1.6897
				θ_2	0.0735	0.1458	0.0083	0.0205	0.0068	0.0230	0.0127	0.0227	1.1393	99%	0.0461	0.0443	0.5936	96%	0.5944	0.5861
			0.7,0.9	α	0.0793	0.6430	0.0061	0.0233	0.0041	0.0308	0.0092	0.0310	4.6582	96.5%	0.1429	0.1430	0.7120	99%	0.7152	0.7123
				θ_1	0.0172	0.9585	0.0031	0.0232	0.0022	0.0304	0.0062	0.0307	4.0936	96.5%	0.1509	0.1509	0.6823	97%	0.6993	0.7019
				θ_2	0.0319	0.0979	0.0026	0.0082	0.0018	0.0084	0.0037	0.0085	1.1880	96%	0.0390	0.0385	0.3608	97%	0.3605	0.3608
	0.9	0.6	0.5,0.7	α	0.1659	0.8079	0.0189	0.1575	0.0026	0.1607	0.0350	0.1558	3.5146	96.5%	0.1162	0.1173	1.5529	99%	1.5720	1.5345
				θ_1	0.1573	1.3022	0.0312	0.2106	0.0179	0.2078	0.0444	0.2148	4.3021	96.5%	0.1390	0.1365	1.7225	96%	1.7825	1.7839
				θ_2	0.1089	0.1724	0.0064	0.0253	0.0094	0.0264	0.0034	0.0244	1.3866	99%	0.0442	0.0446	0.5848	99%	0.6328	0.6121
			0.7,0.9	α	0.1524	0.7729	0.0050	0.0319	0.0024	0.0320	0.0076	0.0319	5.1446	96.5%	0.1616	0.1641	0.6705	96%	0.7016	0.6992
				θ_1	0.0146	1.2540	0.0071	0.0279	0.0053	0.0277	0.0090	0.0281	4.8649	96.5%	0.1585	0.1581	0.6448	96%	0.6512	0.6537
				θ_2	0.0476	0.1045	0.0019	0.0092	0.0010	0.0092	0.0029	0.0092	1.2115	96%	0.0372	0.0372	0.3642	97%	0.3753	0.3755
		0.85	0.5,0.7	α	0.1070	0.7887	0.0148	0.1469	0.0023	0.1569	0.0347	0.1471	3.0646	96.5%	0.2868	0.2528	1.5739	97%	1.6023	1.5819
				θ_1	0.1081	1.1331	0.0304	0.2017	0.0128	0.1921	0.0406	0.2022	3.4540	96.5%	0.3347	0.3447	1.8849	97%	1.8029	1.7966
				θ_2	0.0969	0.1604	0.0061	0.0245	0.0091	0.0257	0.0031	0.0245	1.3772	99%	0.1009	0.0979	0.5858	96%	0.6337	0.6130
			0.7,0.9	α	0.1037	0.6759	0.0050	0.0312	0.0019	0.0309	0.0070	0.0315	5.1147	96.5%	0.3875	0.3700	0.6619	97%	0.6890	0.6928
				θ_1	0.0133	1.1275	0.0049	0.0255	0.0032	0.0254	0.0066	0.0256	4.5294	96.5%	0.3038	0.3098	0.6147	99%	0.6262	0.6268
				θ_2	0.0450	0.1016	0.0015	0.0082	0.0006	0.0082	0.0025	0.0082	1.1267	96.5%	0.0834	0.0835	0.3071	96%	0.3548	0.3554
60,40	0.2	0.6	0.5,0.7	α	0.1699	0.4025	0.0091	0.1478	0.0057	0.1529	0.0239	0.1441	2.2785	96.5%	0.0710	0.0706	1.4135	99%	1.5333	1.4820
				θ_1	0.1381	0.2549	0.0460	0.1909	0.0337	0.1879	0.0582	0.1954	1.6572	99%	0.0503	0.0493	1.6191	96%	1.6795	1.6723
				θ_2	0.0482	0.0470	0.0130	0.0169	0.0147	0.0175	0.0113	0.0163	0.7616	99%	0.0246	0.0237	0.4857	96%	0.5064	0.4930
			0.7,0.9	α	0.0596	0.1830	0.0014	0.0283	0.0039	0.0284	0.0012	0.0282	1.6409	96%	0.0502	0.0502	0.6560	96%	0.6604	0.6591
				θ_1	0.0326	0.1533	0.0025	0.0275	0.0006	0.0275	0.0043	0.0276	1.5141	96%	0.0474	0.0470	0.6322	97%	0.6500	0.6507
				θ_2	0.0176	0.0282	0.0007	0.0040	0.0002	0.0040	0.0012	0.0040	0.6434	96.5%	0.0204	0.0200	0.2416	96%	0.2482	0.2481
		0.85	0.5,0.7	α	0.1623	0.4032	0.0082	0.1274	0.0010	0.1300	0.0231	0.1259	2.2999	96.5%	0.0758	0.0754	1.3038	99%	1.4142	1.3802
				θ_1	0.1196	0.2166	0.0381	0.1895	0.0257	0.1825	0.0504	0.1983	1.5660	99%	0.0488	0.0488	1.6890	97%	1.7088	1.7012
				θ_2	0.0429	0.0242	0.0100	0.0157	0.0117	0.0175	0.0083	0.0165	0.5092	97%	0.0160	0.0163	0.5022	99%	0.5105	0.5001
			0.7,0.9	α	0.0592	0.1631	0.0012	0.0269	0.0002	0.0261	0.0005	0.0251	1.5452	99%	0.0485	0.0485	0.6616	97%	0.6688	0.6680
				θ_1	0.0311	0.1355	0.0018	0.0268	0.0006	0.0267	0.0041	0.0269	1.4071	96%	0.0435	0.0427	0.6210	96%	0.6383	0.6382
				θ_2	0.0167	0.0235	0.0002	0.0040	0.0001	0.0040	0.0004	0.0040	0.6072	96.5%	0.0226	0.0223	0.2519	99%	0.2476	0.2476
	0.9	0.6	0.5,0.7	α	0.0558	0.3452	0.0036	0.1608	0.0130	0.1679	0.0186	0.1550	2.2901	96.5%	0.0721	0.0723	1.4502	99%	1.5332	1.4885
				θ_1	0.0957	0.2190	0.0228	0.1788	0.0213	0.1775	0.0350	0.1817	1.6813	99%	0.0527	0.0539	1.5550	96%	1.6019	1.5919
				θ_2	0.0413	0.0330	0.0139	0.0158	0.0183	0.0164	0.0123	0.0153	0.6511	99%	0.0199	0.0197	0.4534	97%	0.4551	0.4459
			0.7,0.9	α	0.0534	0.1753	0.0004	0.0347	0.0054	0.0349	0.0022	0.0346	1.6346	99%	0.0505	0.0504	0.6977	97%	0.7035	0.7032
				θ_1	0.0387	0.1406	0.0023	0.0358	0.0023	0.0360	0.0042	0.0356	1.4463	96%	0.0481	0.0481	0.6250	97%	0.6509	0.6518
				θ_2	0.0219	0.0181	0.0004	0.0042	0.0018	0.0042	0.0001	0.0042	0.5064	99%	0.0162	0.0159	0.2526	96%	0.2512	0.2511
		0.85	0.5,0.7	α	0.0436	0.3274	0.0024	0.1483	0.0115	0.1531	0.0177	0.1448	2.1379	96.5%	0.1565	0.1571	1.5279	99%	1.6083	1.5438
				θ_1	0.0939	0.2058	0.0203	0.1677	0.0105	0.1673	0.0346	0.1697	1.5848	96.5%	0.1460	0.1391	1.6245	96%	1.6471	1.6356
				θ_2	0.0369	0.0309	0.0127	0.0140	0.0155	0.0144	0.0115	0.0135	0.6099	96%	0.0445	0.0438	0.4593	99%	0.4817	0.4721
			0.7,0.9	α	0.0268	0.1729	0.0004	0.0322	0.0031	0.0322	0.0018	0.0322	1.6034	99%	0.1139	0.1143	0.7316	96%	0.7329	0.7281
				θ_1	0.0340	0.1341	0.0003	0.0276	0.0005	0.0275	0.0016	0.0277	1.4063	97%	0.1029	0.1023	0.7444	96%	0.7459	0.7416
				θ_2	0.0192	0.0181	0.0003	0.0041	0.0010	0.0041	0.0001	0.0041	0.4996	96.5%	0.0355	0.0369	0.2523	96%	0.2547	0.2549

