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ARTICLE

Fuzzy N-Bipolar Soft Sets for Multi-Criteria Decision-Making: Theory and Application

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ABSTRACT: This paper introduces fuzzy N-bipolar soft (FN-BS) sets, a novel mathematical framework designed to enhance multi-criteria decision-making (MCDM) processes under uncertainty. The study addresses a significant limitation in existing models by unifying fuzzy logic, the consideration of bipolarity, and the ability to evaluate attributes on a multinary scale. The specific contributions of the FN-BS framework include: (1) a formal definition and set-theoretic foundation, (2) the development of two innovative algorithms for solving decision-making (DM) problems, and (3) a comparative analysis demonstrating its superiority over established models. The proposed framework is applied to a real-world case study on selecting vaccination programs across multiple countries, showcasing consistent DM outcomes and exceptional adaptability to complex and uncertain scenarios. These results position FN-BS sets as a versatile and powerful tool for addressing dynamic DM challenges.

KEYWORDS: Fuzzy N-bipolar soft sets; N-bipolar soft sets; N-soft sets; MCDM; algorithms

1 Introduction

Handling uncertainty and imprecision in data has been a fundamental challenge in various scientific and engineering disciplines. Several mathematical frameworks have been proposed to address these challenges, each offering unique perspectives. Among them, Zadeh's fuzzy set theory [1] stands out as a pioneering model for representing vagueness. Fuzzy sets allow for degrees of membership rather than binary classification, enabling a nuanced representation of imprecise information. This framework has found applications in control systems, decision-making (DM), and image processing, among others. On the other hand, Pawlak's rough set theory [2] addresses uncertainty arising from indiscernibility or incomplete information. By employing equivalence classes, rough sets approximate imprecise concepts using lower and upper approximations. These models have been instrumental in areas such as feature selection, data mining, and knowledge discovery. Other advancements, such as intuitionistic fuzzy sets [3] and vague sets [4], have further enriched the landscape, each tailored to address specific types of uncertainties.

S-set theory, introduced by Molodtsov [5], provides a parameterized framework for addressing uncertainty in a versatile and straightforward manner. Unlike fuzzy sets and rough sets, which operate within fixed mathematical structures, S-sets pair objects with parameters to model problems flexibly. This approach has been extended in various ways to enhance its applicability. Fuzzy soft (FS) sets, proposed by Maji et al. [6], integrate the concept of fuzzy membership functions into the S-set framework, allowing for the representation of partial truths associated with each parameter. Other extensions include interval-valued FS sets [7], and Einstein q-rung orthopair FS sets [8], which expand the utility of S-sets in diverse applications such as DM [9] and medical diagnosis [10].

The initial work on operations of S-set theory by Maji et al. [11] laid a strong foundation, enabling applications across various domains. Ali et al. [12] further defined several new operations on S-sets. Shabir et al. extended this theory to introduce bipolar soft (BS) sets [13], emphasizing scenarios where attributes can simultaneously express positive and negative aspects. Naz et al. [14] expanded the concept by developing fuzzy BS (FBS) sets, a model incorporating fuzziness and bipolarity to address algebraic structures and real-world problems. These advancements underscored the flexibility and applicability of S-set theory in handling uncertainties. Recent developments have integrated S-set theory with DM strategies to tackle multi-criteria problems. For example, Musa et al. [15] introduced the concept of bipolar hypersoft sets. In a subsequent study [16], they utilized this model in DM applications, demonstrating its practical relevance and effectiveness. Asaad et al. [17] extended this framework to incorporate fuzzy evaluations. Rahman et al. [18] proposed a synergistic multi-criteria decision-making (MCDM) strategy using parameterized single-valued neutrosophic S-sets for selecting sustainable educational institution sites. Similarly, Ihsan et al. [19] applied Pythagorean FS expert sets to MCDM contexts. In supply chain management, Saeed et al. [20] employed possibility single-valued neutrosophic soft settings for efficient DM. Senapati [21] introduced an Aczel-Alsina aggregation-based outranking method for MCDM using single-valued neutrosophic numbers. These contributions demonstrate the potential of S-set theory and its extensions in solving complex DM and uncertainty-related problems. Alcantud et al. [22] provided a systematic literature review of S-set theory, further highlighting its extensive development and applications. Zulqarnain et al. [23] defined interval-valued Pythagorean FS sets and Wang et al. [24] proposed a novel concept of generalized Pythagorean FS sets in decision systems.