Table 6: Multiple-ramp simulation of PS-ALT model for QXG under GPHC where $\alpha = 1.5, \theta_1 = 2, \theta_2 = 2$

K = 4					MLE		SE		Ent 1		Ent 2		AICI		Bootstrap		SE		Ent 1	Ent 2
n_i	p	Q	r_m, r_k		RAB	MSE	RAB	MSE	RAB	MSE	RAB	MSE	LACI	CP	LBP	LBT	LCCU	CP	LCCU	LCCU
20,15,12,10	0.2	0.6	0.5,0.7	α	0.1826	0.7156	0.0285	0.1435	0.0081	0.1461	0.0204	0.1423	2.5092	96.5%	0.0777	0.0780	1.4252	99%	1.4973	1.4571
				θ_1	0.0914	0.8077	0.0259	0.2022	0.0124	0.2038	0.0394	0.2025	3.4081	96.5%	0.1093	0.1098	1.6902	99%	1.7680	1.7377
				θ_2	0.0611	0.0864	0.0094	0.0183	0.0072	0.0185	0.0116	0.0181	1.0489	99%	0.0345	0.0338	0.5387	99%	0.5308	0.5199
			0.7,0.9	α	0.0144	0.6204	0.0060	0.0305	0.0045	0.0305	0.0095	0.0306	3.4858	96.5%	0.1143	0.1148	0.6668	96%	0.6844	0.6839
				θ_1	0.0140	0.7677	0.0032	0.0300	0.0013	0.0317	0.0052	0.0317	3.4345	96.5%	0.1082	0.1083	0.6659	96%	0.6979	0.6970
				θ_2	0.0287	0.0581	0.0022	0.0065	0.0035	0.0064	0.0049	0.0065	0.9178	96%	0.0283	0.0277	0.3068	97%	0.3134	0.3140
		0.85	0.5,0.7	α	0.1577	0.6641	0.0062	0.1387	0.0051	0.1418	0.0195	0.1389	2.0208	96.5%	0.0929	0.0928	1.4564	99%	1.4723	1.4322
				θ_1	0.0911	0.7392	0.0152	0.1930	0.0121	0.1911	0.0315	0.1965	3.2002	96.5%	0.1033	0.1035	1.6442	97%	1.7062	1.6990
				θ_2	0.0310	0.0723	0.0084	0.0172	0.0063	0.0172	0.0104	0.0172	1.0264	99%	0.0325	0.0326	0.4885	99%	0.5251	0.5141
			0.7,0.9	α	0.0089	0.6101	0.0059	0.0228	0.0035	0.0275	0.0083	0.0277	3.3046	96.5%	0.1071	0.1056	0.6324	96%	0.6496	0.6511
				θ_1	0.0125	0.7232	0.0028	0.0203	0.0012	0.0302	0.0050	0.0305	3.3294	96.5%	0.1084	0.1083	0.6854	96%	0.6805	0.6805
				θ_2	0.0209	0.0566	0.0024	0.0062	0.0017	0.0062	0.0040	0.0063	0.9095	99%	0.0316	0.0316	0.2999	96%	0.3017	0.3020
	0.9	0.6	0.5,0.7	α	0.1518	0.7625	0.0163	0.1371	0.0022	0.1387	0.0472	0.1367	3.2738	96.5%	0.1407	0.1415	1.4079	96%	1.4578	1.4241
				θ_1	0.1503	0.7637	0.0101	0.1995	0.0034	0.2028	0.0238	0.1979	3.1810	96.5%	0.1413	0.1406	1.7295	97%	1.7670	1.7356
				θ_2	0.0828	0.1155	0.0019	0.0183	0.0026	0.0188	0.0017	0.0179	1.1644	99%	0.0506	0.0533	0.5128	99%	0.5380	0.5247
			0.7,0.9	α	0.0667	0.7185	0.0045	0.0314	0.0021	0.0312	0.0061	0.0316	3.6008	96.5%	0.1618	0.1637	0.6968	96%	0.6898	0.6914
				θ_1	0.0095	0.7086	0.0041	0.0253	0.0022	0.0306	0.0060	0.0308	5.3530	96.5%	0.1427	0.2494	0.6769	96%	0.6866	0.6875
				θ_2	0.0321	0.0720	0.0017	0.0061	0.0010	0.0061	0.0014	0.0061	1.0220	99%	0.0461	0.0452	0.3099	96%	0.3075	0.3070
		0.85	0.5,0.7	α	0.0916	0.5926	0.0138	0.1306	0.0021	0.1257	0.0331	0.1254	2.7585	96.5%	0.1266	0.1274	1.5151	96%	1.5564	1.5293
				θ_1	0.0991	0.7176	0.0094	0.1901	0.0025	0.1868	0.0205	0.1950	3.0369	96.5%	0.1486	0.1477	1.6458	96%	1.6847	1.6865
				θ_2	0.0743	0.0899	0.0015	0.0169	0.0026	0.0176	0.0013	0.0164	1.0216	99%	0.0459	0.0458	0.4881	99%	0.5167	0.5020
			0.7,0.9	α	0.0366	0.4689	0.0048	0.0311	0.0020	0.0302	0.0109	0.0302	3.6046	96.5%	0.1608	0.1573	0.6692	99%	0.6988	0.6969
				θ_1	0.0085	0.6718	0.0031	0.0249	0.0019	0.0230	0.0051	0.0230	3.2967	96.5%	0.1474	0.1446	0.6939	96%	0.7000	0.7022
				θ_2	0.0304	0.0511	0.0015	0.0055	0.0005	0.0055	0.0012	0.0055	0.8287	96%	0.0369	0.0375	0.2648	96%	0.2896	0.2900
40,30,20,10	0.2	0.6	0.5,0.7	α	0.1524	0.4003	0.0062	0.1448	0.0031	0.1487	0.0328	0.1421	2.1721	96.5%	0.0981	0.0980	1.3895	99%	1.5129	1.4666
				θ_1	0.1061	0.1929	0.0305	0.1790	0.0309	0.1751	0.0623	0.1842	1.5089	99%	0.0717	0.0680	1.6048	99%	1.6140	1.6118
				θ_2	0.0342	0.0222	0.0037	0.0140	0.0053	0.0144	0.0021	0.0137	0.5197	95%	0.0235	0.0234	0.4609	96%	0.4689	0.4588
			0.7,0.9	α	0.0499	0.1837	0.0011	0.0280	0.0025	0.0279	0.0008	0.0282	1.5527	96%	0.0702	0.0692	0.6312	96%	0.6551	0.6573
				θ_1	0.0316	0.1466	0.0024	0.0270	0.0005	0.0299	0.0042	0.0299	1.5722	97%	0.0689	0.0690	0.6389	97%	0.6782	0.6777
				θ_2	0.0142	0.0189	0.0007	0.0034	0.0009	0.0041	0.0011	0.0041	0.5142	99%	0.0244	0.0239	0.2414	96%	0.2524	0.2522
		0.85	0.5,0.7	α	0.1499	0.3349	0.0058	0.1198	0.0009	0.1343	0.0305	0.1310	2.0354	96.5%	0.1016	0.0990	1.3763	99%	1.4338	1.3918
				θ_1	0.1013	0.1823	0.0303	0.1757	0.0231	0.1737	0.0541	0.1790	1.3653	96.5%	0.0736	0.0731	1.5246	96%	1.6171	1.6047
				θ_2	0.0252	0.0205	0.0027	0.0125	0.0051	0.0127	0.0020	0.0124	0.5053	96%	0.0236	0.0241	0.4571	99%	0.4904	0.4801
			0.7,0.9	α	0.0473	0.1621	0.0011	0.0260	0.0002	0.0262	0.0004	0.0258	1.5316	96%	0.0721	0.0708	0.6198	97%	0.6342	0.6304
				θ_1	0.0271	0.1257	0.0017	0.0258	0.0004	0.0219	0.0032	0.0266	1.5423	97%	0.0663	0.0659	0.6706	96%	0.6960	0.7007
				θ_2	0.0140	0.0177	0.0002	0.0039	0.0007	0.0041	0.0002	0.0041	0.5027	96%	0.0217	0.0217	0.2389	96%	0.2524	0.2516
	0.9	0.6	0.5,0.7	α	0.0511	0.3328	0.0035	0.1371	0.0102	0.1426	0.0207	0.1327	2.1636	96.5%	0.0956	0.0991	1.4370	99%	1.4807	1.4245
				θ_1	0.0412	0.2141	0.0204	0.1709	0.0193	0.1669	0.0539	0.1765	1.5730	99%	0.0705	0.0707	1.5089	97%	1.5866	1.5931
				θ_2	0.0329	0.0249	0.0105	0.0150	0.0165	0.0155	0.0132	0.0144	0.5191	96%	0.0235	0.0233	0.4514	99%	0.4715	0.4600
			0.7,0.9	α	0.0404	0.1703	0.0004	0.0290	0.0029	0.0289	0.0018	0.0291	1.6018	97%	0.0693	0.0699	0.6612	97%	0.6667	0.6677
				θ_1	0.0365	0.1344	0.0008	0.0292	0.0021	0.0292	0.0011	0.0293	1.4634	97%	0.0713	0.0711	0.6736	99%	0.6702	0.6721
				θ_2	0.0189	0.0182	0.0004	0.0041	0.0006	0.0041	0.0001	0.0041	0.5375	96.5%	0.0234	0.0234	0.2477	96%	0.2518	0.2518
		0.85	0.5,0.7	α	0.0302	0.3042	0.0025	0.1292	0.0100	0.1284	0.0202	0.1290	2.0179	96.5%	0.0971	0.0951	1.4156	97%	1.5604	1.5101
				θ_1	0.0409	0.2038	0.0203	0.1597	0.0101	0.1483	0.0452	0.1598	1.4896	99%	0.0762	0.0741	1.6557	99%	1.7401	1.7079
				θ_2	0.0296	0.0229	0.0102	0.0137	0.0135	0.0141	0.0103	0.0128	0.4571	96%	0.0255	0.0261	0.4610	96%	0.4869	0.4783
			0.7,0.9	α	0.0261	0.1529	0.0002	0.0290	0.0023	0.0289	0.0016	0.0291	1.4922	96%	0.0684	0.0692	0.6791	96%	0.6671	0.6691
				θ_1	0.0299	0.1324	0.0003	0.0268	0.0002	0.0210	0.0010	0.0271	1.4471	96%	0.0636	0.0636	0.6633	96%	0.6786	0.6762
				θ_2	0.0121	0.0179	0.0002	0.0038	0.0005	0.0038	0.0001	0.0038	0.4972	99%	0.0240	0.0229	0.2410	97%	0.2414	0.2411