Building on these foundations, Fatimah et al. [25] introduced N-soft (N-S) set to enable multi-level graded evaluations. This framework is particularly suitable for problems requiring discrete grading systems, such as academic grading, performance evaluation, and ranking systems. The N-S set framework has been significantly extended to address diverse DM challenges. Akram et al. [26] introduced fuzzy N-soft set (FN-S) set, incorporating fuzziness to accommodate partial memberships under uncertainty. Additionally, Korkmaz et al. [27] proposed a novel approach to FN-S sets, applying them to the identification and sanctioning of cyber harassment on social media platforms, demonstrating their applicability in digital DM contexts. Rehman et al. [28] introduced picture FN-S set, extending fuzziness by including neutral and abstain membership values. Musa et al. [29] proposed N-hypersoft set, while Musa [30] introduced N-bipolar hypersoft set, and Musa et al. [31] defined N-bipolar hypersoft topology. Rehman et al. [32] suggested complex intuitionistic FN-S set, and Akram et al. [33] introduced complex q-rung orthopair FN-S set, enhancing DM with advanced fuzzy systems. Farooq et al. [34] developed complex bipolar FN-S set emphasizing bipolarity in collaborative and group DM.

Recently, Shabir et al. [35] proposed N-bipolar soft (N-BS) set. Furthermore, Riaz et al. [36,37] introduced M-parametrized N-S set, which provide a versatile approach to MCDM by incorporating an additional layer of parameterization. Building on this, Musa et al. [38] defined bipolar M-parametrized N-S set, enhancing the framework to address problems involving both positive and negative evaluations in

DM contexts. In addition, Kamaci et al. [39] presented m-polar N-S set, demonstrating its applicability in MCDM. Khan et al. [40] presented a synergistic method for evaluating educational institutions using similarity measures of possibility Pythagorean fuzzy hypersoft sets, contributing to the broader application of advanced fuzzy systems in MCDM. Alballa et al. [41] proposed a solid waste management approach utilizing fuzzy parameterized possibility single-valued neutrosophic hypersoft expert settings, highlighting the use of fuzzy and neutrosophic sets in practical MCDM applications. Gul [42] extended the VIKOR approach for MCDM (VIKOR stands for "VIsekriterijumsko KOmpromisno Rangiranje"), incorporating a bipolar fuzzy preference δ-covering-based bipolar fuzzy rough set model, enhancing the DM process in complex environments with bipolar attributes. Wang et al. [43] introduced group DM methods based on probabilistic hesitant N-S sets, further refining the probabilistic approach in S-set-based DM methodologies. Musa et al. [44] proposed N-BS expert sets, expanding the DM paradigm in robust MCDM applications. These contributions further advance the field of MCDM, integrating new methodologies and frameworks that enhance the handling of uncertainty in real-world scenarios.

The FN-BS set framework provides a significant advancement over existing uncertainty theories, such as fuzzy sets, S-sets, and their various extensions. Traditional uncertainty models, while useful, often exhibit key limitations in handling the complexities of real-world DM scenarios. Specifically, many of these models struggle to simultaneously account for both positive and negative aspects (bipolarity) and to deal with evaluations involving more than two states (multinary evaluations). These shortcomings are particularly problematic in fields like MCDM, where decision-makers are frequently confronted with conflicting, uncertain, and imprecise data.

The FN-BS set model addresses these issues by integrating three critical components: fuzziness, bipolarity, and multi-graded levels of assessment. Fuzziness allows for modeling uncertain, imprecise information; bipolarity enables the capture of both positive and negative factors that influence decisions; and multinary evaluations provide the flexibility to consider more than just binary outcomes, thus reflecting the complexities of real-world decision contexts. For instance, in a healthcare MCDM scenario, FN-BS sets facilitate the simultaneous modeling of positive attributes such as treatment effectiveness and negative attributes like adverse effects, each with varying degrees of certainty and importance. This capability is particularly crucial in healthcare, where both the benefits and risks must be carefully weighed, and the evaluation of treatments or interventions involves more than just a yes/no decision.

Moreover, FN-BS sets allow for the inclusion of expert input, which can be essential for navigating complex DM processes where quantitative data may be scarce or difficult to interpret. The model's ability to handle varying degrees of expert judgment further enhances its applicability in uncertain environments. By offering a more nuanced, comprehensive framework for DM, FN-BS sets provide decision-makers with a clearer understanding of the trade-offs involved in their choices, improving the reliability and robustness of their decisions.

These innovative features make the FN-BS framework highly relevant to modern research on uncertainty theories and their applications in DM. Unlike traditional models, which may be limited to simpler DM contexts, FN-BS sets are particularly suited for complex, uncertain, and multi-dimensional problems. Their ability to simultaneously incorporate multiple sources of uncertainty, such as both positive and negative factors and varying levels of expert input, provides a more holistic and realistic approach to DM under uncertainty. This novel approach represents a significant step forward in the development of DM models, demonstrating the potential of FN-BS sets to transform MCDM in areas such as healthcare, finance, and beyond.

1.1 Objectives

The primary objectives of this research are as follows:

- To develop a robust and generalized framework for MCDM under uncertainty by integrating fuzziness, bipolarity, and multi-level graded evaluations.
- To address the limitations of existing uncertainty theories by proposing a novel model, FN-BS sets, which combines the strengths of fuzzy logic, bipolar evaluations, and parameterized frameworks.
- To establish a solid theoretical foundation for FN-BS sets by defining their operations, properties, and algebraic structures.
- To demonstrate the practical utility of FN-BS sets through a real-world case study, showcasing their effectiveness in handling complex decision scenarios with both positive and negative attributes.
- To provide comparative analysis and sensitivity evaluation of the FN-BS framework against existing approaches, highlighting its advantages and applicability in diverse MCDM contexts.