The entire study is carried out for the first ramp-stress is Small sample: $(n_1 = 20)$, $(n_2 = 15)$, large sample: $(n_1 = 60)$, $(n_2 = 40)$; for the second ramp-stress is Small sample: $(n_1 = 20)$, $(n_2 = 15)$, $(n_3 = 12)$, $(n_4 = 10)$, large sample: $(n_1 = 40)$, $(n_2 = 30)$, $(n_3 = 20)$, $(n_4 = 10)$ and the ratio of censored sample $m_i = r_m * n_i$, and $k_i = r_k * n_i$. The resulting MLE and Bayesian estimates, along with the ACI and HPD interval lengths, are presented in Tables 3 and 4 for each of the two sets of (n, m) values. The RAB, and MSE are shown in brackets beneath the estimated values, while the coverage probabilities are displayed in brackets beneath the interval lengths. All results are based on 1000 simulation iterations.

The results shown in Tables 1–6 indicate that Bayes estimates of $\alpha, \theta_1, \theta_2$, using an informative gamma prior, outperform MLE estimates in terms of RAB, and MSEs across all two censoring schemes and two different ramp stress ALTs. However, Bayes estimates of parameters α, θ_1 and θ_2 based on asymmetric entropy loss function generally perform better than MLE in all cases. Comparing the results for the two sample size sets (small and large), it is evident that the estimation methods point estimates for large sample size perform better across all three ramp-stress ALTs.

Regarding the confidence intervals, Bayesian credible intervals for α, θ_1 and θ, under ramp-stress ALTs with two or four stress levels, exhibit shorter lengths compared to asymptotic intervals. Similarly, Bayesian intervals for α, θ_1 and θ_2 outperform both boot-p and boot-t intervals. In terms of coverage probabilities (CP), MLE and Bayesian intervals for α, θ_1 and θ_2 consistently achieve probabilities above the nominal level of 95%.

For comparison, consider the censoring scheme using a binomial removal, where items are withdrawn immediately after the first failure, vs. predicted values for items withdrawn at the r-th failure. The predicted estimates under the first censoring ($p = 0.2$) scheme are smaller than those under the second censoring scheme ($p = 0.9$). In ramp-stress ALTs with two stress levels, higher stress levels correspond to smaller predicted estimates, while lower stress levels lead to shorter lifetime values, indicating that higher stress levels result in earlier failure. Similar trends are observed in ramp-stress ALTs with four stress levels.

6 Application

To illustrate the proposed methodologies, we analyze an accelerated life test data set from [59,60].

First data set: The data presented in Chapter 5 of [59] were obtained from ramp-voltage tests conducted on miniature light bulbs designed to operate at a stress (voltage) of 2 V. In these tests, 62 miniature light bulbs were tested at a ramp rate of 2.01 V/h, while 61 bulbs were tested at a ramp rate of 2.015 V/h. The lifetime data from these ramp-voltage tests are provided by [36].

Second data includes failure times for the time-dependent dielectric breakdown of metal-oxide-semiconductor integrated circuits. The test was conducted at three elevated temperatures: 170°C, 200°C, and 250°C. For this study, we focus on the data collected at 170°C and 200°C.

To evaluate the suitability of the data for a QXG distribution, we calculate AIC, BIC, the Kolmogorov-Smirnov distance (KSD), and the corresponding p-values for each stress level. These results, along with the each data set, are presented in Tables 7 and 8, respectively. The QXG distribution is contrasted with four rival models, including the log-logistic (LogLog) [38], exponentiated Weibull (EW) [14], extended exponential (ExEx) distribution [36], and the logistic exponential (LogEx) distribution [61]. These models are evaluated using statistical measures such as AIC, BIC, KSD, and p-values.

Table 7: MLE estimates of parameters for competing models and some goodness of fit measures: Data I

		α	θ_1	θ_2	δ	**AIC**	**BIC**	**KSD**	**PVKS**
QXG	$t_{2.01V/h}$	0.6171	0.00006740	1.5200		429.2446	434.8582	0.1127	0.5380
		0.0673	0.00000001	0.0547					
	$t_{2.015V/h}$	0.7832	0.00003158	1.8880		390.3940	396.0076	0.1283	0.4079
		0.0515	0.00003249	0.2362					
ExEx	$t_{2.01V/h}$	0.00001230	1.3342	1.6067		434.5863	440.1999	0.1221	0.4371
		7.55989 E-11	0.1053	0.0138					
	$t_{2.015V/h}$	0.00001057	1.7046	1.8014		392.4747	398.0883	0.1515	0.2207
		1.06373 E-11	0.4419	0.0133					
LogLog	$t_{2.01V/h}$	0.5499	0.00000178	2.1918		475.2910	480.9046	0.3173	0.0001
		0.0052	0.00000000	0.0327					

(Continued)

Table 7 (continued)

		α	θ_1	θ_2	δ	AIC	BIC	KSD	PVKS
	$t_{2.015V/h}$	4.8161	0.05351258	0.2135		410.2181	415.8317	0.1627	0.1576
		55.8147	0.03893432	1.4751					
EW	$t_{2.01V/h}$	0.2833	0.00165428	1.2577	0.8761	563.6301	571.1149	0.6252	0.0000
		0.0016	0.00000020	0.0197	0.0160				
	$t_{2.015V/h}$	2.2613	0.00001072	1.8285	0.3807	390.4715	397.8563	0.1287	0.4045
		0.5060	0.00000000	0.0017	0.0253				
LogEx	$t_{2.01V/h}$	0.9467	0.00001026	1.7462		435.7660	441.3796	0.1426	0.2579
		0.0178	0.00000000	0.0186					
	$t_{2.015V/h}$	0.9558	0.00000894	1.9933		395.9844	401.5980	0.1254	0.4372
		0.0163	0.00000000	0.0265					

Table 8: MLE estimates of parameters for competing models and some goodness of fit measures: Data II

		α	θ_1	θ_2	δ	AIC	BIC	KSD	PVKS
QXG	t_{170C}	2.0911	0.00001696	0.4221		234.0681	236.1923	0.1783	0.7272
		4.4453	1.53758E-10	0.0034					
	t_{200C}	3.2062	0.00008580	0.2422		320.8534	323.8406	0.1117	0.9407
		1.2596	0.00044223	0.4049					
ExEx	t_{170C}	0.00000901	0.6000	0.4989		235.2229	237.3470	0.1796	0.7203
		2.42826E-10	0.0577	0.0148					
	t_{200C}	0.00001442	0.5327	0.4509		321.4052	323.5293	0.1141	0.9361
		1.14738E-09	0.0353	0.0375					
LogLog	t_{170C}	0.1438	0.00326368	0.1537		293.5487	295.6729	0.5289	0.0005
		0.0015	3.65859E-05	0.0311					
	t_{200C}	1.0250	0.00000679	0.4754		323.6021	325.7263	0.1161	0.9304
		0.0364	6.49674E-11	0.0084					
EW	t_{170C}	0.1585	0.00055059	0.1111	0.4997	295.4814	298.3136	0.7435	0.0000
		0.0018	0.00000047	0.0180	0.0180				
	t_{200C}	0.4478	0.00001120	0.5273	1.4218	331.6423	334.4745	0.3331	0.0177
		0.0343	2.10565E-11	0.0110	0.0539				
LogEx	t_{170C}	0.9643	0.00001652	0.3794		234.7455	236.8696	0.1652	0.8078
		0.0483	2.72171E-10	0.0072					
	t_{200C}	0.9439	0.00005045	0.2552		321.2075	323.3316	0.1193	0.9067
		0.0018	2.24688E-08	0.0660					

Tables 7 and 8 identify the best-fitting model as the one with the lowest AIC, BIC, and KSD values, combined with the highest p-values, indicating a superior fit to the QXG distribution.

Fig. 3 displays the likelihood profiles for the parameters α, θ_1, and θ_2 of the QXG model based on PSALT using data II. Each plot shows the log-likelihood ($l(\alpha, \theta_1, \theta_2)$) as a function of one parameter, with the others fixed at their optimal values. The red dots represent the MLEs, with the profiles exhibiting distinct peaks, indicating well-defined and identifiable parameter estimates.

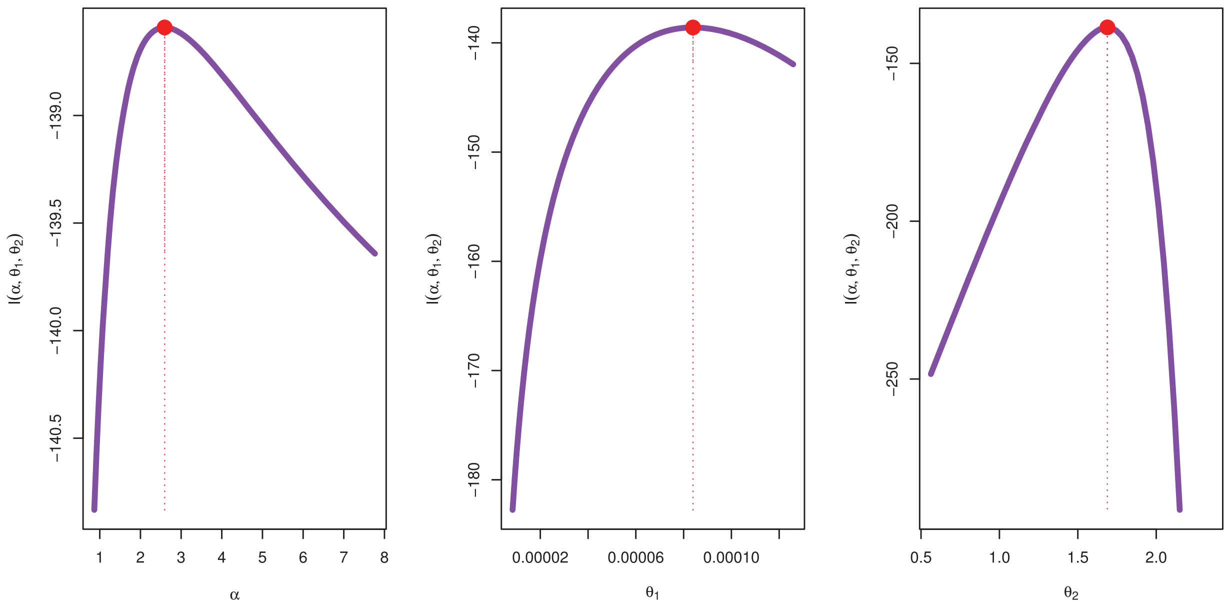

Figure 3: Likelihood profile for parameters of QXG based on PSALT with Data I

The results presented in Tables 9 and 10 summarize the MLE of parameters for various models under the PSALT framework, evaluated for two distinct datasets: Data I and Data II. The models compared include QXG, ExEx, LogLog, EW, and LogEx, with model performance assessed using the AIC and BIC. Lower values of AIC and BIC generally indicate better model fit for QXG under the PSALT model.

Table 9: MLE of parameters based on PSALT model for Data I

		α	θ_1	θ_2	δ	**AIC**	**BIC**
QXG	Estimates	0.915682	0.000196	1.336384		841.0324	846.6460
	StEr	0.005414	7.25491E-08	0.086160			
ExEx	Estimates	0.000003	1.714760	1.838152		885.5127	891.1263
	StEr	0.000000	0.033201	0.020711			
LogLog	Estimates	1.469887	0.000190	1.261741		867.3418	872.9554
	StEr	0.283287	0.000000	0.739859			
EW	Estimates	2.801877	0.000005	1.756125	0.229103	852.7938	863.0511
	StEr	0.000536	0.000000	0.003890	0.000702		
LogEx	Estimates	0.852398	0.000012	1.806744		844.3204	852.0134
	StEr	0.006461	0.000000	0.013624			

Table 10: MLE of parameters based on PSALT model for Data II

		α	θ_1	θ_2	δ	**AIC**	**BIC**
QXG	Estimates	2.770903	0.000077	0.266091		550.5654	555.2314
	StEr	11.327614	1.17006E-08	0.011216			
ExEx	Estimates	0.000012	0.582137	0.472000		551.0718	555.7378

(Continued)

Table 10 (continued)

		α	θ_1	θ_2	δ	AIC	BIC
	StEr	0.000000	0.048746	0.001812			
LogLog	Estimates	1.288557	0.000022	0.391893		553.8865	558.5525
	StEr	0.029345	0.000000	0.005007			
EW	Estimates	0.456682	0.000003	0.566157	1.419044	559.7851	566.0065
	StEr	0.005822	0.000000	0.000549	0.147713		
LogEx	Estimates	0.973619	0.000057	0.256971		551.1106	555.7766
	StEr	0.019381	0.000000	0.006786			

For Data I (Table 9), the QXG model emerges as the best-fitting model, with the lowest AIC (841.0324) and BIC (846.6460). The LogEx model performs competitively, but its AIC (844.3204) and BIC (852.0134) remain slightly higher than QXG. The ExEx model performs the worst, as evidenced by its significantly higher AIC (885.5127) and BIC (891.1263). The LogLog and EW models provide moderate fits, but neither outperforms QXG. Notably, the parameter estimates under QXG show relatively small standard errors, indicating precise and reliable estimates.

For Data II (Table 10), the QXG model again demonstrates superior performance, achieving the lowest AIC (550.5654) and BIC (555.2314). The ExEx and LogEx models follow closely, with AIC and BIC values that are only slightly higher, suggesting these models also offer reasonable fits for Data II. The EW model, despite its added parameter (δ), performs the worst, with the highest AIC (559.7851) and BIC (566.0065). As with Data I, the parameter estimates under QXG for Data II show small standard errors, indicating precision and stability in the estimation process.

Table 11 presents progressively censored data for a specific dataset of amp-voltage tests conducted on miniature light bulbs designed to operate at a stress. The table shows different censoring schemes defined by k_1, k_2, and censoring probability (p). For each scheme, the table lists the observed failure times (x_1) and the censored observations (R, where $R > 0$ indicates a censored observation). This data is likely used to study the effects of different censoring schemes on statistical analyzes, such as parameter estimation and survival analysis.