1.2 Motivation

DM in uncertain and complex environments has become a critical area of research due to its wide-ranging applications. Traditional DM models often fall short in addressing uncertainty, imprecision, and the intricate interplay of attributes, especially in scenarios where attributes exhibit opposing (bipolar) effects or require nuanced evaluations beyond binary classifications. The significance of this research lies in addressing these limitations by developing a more generalized and robust framework for MCDM.

FN-BS sets are important because they bridge these gaps by integrating fuzzy logic, bipolar evaluations, and non-binary assessments, offering a powerful tool for representing real-world problems more accurately. For instance, FN-BS sets enable simultaneous modeling of positive factors and negative factors, while incorporating graded levels of evaluation to reflect the complexities of outcomes. Such an approach is crucial for advancing MCDM methodologies and ensuring more reliable and informed decisions in uncertain environments.

1.3 Main Contributions

This paper introduces FN-BS sets as a novel and comprehensive framework designed to enhance MCDM processes in uncertain and complex environments. The key contributions of this work are as follows:

- **Addressing the Limitations of Existing Models:** FN-BS sets extend traditional S-sets and other uncertainty theories by incorporating fuzziness, bipolarity, and multi-level graded evaluations, which are critical for capturing the nuanced nature of real-world decision problems.
- **Theoretical Development:** This paper formally defines FN-BS sets, introduces their set-theoretic operations, and explores their algebraic properties, providing a solid theoretical foundation for further research.
- **Novel Applications:** FN-BS sets are applied to a practical case study, demonstrating their effectiveness in modeling complex scenarios with both positive and negative attributes. Two algorithms are proposed for evaluating and selecting optimal choices in DM scenarios, showcasing the practical utility of the FN-BS framework.
- **Comparative Analysis:** The FN-BS model is compared against existing approaches, highlighting its advantages in handling binary and non-binary evaluations, bipolar settings, and membership degrees.

These contributions collectively establish the FN-BS framework as a significant advancement in the field of uncertainty theories and MCDM, addressing critical gaps and enabling more robust DM processes.

1.4 Structure of the Paper

The structure of this paper is organized as follows:

- Section 2: Reviews the foundational concepts of fuzzy sets, S-sets, and their extensions, providing the theoretical background necessary for understanding FN-BS sets.
- Section 3: Introduces the novel framework of FN-BS sets, starting with their formal definition and foundational concepts. This section also explores set-theoretic operations and algebraic properties related to FN-BS sets.
- Section 4: Demonstrates the application of FN-BS sets in MCDM, presenting two algorithms for evaluating and selecting optimal choices in decision scenarios. A case study of selecting the best vaccination program is used to illustrate these algorithms.
- Section 5: Compares and discusses the outcomes of the two proposed algorithms, highlighting their strengths and the consistency of results across different approaches.
- Section 6: Presents a sensitivity analysis of the FN-BS model against existing approaches, emphasizing its advantages in handling binary and non-binary evaluations, bipolar settings, and membership degrees.
- Section 7: Provides the conclusions, summarizing the findings of the paper, discussing limitations of the FN-BS model, and suggesting directions for future work, including possible extensions of the FN-BS framework and its integration with emerging technologies.

Through this structure, the paper systematically presents the theoretical development, application, and comparative evaluation of FN-BS sets, providing a comprehensive perspective on their usefulness for MCDM.

2 Preliminaries

This section reviews the foundational concepts of fuzzy sets, S-sets, and their extensions, which underpin the study. Here, $\mathcal{S}$ denotes a universal set of objects, A represents a set of attributes (parameters), and $G = \{0, 1, \ldots, N-1\}$ a set of ordered grades, where $N \in \{2, 3, \ldots\}$.

Definition 1. *[1] A fuzzy set μ over $\mathcal{S}$ is defined by a membership function $\mu : \mathcal{S} \to [0, 1]$, which assigns to each element $s \in \mathcal{S}$ a membership degree in the interval $[0, 1]$. The value μ_s represents the degree to which s belongs to $\mathcal{S}$. The collection of all fuzzy sets on $\mathcal{S}$ is denoted by $\mathcal{F}(\mathcal{S})$.*

Definition 2. *[5] An S-set is represented as a pair (Γ, A), where $\Gamma : A \to P(\mathcal{S})$, and $P(\mathcal{S})$ denotes the power set of $\mathcal{S}$, i.e., the set of all crisp subsets of $\mathcal{S}$.*

Definition 3. *[6] A pair (Λ, A) is called an FS set, where $\Lambda : A \to \mathcal{F}(\mathcal{S})$.*