Table 11: Create progressive censored samples for Data I

k_1, k_2	p		Censored data
15,18	0.2	x_1	13.57 14.51 15.61 15.85 17.73 19.65 19.92 21.05 31.16 34.00 42.06 46.26 46.41 50.60 56.43
			60.13 62.48 62.81 65.00 65.86 66.15 66.41 69.91 72.60 75.43 75.85 76.20 77.78 82.65 90.33
		R	4 2 4 3 0 1 2 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
		x_2	8.85 11.31 11.83 14.50 14.83 17.73 19.30 26.50 27.42 28.62 30.62 34.42 34.81 35.30 38.30
			40.52 43.00 43.08 43.83 45.63 46.33 50.66 50.96 51.03 51.27 52.36 53.25 54.25 55.47 56.83
		R	1 3 5 3 2 1 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

(Continued)

Table 11 (continued)

k_1, k_2	p		Censored data
		x_1	13.57 14.51 15.61 15.85 17.73 19.65 21.05 34.00 42.06 46.26 47.88 50.60 54.21 54.55 55.85
	0.9		56.43 56.76 56.85 58.86 60.13 64.18 65.86 66.15 66.20 69.91 72.60 75.43 75.85 80.65 82.65
		R	1 16 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
		x_2	8.85 11.31 11.83 14.50 14.83 17.73 19.30 23.50 27.42 30.62 31.00 35.30 35.48 36.03 38.30
			40.52 43.00 43.83 45.63 46.03 49.62 49.65 49.86 50.66 51.27 51.73 51.95 55.47 56.17 56.83
		R	4 13 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
		x_1	13.57 14.51 15.61 15.85 17.73 19.65 19.92 21.05 31.16 34.00 42.06 46.26 46.41 50.60 56.43
	0.2		60.13 62.48 62.81 65.00 65.86 66.15 66.41 69.91 72.60 75.43 75.85 76.20 77.78 82.65 90.33
		R	4 2 4 3 0 1 2 0 0 1 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
25,25		x_2	8.85 11.31 11.83 14.50 14.83 17.73 19.30 26.50 27.42 28.62 30.62 34.42 34.81 35.30 38.30
			40.52 43.00 43.08 43.83 45.63 46.33 50.66 50.96 51.03 51.27 52.36 53.25 54.25 55.47 56.83
		R	1 3 5 3 2 1 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
		x_1	13.57 14.51 15.61 15.85 17.73 19.65 21.05 34.00 42.06 46.26 47.88 50.60 54.21 54.55 55.85
	0.9		56.43 56.76 56.85 58.86 60.13 64.18 65.86 66.15 66.20 69.91 72.60 75.43 75.85 80.65 82.65
		R	1 16 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
		x_2	8.85 11.31 11.83 14.50 14.83 17.73 19.30 23.50 27.42 30.62 31.00 35.30 35.48 36.03 38.30
			40.52 43.00 43.83 45.63 46.03 49.62 49.65 49.86 50.66 51.27 51.73 51.95 55.47 56.17 56.83
		R	4 13 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0

Figs. 4 and 5 show the MCMC diagnostic plots for the parameters α, θ_1, and θ_2 of the QXG model using PSALT under GPHCS conditions ($P = 0.2$, $k_1 = 15$, $k_2 = 18$), and $k_1 = 25$, $k_2 = 25$) with data I. Each row corresponds to one parameter and includes a trace plot (left), running mean plot (middle), and posterior density plot (right). The trace plots indicate good mixing, as the chains fluctuate around stable means without visible trends, while the running mean plots demonstrate convergence by stabilizing over iterations. The posterior density plots show well-defined unimodal distributions, highlighting the reliability of the parameter estimates. These diagnostics confirm that the MCMC sampling converged effectively, providing robust posterior samples for α, θ_1, and θ_2.

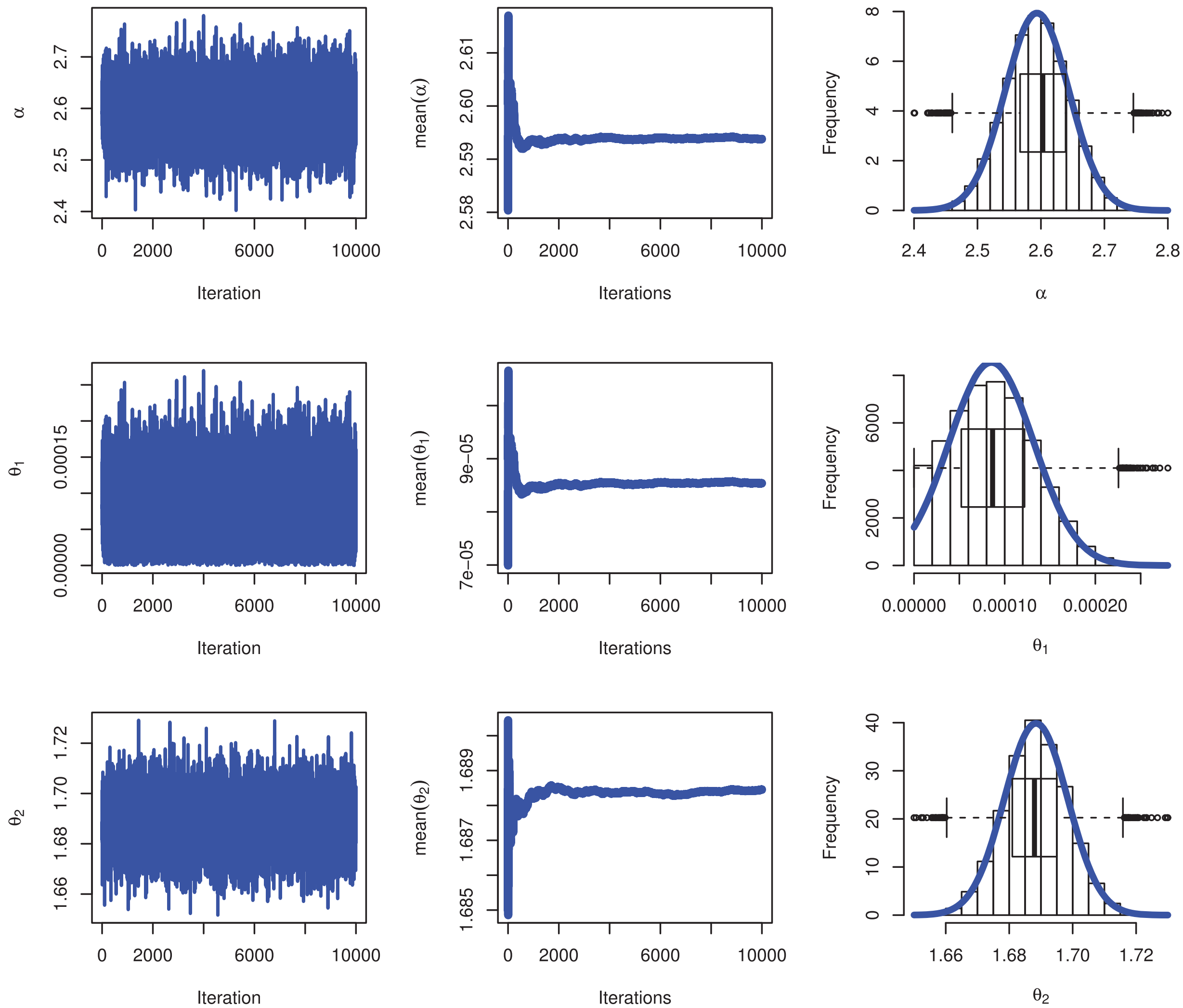

Figure 4: MCMC plots for parameters of QXG based on PSALT under GPHCS $(P = 0.2, k_1 = 15, k_2 = 18)$ with Data I

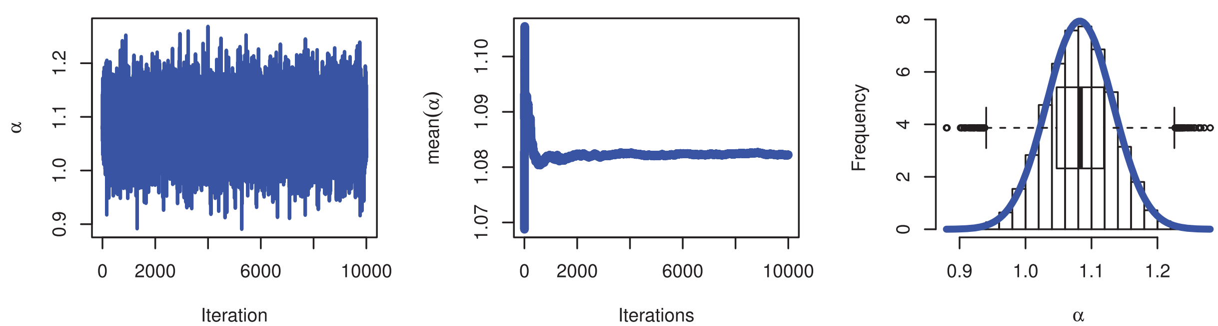

Figure 5: (Contiuned)