Definition 4. *[25] An N-S set is represented as a triple (Θ, A, N), where $\Theta : A \to P(\mathcal{S} \times G)$ satisfies the condition that for each $a \in A$, there exists a unique pair $(s, g_a) \in \mathcal{S} \times G$ such that $(s, g_a) \in \Theta(a)$ or equivalently, $\Theta(a)(s) = g_a$, where $s \in \mathcal{S}$ and $g_a \in G$. Here, $P(\mathcal{S} \times G)$ denotes the set of all crisp subsets of $\mathcal{S} \times G$.*

Definition 5. *[26] An FN-S set is expressed as a triple (Ω, A, N), where $\Omega : A \to \mathcal{F}(\mathcal{S} \times G)$, and for each $a \in A$, $\langle (s, g_a), \Omega_{g_a} \rangle \in \Omega(a)$ or equivalently, $\Omega(a)(s) = \langle g_a, \Omega_{g_a} \rangle$. Here, $s \in \mathcal{S}$, $g_a \in G$, $\Omega_{g_a} \in [0, 1]$, and $\mathcal{F}(\mathcal{S} \times G)$ represents the collection of all fuzzy subsets of $\mathcal{S} \times G$.*

Definition 6. *[11] For a set of attributes $A = \{a_1, a_2, \ldots, a_n\}$, the NOT set of A, denoted $\neg A$, is defined as $\neg A = \{\neg a_1, \neg a_2, \ldots, \neg a_n\}$, where $\neg a_i$ indicates the negation of a_i for $i = 1, 2, \ldots, n$.*

Definition 7. *[13] A BS set is a triple (ψ, ζ, A), where $\psi : A \to P(\mathcal{S})$ and $\zeta : \neg A \to P(\mathcal{S})$ such that, for all $a \in A$, $\psi(a) \cap \zeta(\neg a) = \varnothing$, where $\psi(a), \zeta(\neg a) \in \mathcal{S}$.*

Definition 8. *[14] An FBS set is represented by a triple (γ, λ, A), where $\gamma : A \to \mathcal{F}(\mathcal{S})$ and $\lambda : \neg A \to \mathcal{F}(\mathcal{S})$ such that, for all $a \in A$, the condition $0 \leq \gamma(a)(s) + \lambda(\neg a)(s) \leq 1$ holds, where $s \in \mathcal{S}$ and $\gamma(a)(s), \lambda(\neg a)(s) \in [0, 1]$.*

Definition 9. *[35] An N-BS set is a quadruple $(f, \hbar, A, N)$, where $f : A \to P(\mathcal{S} \times G)$ and $\hbar : \neg A \to P(\mathcal{S} \times G)$ satisfy the following conditions: For each $a \in A$, there exists a unique pair $(s, g_a) \in \mathcal{S} \times G$ such that $(s, g_a) \in f(a)$, or equivalently, $f(a)(s) = g_a$ and for each $\neg a \in \neg A$, there exists a unique pair $(s, g_{\neg a}) \in \mathcal{S} \times G$ such that $(s, g_{\neg a}) \in \hbar(\neg a)$, or equivalently, $\hbar(\neg a)(s) = g_{\neg a}$, subject to the condition $g_a + g_{\neg a} \leq N - 1$, where $s \in \mathcal{S}$ and $g_a, g_{\neg a} \in G$.*

3 Fuzzy N-Bipolar Soft Sets

This section introduces and systematically analyzes the novel framework of FN-BS sets. It is structured into three subsections: foundational concepts, set-theoretic operations, and algebraic properties. The aim is to establish a robust mathematical basis for this model.

3.1 Definition and Fundamental Concepts of Fuzzy N-Bipolar Soft Sets

In this subsection, the novel hybrid model of FN-BS sets is formally defined. Key foundational concepts are introduced, including the representation of FN-BS sets and their components, providing a comprehensive framework for their application in DM scenarios.

Definition 10. *An FN-BS set on a universe of objects $\mathcal{S}$, a set of attributes $A \subseteq E$, and a set of ordered grades G is defined as a quadruple (μ, η, A, N), where $\mu : A \to \mathcal{F}(\mathcal{S} \times G)$ and $\eta : \neg A \to \mathcal{F}(\mathcal{S} \times G)$, such that for each $a \in A$ and $s \in \mathcal{S}$, $\langle (s, g_a), \mu_{g_a} \rangle \in \mu(a)$, or equivalently, $\mu(a)(s) = \langle g_a, \mu_{g_a} \rangle$ and for each $\neg a \in \neg A$ and $s \in \mathcal{S}$, $\langle (s, g_{\neg a}), \eta_{g_{\neg a}} \rangle \in \eta(\neg a)$, or equivalently, $\eta(\neg a)(s) = \langle g_{\neg a}, \eta_{g_{\neg a}} \rangle$, with the conditions:*

$$g_a + g_{\neg a} \leq N - 1, \tag{1}$$

$$0 \leq \mu_{g_a} + \eta_{g_{\neg a}} \leq 1, \tag{2}$$

where $g_a, g_{\neg a} \in G$, and $\mu_{g_a}, \eta_{g_{\neg a}} \in [0, 1]$.