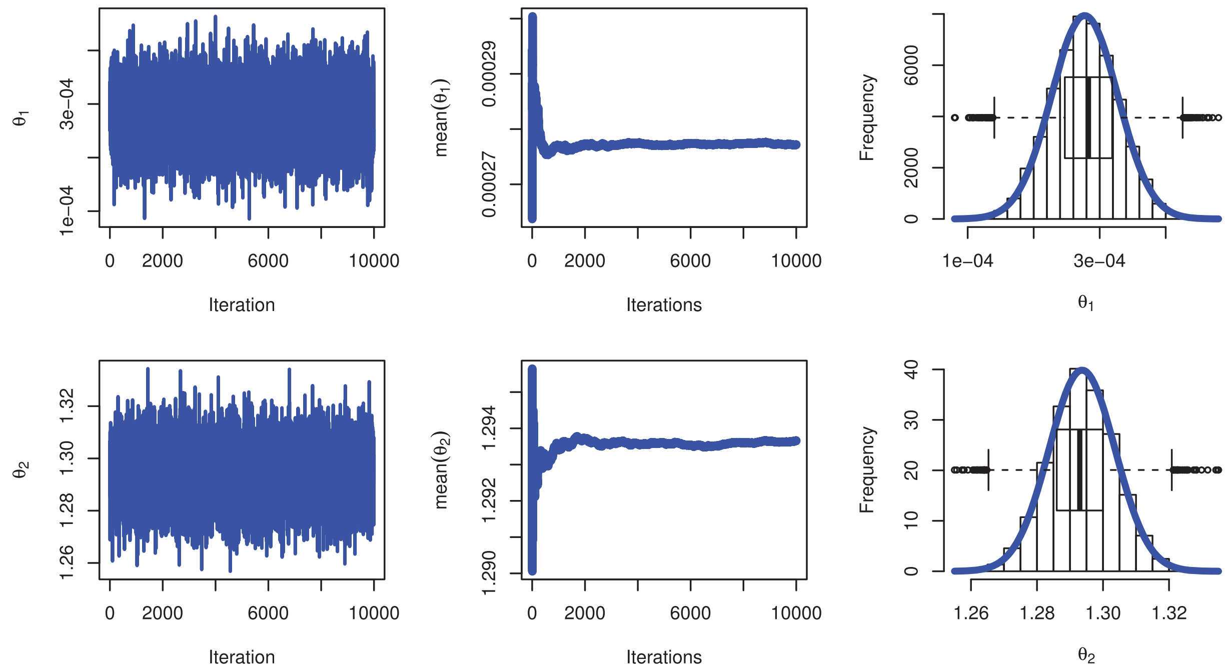

Figure 5: MCMC plots for parameters of QXG based on PSALT under GPHCS $(P = 0.2, k_1 = 25, k_2 = 25)$ with Data I

Table 12 presented the results of parameter estimation for a dataset (Data I) under different stress levels, using both MLE and Bayesian methods with varying loss functions (SE, Ent 1, Ent 2). The table shows estimates for parameters $\alpha, \theta_1, \theta_2, S_1$, and S_2 under different experimental conditions, allowing for a comparison of the estimation methods and their sensitivity to different loss functions.

Table 12: MLE, Baesian estimation with different loss function for parameters and survival of stress level: Data I

T	m_1, m_2	k_1, k_2	p		MLE	SE	Ent 1	Ent 2
57, 42	30, 30	15, 18	0.2	α	2.59436	2.59379	2.59442	2.59316
				θ_1	0.00008393	0.00008539	0.00008539	0.00008539
				θ_2	1.68847	1.68846	1.68848	1.68843
				S_1	0.03111	0.02908	0.02906	0.02910
				S_2	0.23010	0.22439	0.22433	0.22445
			0.9	α	1.62353	1.62296	1.62359	1.62233
				θ_1	0.00014055	0.00014007	0.00014007	0.00014007
				θ_2	1.54667	1.54666	1.54668	1.54663
				S_1	0.05889	0.05958	0.05955	0.05961
				S_2	0.30254	0.30388	0.30379	0.30396
57, 42	30, 30	25, 25	0.2	α	1.08278	1.08221	1.08284	1.08158
				θ_1	0.00027774	0.00027717	0.00027717	0.00027717
				θ_2	1.29367	1.29366	1.29368	1.29363
				S_1	0.20575	0.20662	0.20652	0.20671
				S_2	0.48459	0.48544	0.48531	0.48556

(Continued)

Table 12 (continued)

T	m_1, m_2	k_1, k_2	p		MLE	SE	Ent 1	Ent 2
			0.9	α	0.95146	0.95089	0.95152	0.95025
				θ_1	0.00015575	0.00015521	0.00015521	0.00015521
				θ_2	1.45850	1.45848	1.45851	1.45846
				S_1	0.16461	0.16596	0.16587	0.16604
				S_2	0.47194	0.47337	0.47324	0.47351

Fig. 6 illustrates the likelihood profiles for the parameters α, θ_1, and θ_2 of the QXG model based on PSALT under GPHCS conditions ($P = 0.2$, $k_1 = 25$, $k_2 = 25$) using data I. Each plot represents the log-likelihood ($l(\alpha, \theta_1, \theta_2)$) as a function of one parameter while keeping the others fixed at their optimal values. The red dots mark the MLEs, corresponding to the parameter values that maximize the log-likelihood. The profiles show clear peaks, indicating well-defined MLEs for all three parameters, and the smooth curvature suggests good identifiability of the parameters within the model.

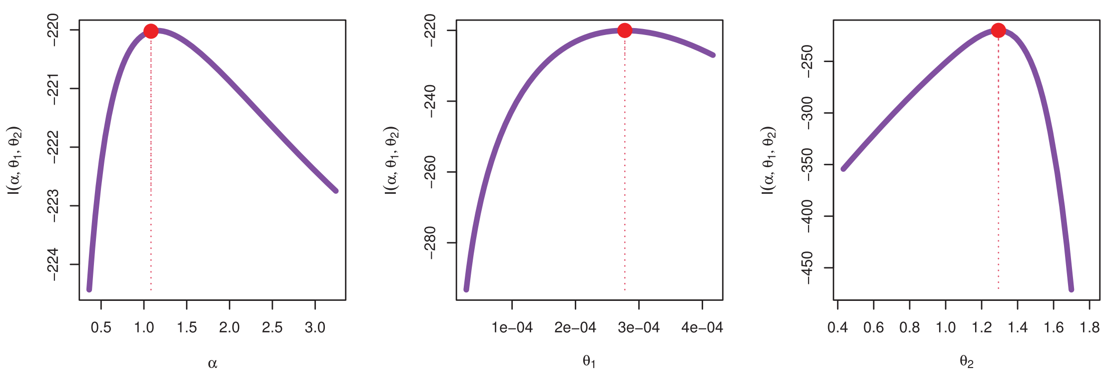

Figure 6: Likelihood profile for parameters of QXG based on PSALT under GPHCS ($P = 0.2, k_1 = 25, k_2 = 25$) with Data I

Table 13 presents progressively censored time-dependent dielectric breakdown of metal oxide-semiconductor integrated circuits data, temperatures 170°C and 200°C, censoring schemes, and their impact on statistical analysis. Table 14 discussed the results of parameter estimation for a dataset (Data II) under different stress levels (T_1 and T_2), censoring schemes, and estimation methods. It compares MLE with Bayesian estimation using various loss functions (SE, Ent 1, Ent 2) with values of survival (S_1 and S_2). The table shows estimates for parameters α, θ_1, and θ_2. By analyzing these results, researchers can assess the impact of stress levels, censoring, and the choice of loss function on the accuracy and precision of parameter estimates, and compare the performance of MLE and Bayesian approaches.