It is assumed that both $\mathcal{S}$ and A are finite unless otherwise specified. In such cases, the FN-BS set can be represented in a tabular format. Here, $\langle g_{ij}^{+}, \mu_{ij} \rangle$ and $\langle g_{ij}^{-}, \eta_{ij} \rangle$ represent $\langle (s_i, g_{ij}^{+}), \mu_{ij} \rangle \in \mu(a_j)$ and $\langle (s_i, g_{ij}^{-}), \eta_{ij} \rangle \in \eta(\neg a_j)$, respectively. This tabular representation is shown in Table 1.

Table 1: Tabular form of FN-BS set (μ, η, A, N)

(μ, A, N)	a_1	a_2	...	a_n
s_1	$\langle g_{11}^{+}, \mu_{11} \rangle$	$\langle g_{12}^{+}, \mu_{12} \rangle$	...	$\langle g_{1n}^{+}, \mu_{1n} \rangle$
s_2	$\langle g_{21}^{+}, \mu_{21} \rangle$	$\langle g_{22}^{+}, \mu_{22} \rangle$	...	$\langle g_{2n}^{+}, \mu_{2n} \rangle$
...	...	...	...	...
s_m	$\langle g_{m1}^{+}, \mu_{m1} \rangle$	$\langle g_{m2}^{+}, \mu_{m2} \rangle$	...	$\langle g_{mn}^{+}, \mu_{mn} \rangle$
$(\eta, \neg A, N)$	$\neg a_1$	$\neg a_2$	...	$\neg a_n$
s_1	$\langle g_{11}^{-}, \eta_{11} \rangle$	$\langle g_{12}^{-}, \eta_{12} \rangle$	...	$\langle g_{1n}^{-}, \eta_{1n} \rangle$
s_2	$\langle g_{21}^{-}, \eta_{21} \rangle$	$\langle g_{22}^{-}, \eta_{22} \rangle$	...	$\langle g_{2n}^{-}, \eta_{2n} \rangle$
...	...	...	...	...
s_m	$\langle g_{m1}^{-}, \eta_{m1} \rangle$	$\langle g_{m2}^{-}, \eta_{m2} \rangle$	...	$\langle g_{mn}^{-}, \eta_{mn} \rangle$

Remark 1. *The following points require attention in relation to Definition 10:*

1. *The condition* $g_a + g_{\neg a} \leq N - 1$ *introduces a restriction to prevent contradictions or inconsistent evaluations within the model. In a bipolar evaluation system, an object cannot be both fully present (for a given attribute a) and fully absent (for its negation* $\neg a$*) to the same extreme degree. For example, if an attribute's grade is very high, the negation of the attribute must have a relatively low grade to maintain this balance.*
2. *The condition* $0 \leq \mu_{g_a} + \eta_{g_{\neg a}} \leq 1$ *ensures that the total membership degree for an object with respect to an attribute a and its negation* $\neg a$ *does not exceed 1. Essentially, it says that for each attribute a, if an object is highly present with respect to the attribute a (i.e., a high membership degree for* g_a*), it should be less present with respect to the negation of that attribute (i.e., a low membership degree for* $g_{\neg a}$*), and vice versa.*
3. *The relationship* $g_a \cap g_{\neg a} = \varnothing$ *may not hold in general. This is because, in certain cases, an object might have evaluations for both the attribute a and its negation* $\neg a$ *that are not mutually exclusive. For example, an object might have a certain degree of presence or relevance for both the attribute a and its negation* $\neg a$*. In such cases, the evaluations for both the attribute and its negation coexist and overlap, meaning the intersection of* g_a *and* $g_{\neg a}$ *is not empty.*
4. *The grades and their corresponding values are considered proportional and may vary. This variation in the grades is due to the context and sensitivity of the evaluation process, where the assignment of grades and their corresponding values can adapt to different scales or evaluation requirements.*

To better understand the essential aspects of our new model, consider the following example.

Example 1. *Consider a medical institution evaluating patients* $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$ *for a specific treatment based on the set of attributes* $A = \{a_1 = good\ health, a_2 = high\ mobility, a_3 = strong\ immune\ system\}$ *and their negations* $\neg A = \{\neg a_1 = poor\ health, \neg a_2 = low\ mobility, \neg a_3 = weak\ immune\ system\}$*. From Definition 9, the 4-BS set can be obtained from Table 2, where*

- *One square "□" represents least favorable.*
- *One triangle "△" represents slightly favorable.*
- *Two triangles " △ △" represent moderately favorable.*
- *Three triangles " △ △△" represent most favorable.*

Table 2: Evaluation of patients based on positive attributes and their negations in Example 1

$\mathcal{S}\backslash A$	a_1	a_2	a_3
s_1	△ △ △	△△	△
s_2	△△	△	□
s_3	□	△	△△
s_4	△	□	△△
$\mathcal{S}\backslash \neg A$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	□	△△	△
s_2	△	△△	△△
s_3	△ △ △	△	△△
s_4	△△	□	□

This graded evaluation by symbols can easily be identified with numbers as in $G = \{0, 1, 2, 3\}$ *where*

- 0 *corresponds to* □.