Table 13: Create progressive censored samples for Data II

m_1, m_2	p		Censored data
10, 12	0.2	x_1	164 164 218 230 467 538 639 1148 1678 1678
		R	2 0 1 1 0 0 1 0 0 0
		x_2	76 82 210 385 412 491 504 522 775 1131 1446 3077

(Continued)

Table 13 (continued)

m_1, m_2	p		Censored data
		R	1 3 1 1 0 0 1 0 0 0 1 0
	0.9	x_1	164 164 218 263 467 669 1148 1678 1678 1678
		R	0 4 1 0 0 0 0 0 0 0
		x_2	76 82 210 315 412 504 522 646 775 2248 2385 3077
		R	1 7 0 0 0 0 0 0 0 0 0 0
13, 16	0.2	x_1	164 164 218 230 263 639 669 917 1148 1678 1678 1678 1678
		R	1 0 1 0 0 0 0 0 0 0 0 0 0
		x_2	76 82 210 315 385 412 491 504 646 775 884 1131 1824 2248 2385 3077
		R	0 1 1 0 0 1 0 0 0 0 1 0 0 0 0 0
	0.9	x_1	164 164 218 263 467 538 669 917 1148 1678 1678 1678 1678
		R	0 2 0 0 0 0 0 0 0 0 0 0 0
		x_2	76 82 210 315 385 504 522 646 678 1131 1446 1824 1827 2248 2385 3077
		R	0 3 0 1 0 0 0 0 0 0 0 0 0 0 0 0

Table 14: MLE, Baesian estimation with different loss function for parameters and survival of stress level: Data II

T_1, T_2	m_1, m_2	k_1, k_2	p		MLE	SE	Ent 1	Ent 2
800, 995	10, 12	8, 10	0.2	α	3.69607	3.69240	3.71822	3.66652
				θ_1	0.00000070	0.00000070	0.00000070	0.00000070
				θ_2	0.74901	0.74877	0.75879	0.73877
				S_1	0.21292	0.21423	0.17912	0.25111
				S_2	0.07918	0.08004	0.05790	0.10617
			0.9	α	8.54058	8.53691	8.56273	8.51103
				θ_1	0.00005563	0.00005556	0.00005556	0.00005556
				θ_2	0.31529	0.31526	0.31551	0.31501
				S_1	0.30668	0.30724	0.30613	0.30835
				S_2	0.19639	0.19689	0.19587	0.19790
800, 995	13, 16	11, 12	0.2	α	2.91956	2.91590	2.94172	2.89002
				θ_1	0.00001040	0.00001039	0.00001039	0.00001039
				θ_2	0.48328	0.48322	0.48380	0.48263
				S_1	0.32975	0.33051	0.32730	0.33375
				S_2	0.19515	0.19583	0.19296	0.19873
			0.9	α	3.16147	3.15781	3.18362	3.13192
				θ_1	0.00002194	0.00002192	0.00002192	0.00002192
				θ_2	0.40419	0.40414	0.40455	0.40373
				S_1	0.36722	0.36793	0.36549	0.37037
				S_2	0.24061	0.24128	0.23896	0.24361

Tables 12 and 13 present the application of the proposed PSALT model under the Generalized Progressive Hybrid Censoring (GPHC) scheme to two distinct real-world datasets. These applications are intended to complement the simulation study by demonstrating the practical utility and flexibility of the proposed methodology. The analysis includes model fitting comparisons using criteria such as AIC, BIC, and KSD, confirming that the quasi Xgamma distribution under GPHC performs well relative to alternative models. These results provide empirical support for the robustness of the proposed approach in real-life accelerated life testing scenarios, as highlighted in the reviewer's comment.

Figs. 7 and 8 present the likelihood profiles for the parameters α, θ_1, and θ_2 of the QXG model, derived from PSALT using Data II. Each graph illustrates the log-likelihood ($l(\alpha, \theta_1, \theta_2)$) as a function of a single parameter, while the remaining parameters are held constant at their optimal values. The red dots indicate the MLEs, with the profiles showing clear peaks, demonstrating that the parameter estimates are well-defined and identifiable. Figs. 9 and 10 present MCMC diagnostic plots for parameters of QXG based on PSALT under GPHCS with specific parameter values. Each row corresponds to a different parameter, displaying its trace plot (left), running mean plot (middle), and posterior distribution (right). The trace plots indicate good mixing, while the running mean plots show convergence to a stable value. The posterior distributions are well-defined, suggesting effective sampling. This analysis validates the robustness of the MCMC process in estimating the model parameters using data II.

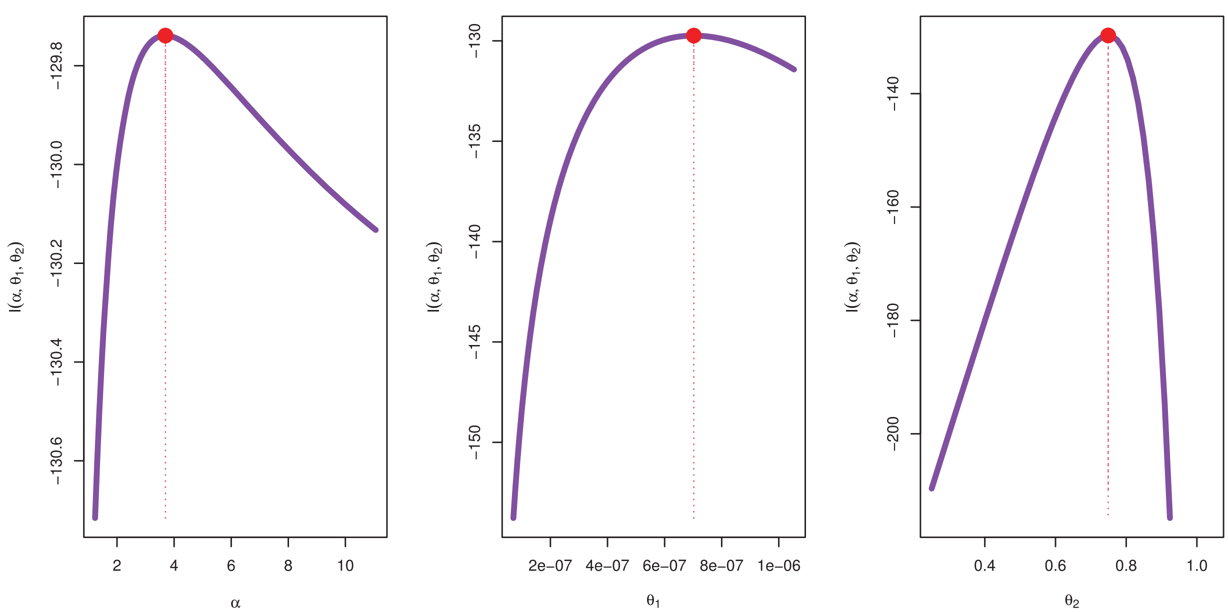

Figure 7: Likelihood profile for parameters of QXG based on PSALT under GPHCS ($P = 0.2, m_1 = 10, m_2 = 12$) with Data II

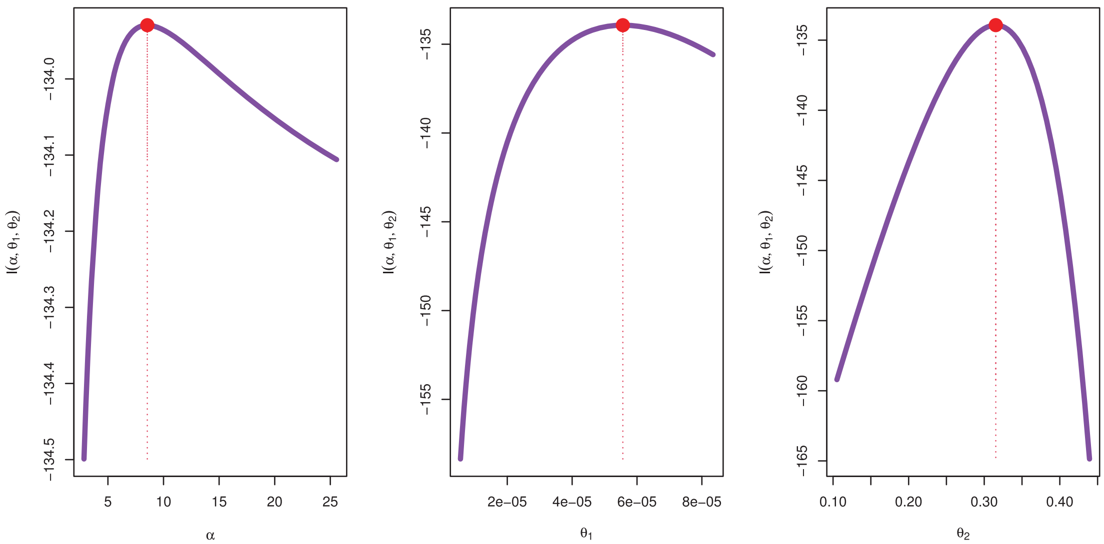

Figure 8: Likelihood profile for parameters of QXG based on PSALT under GPHCS $(P = 0.9, m_1 = 10, m_2 = 12)$ with Data II