- 1 *corresponds to* △.
- 2 *corresponds to* △△.
- 3 *corresponds to* △ △ △.

The tabular representation of the 4-BS set $(\mu', \eta', A, 4)$ *is shown in* Table 3.

Table 3: Tabular form of 4-BS set $(\mu', \eta', A, 4)$ in Example 1

$(\mu', A, 4)$	a_1	a_2	a_3
s_1	3	2	1
s_2	2	1	0
s_3	0	1	2
s_4	1	0	2
$(\eta', \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	0	2	1
s_2	1	2	2
s_3	3	1	2
s_4	2	0	0

This information is sufficient when derived from precise data; however, when the data is vague or uncertain, the FN-BS set becomes essential to provide insights into how these grades are assigned to patients. The selection panel assigns membership values based on the evaluation grade of the patients as follows:

For positive membership values (μ_{g_a})*:*

$$\begin{aligned} &0 \le \mu_{g_a} < 0.2 && \text{when } g_a = 0,\\ &0.2 \le \mu_{g_a} < 0.5 && \text{when } g_a = 1,\\ &0.5 \le \mu_{g_a} < 0.8 && \text{when } g_a = 2,\\ &0.8 \le \mu_{g_a} \le 1 && \text{when } g_a = 3. \end{aligned} \tag{3}$$

For negative membership values ($\eta_{g_{\neg a}}$)*:*

$$\begin{aligned} &0 \le \eta_{g_{\neg a}} < 0.2 && \text{when } g_{\neg a} = 0,\\ &0.2 \le \eta_{g_{\neg a}} < 0.5 && \text{when } g_{\neg a} = 1,\\ &0.5 \le \eta_{g_{\neg a}} < 0.8 && \text{when } g_{\neg a} = 2,\\ &0.8 \le \eta_{g_{\neg a}} \le 1 && \text{when } g_{\neg a} = 3. \end{aligned} \tag{4}$$

Then, the F4-BS set $(\mu, \eta, A, 4)$ *for the same patients and attributes is presented in* Table 4. *To illustrate, for* s_1*, the grade* $g_{a_1} = 3$ *reflects excellent health, with the membership value* $\mu_{g_{a_1}} = 0.8$ *quantifying its strong alignment. In contrast, for* $\neg a_1$*, the grade* $g_{\neg a_1} = 0$ *indicates minimal concern for poor health, with* $\eta_{g_{\neg a_1}} = 0.1$ *confirming this minimal association.*

Remark 2. *The grade* $0 \in G$*, as defined in Definition 10, does not signify a lack of information or incompleteness. Instead, it denotes the minimum grade within the hierarchy of ordered grades.*

Remark 3. *Any FN-BS set can naturally be viewed as an F(N + 1)-BS set or, more generally, as an FM-BS set where M > N.*

Table 4: Tabular form of F4-BS set $(\mu, \eta, A, 4)$ in Example 1

$(\mu, A, 4)$	a_1	a_2	a_3
s_1	$\langle 3, 0.9\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.4\rangle$
s_2	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.6\rangle$
$(\eta, \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	$\langle 0, 0.1\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.2\rangle$
s_2	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$
s_3	$\langle 3, 0.8\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$
s_4	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$

The rationale for this concept lies in the fact that, in some situations, the highest grades may appear in (μ, A, N) but not in $(\eta, \neg A, N)$, and vice versa. In certain cases, the highest grades might be found in both sets. Based on this insight, the following definitions are provided.

Definition 11. *An FN-BS set (μ, η, A, N) is called positively efficient if, for some $a \in A$ and $s \in \mathcal{S}$, $\mu(a)(s) = \langle N-1, \mu_{N-1}\rangle$, and for some $\neg a \in \neg A$ and $s \in \mathcal{S}$, $\eta(\neg a)(s) = \langle 0, \eta_0\rangle$, where $\mu_{N-1}, \eta_0 \in [0,1]$.*

Definition 12. *An FN-BS set (μ, η, A, N) is termed negatively efficient if, for some $a \in A$ and $s \in \mathcal{S}$, $\mu(a)(s) = \langle 0, \mu_0\rangle$, and for some $\neg a \in \neg A$ and $s \in \mathcal{S}$, $\eta(\neg a)(s) = \langle N-1, \eta_{N-1}\rangle$, where $\mu_0, \eta_{N-1} \in [0,1]$.*

Definition 13. *An FN-BS set (μ, η, A, N) is said to be totally efficient if it is both positively efficient and negatively efficient.*

Definition 14. *If (μ, η, A, N) is not a totally efficient FN-BS set, then its minimized FN-BS set, denoted by (μ^m, η^m, A, M), is defined as follows: for every $a \in A$, $\neg a \in \neg A$, and $s \in \mathcal{S}$, $M = \max\{g_a, g_{\neg a}\} + 1$, $g_a^m = g_a$, $g_{\neg a}^m = g_{\neg a}$, $\mu^m_{g_a^m} = \mu_{g_a}$, and $\eta^m_{g_{\neg a}^m} = \eta_{g_{\neg a}}$, where $\langle g_a, \mu_{g_a}\rangle \in \mu(a)(s)$, $\langle g_a^m, \mu^m_{g_a^m}\rangle \in \mu^m(a)(s)$, $\langle g_{\neg a}, \eta_{g_{\neg a}}\rangle \in \eta(\neg a)(s)$, and $\langle g_{\neg a}^m, \eta^m_{g_{\neg a}^m}\rangle \in \eta^m(\neg a)(s)$.*

Remark 4. *Any minimized FN-BS set (μ^m, η^m, A, M) of a non-totally efficient FN-BS set (μ, η, A, N) is always totally efficient.*

Remark 5. *For each value of $0 < \mathcal{T} < N$, there exists an associated FBS set for every FN-BS set.*

Definition 15. *Given a threshold $0 < \mathcal{T} < N$ and an FN-BS set (μ, η, A, N), the corresponding FBS set is denoted as $(\mu^{\mathcal{T}}, \eta^{\mathcal{T}}, A)$ and is defined by the following expressions, for all $a \in A$ and $s \in \mathcal{S}$:*

$$\mu^{\mathcal{T}}(a)(s) = \begin{cases} \mu_{g_a}, & if\ \langle g_a, \mu_{g_a}\rangle \in \mu(a)(s)\ such\ that\ g_a \geq \mathcal{T} \\ 0, & otherwise \end{cases} \tag{5}$$

and for all $\neg a \in \neg A$ and $s \in \mathcal{S}$:

$$\eta^{\mathcal{T}}(\neg a)(s) = \begin{cases} \eta_{g_{\neg a}}, & if\ \langle g_{\neg a}, \eta_{g_{\neg a}}\rangle \in \eta(\neg a)(s)\ such\ that\ g_{\neg a} \geq N - \mathcal{T} \\ 0, & otherwise \end{cases} \tag{6}$$

In particular, we define $(\mu^{\mathcal{T}=1}, \eta^{\mathcal{T}=1}, A)$ as the bottom FBS set corresponding to (μ, η, A, N), while $(\mu^{\mathcal{T}=N-1}, \eta^{\mathcal{T}=N-1}, A)$ is known as the top FBS set associated with (μ, η, A, N).

Definition 16. *An FN-BS set* $(\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A, N)$ *is called a relative null FN-BS set if, for every* $a \in A$ *and* $s \in \mathcal{S}$, $\mu^{\mathbb{O}}(a)(s) = \langle 0, 0\rangle$, *and for every* $\neg a \in \neg A$ *and* $s \in \mathcal{S}$, $\eta^{\mathbb{O}}(\neg a)(s) = \langle N-1, 1\rangle$.

Definition 17. *An FN-BS set* $(\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A, N)$ *is referred to as a relative whole FN-BS set if, for all* $a \in A$ *and* $s \in \mathcal{S}$, $\mu^{\mathbb{W}}(a)(s) = \langle N-1, 1\rangle$, *and for all* $\neg a \in \neg A$ *and* $s \in \mathcal{S}$, $\eta^{\mathbb{W}}(\neg a)(s) = \langle 0, 0\rangle$.

3.2 Set-Theoretic Operations on Fuzzy N-Bipolar Soft Sets

This subsection investigates various set-theoretic operations within the framework of FN-BS sets, such as complement, subset, union, intersection, and others. The properties of these operations are analyzed, with examples to illustrate their behavior and potential applications.

In the context of FN-BS sets, there are four types of complementary operations, starting with the most significant one as follows:

Definition 18. *The FN-BS complement of* (μ, η, A, N), *denoted by* $(\mu, \eta, A, N)^{\tilde{c}}$, *is defined as* $(\mu, \eta, A, N)^{\tilde{c}} = (\mu^{\tilde{c}}, \eta^{\tilde{c}}, A, N)$, *such that, for all* $a \in A$ *and* $s \in \mathcal{S}$, $g_a^{\tilde{c}} = g_{\neg a}$ *and* $\mu^{\tilde{c}}_{g_a^{\tilde{c}}} = \eta_{g_{\neg a}}$, *and for all* $\neg a \in \neg A$ *and* $s \in \mathcal{S}$, $g_{\neg a}^{\tilde{c}} = g_a$ *and* $\eta^{\tilde{c}}_{g_{\neg a}^{\tilde{c}}} = \mu_{g_a}$.