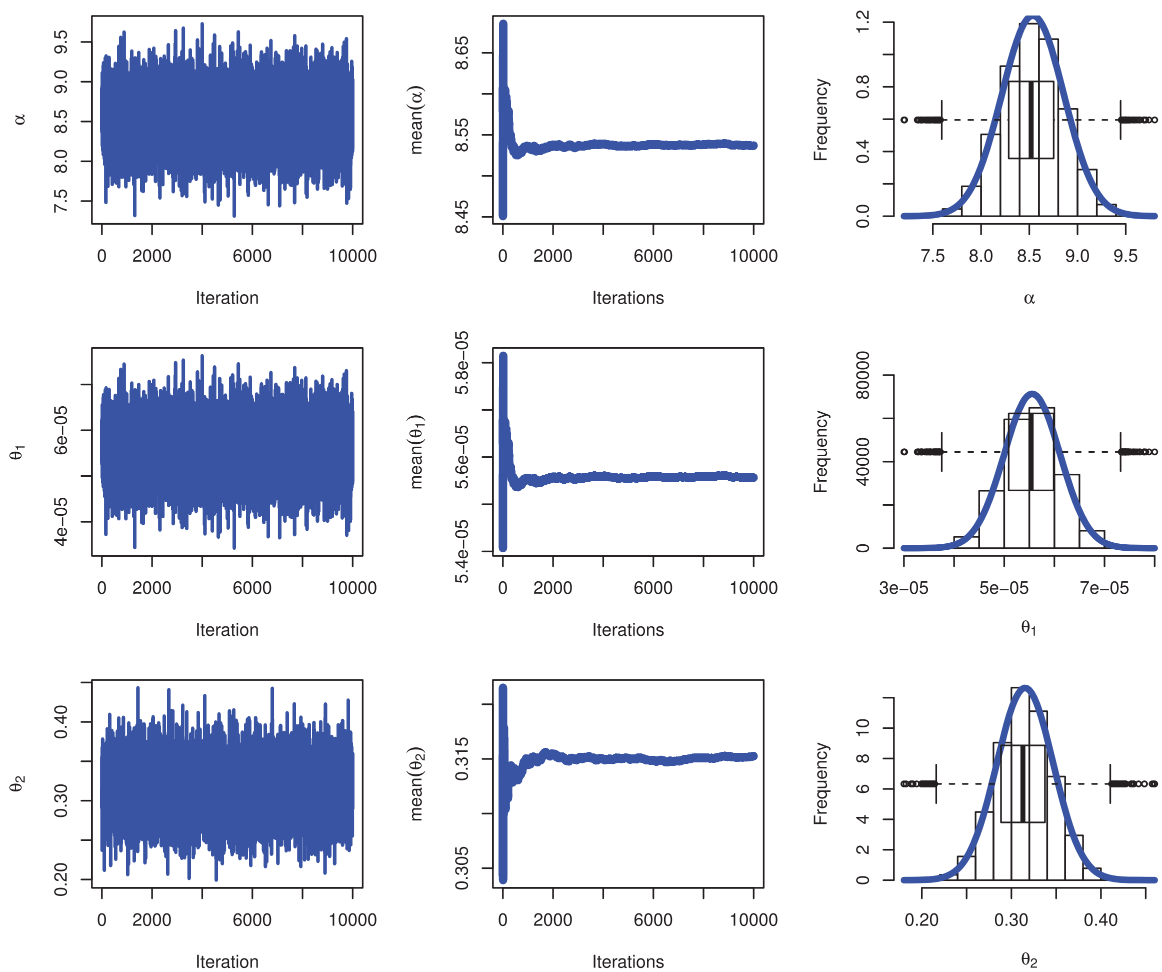

Figure 9: MCMC plots for parameters of QXG based on PSALT under GPHCS $(P = 0.2, m_1 = 10, m_2 = 12)$ with Data II

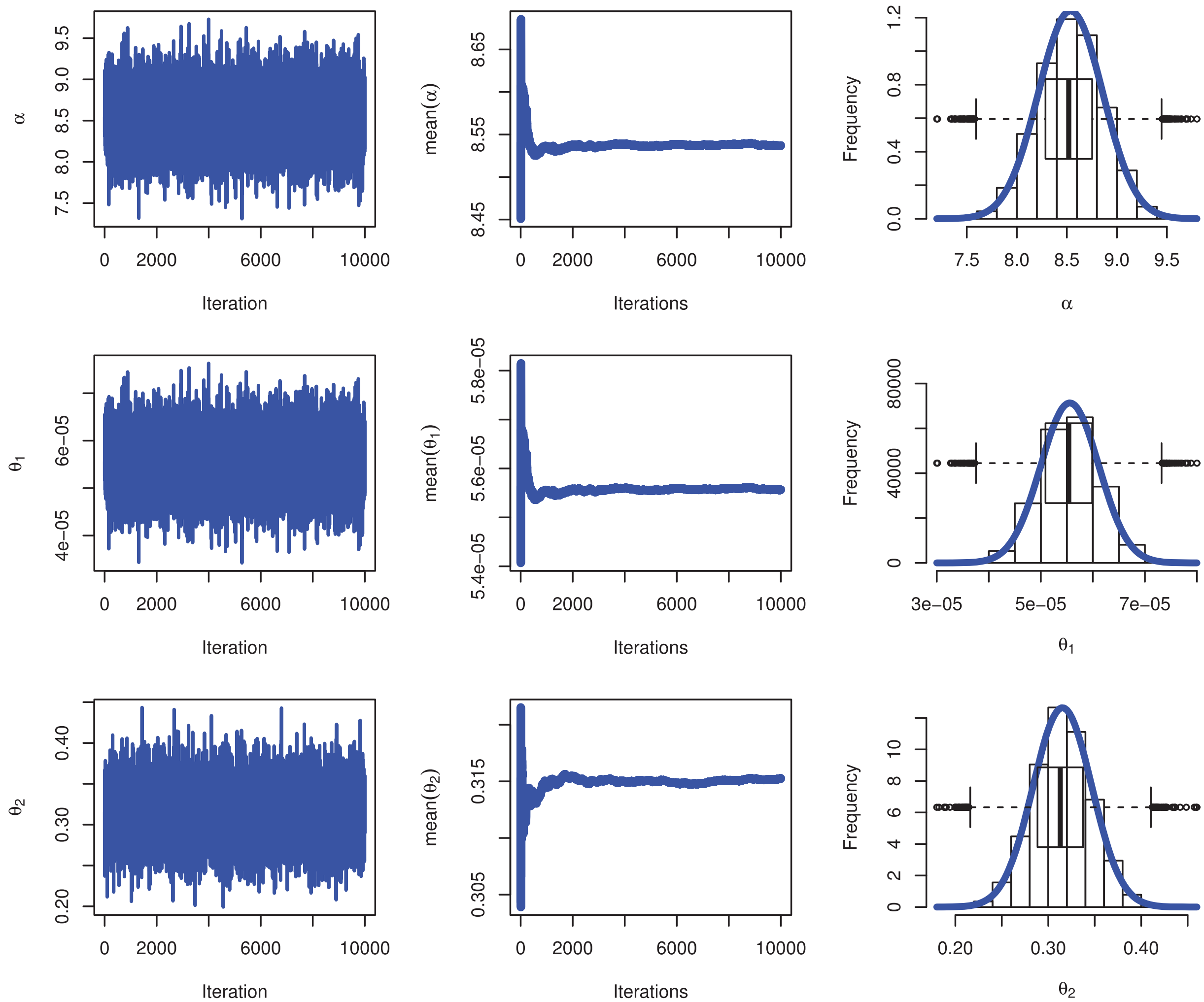

Figure 10: MCMC plots for parameters of QXG based on PSALT under GPHCS $(P = 0.9, m_1 = 10, m_2 = 12)$ with Data II

7 Conclusion

This paper investigates progressive-stress ALT for the lifetime quasi xgamma distribution under a generalized progressive hybrid censoring scheme, demonstrating the proposed methods through a practical example. MLEs and Bayesian estimates of the parameters were derived for various sample sizes and two binomial parameter schemes for random removal, with their performance assessed through MSEs. Confidence intervals based on the asymptotic distribution of the MLEs were compared with those derived from posterior distributions. We constructed various confidence intervals, including asymptotic, boot-p, boot-t, and Bayesian intervals, within the PSALT framework. The simulation results suggest that Bayesian estimates generally outperform their classical counterparts, with p-boot intervals showing satisfactory coverage probabilities. The simulation study results indicate that Bayesian point estimates and HPD credible intervals outperform classical point estimates and confidence intervals with bootstrap. This trend was similarly reflected in two real-life data analyses. The QXG distribution based on progressive-stress ALT is compared to four competing models: log-logistic, exponentiated Weibull, extended exponential, and logistic exponential distributions based on progressive-stress ALT. The comparison, based on statistical measures such as AIC, BIC, KSD, and p-values, demonstrates that the QXG distribution based on progressive-stress ALT outperforms the rival models.

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