Definition 19. *The FN-BS weak complement of* (μ, η, A, N) *is any FN-BS set* $(\mu, \eta, A, N)^{\tilde{\omega}} = (\mu^{\tilde{\omega}}, \eta^{\tilde{\omega}}, A, N)$, *where, for all* $a \in A$ *and* $s \in \mathcal{S}$, $g_a^{\tilde{\omega}} \cap g_a = \varnothing$ *and* $\mu^{\tilde{\omega}}_{g_a^{\tilde{\omega}}} \cap \mu_{g_a} = \varnothing$, *and for all* $\neg a \in \neg A$ *and* $s \in \mathcal{S}$, $g_{\neg a}^{\tilde{\omega}} \cap g_{\neg a} = \varnothing$ *and* $\eta^{\tilde{\omega}}_{g_{\neg a}^{\tilde{\omega}}} \cap \eta_{g_{\neg a}} = \varnothing$.

Definition 20. *The FN-BS top weak complement of* (μ, η, A, N) *is* $(\mu, \eta, A, N)^{\tilde{t}} = (\mu^{\tilde{t}}, \eta^{\tilde{t}}, A, N)$ *where*

$$\mu^{\tilde{t}}(a)(s) = \begin{cases} \langle N-1, \mu_{g_a}\rangle, & \text{if } g_a < N-1 \\ \langle 0, \mu_{g_a}\rangle, & \text{if } g_a = N-1 \end{cases} \tag{7}$$

and

$$\eta^{\tilde{t}}(\neg a)(s) = \begin{cases} \langle 0, \eta_{g_{\neg a}}\rangle, & if\ g_{\neg a} > 0 \\ \langle 0, \eta_{g_{\neg a}}\rangle, & if\ g_{\neg a} = 0\ and\ g_a < N-1 \\ \langle N-1, \eta_{g_{\neg a}}\rangle, & otherwise \end{cases} \tag{8}$$

Here, $\mu_{g_a}, \eta_{g_{\neg a}} \in [0, 1]$.

Definition 21. *The FN-BS bottom weak complement of* (μ, η, A, N) *is* $(\mu, \eta, A, N)^{\tilde{b}} = (\mu^{\tilde{b}}, \eta^{\tilde{b}}, A, N)$ *where*

$$\mu^{\tilde{b}}(a)(s) = \begin{cases} \langle 0, \mu_{g_a}\rangle, & \text{if } g_a > 0 \\ \langle 0, \mu_{g_a}\rangle, & \text{if } g_a = 0 \text{ and } g_{\neg a} < N-1 \\ \langle N-1, \mu_{g_a}\rangle, & \text{otherwise} \end{cases} \tag{9}$$

and

$$\mu^{\tilde{b}}(\neg a)(s) = \begin{cases} \langle N-1, \eta_{g_{\neg a}}\rangle, & \text{if } g_{\neg a} < N-1 \\ \langle 0, \eta_{g_{\neg a}}\rangle, & \text{if } g_{\neg a} = N-1 \end{cases} \tag{10}$$

Here, $\mu_{g_a}, \eta_{g_{\neg a}} \in [0, 1]$.

Example 2. *Consider the F4-BS set* $(\mu, \eta, A, 4)$ *as in Example 1. The F4-BS complement, weak complement, top weak complement, and bottom weak complement of this set are presented in tabular form in Tables 5–8.*

Table 5: Tabular form of F4-BS complement of $(\mu, \eta, A, 4)$ in Example 1

$(\mu^{\tilde{c}}, A, 4)$	a_1	a_2	a_3
s_1	$\langle 0, 0.1\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.2\rangle$
s_2	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$
s_3	$\langle 3, 0.8\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$
s_4	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$
$(\eta^{\tilde{c}}, \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	$\langle 3, 0.9\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.4\rangle$
s_2	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.6\rangle$

Table 6: Tabular form of one of the F4-BS weak complement of $(\mu, \eta, A, 4)$ in Example 1

$(\mu^{\tilde{\omega}}, A, 4)$	a_1	a_2	a_3
s_1	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.6\rangle$
s_2	$\langle 3, 0.8\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.2\rangle$
s_3	$\langle 1, 0.4\rangle$	$\langle 3, 1\rangle$	$\langle 1, 0.3\rangle$
s_4	$\langle 2, 0.7\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$
$(\eta^{\tilde{\omega}}, \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	$\langle 1, 0.2\rangle$	$\langle 3, 1\rangle$	$\langle 0, 0.1\rangle$
s_2	$\langle 0, 0.1\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.2\rangle$
s_3	$\langle 2, 0.5\rangle$	$\langle 0, 0\rangle$	$\langle 1, 0.4\rangle$
s_4	$\langle 1, 0.3\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$

Table 7: Tabular form of F4-BS top weak complement of $(\mu, \eta, A, 4)$ in Example 1

$(\mu^{\tilde{t}}, A, 4)$	a_1	a_2	a_3
s_1	$\langle 0, 0.9\rangle$	$\langle 3, 0.5\rangle$	$\langle 3, 0.4\rangle$
s_2	$\langle 3, 0.5\rangle$	$\langle 3, 0.3\rangle$	$\langle 3, 0.1\rangle$
s_3	$\langle 3, 0.1\rangle$	$\langle 3, 0.4\rangle$	$\langle 3, 0.5\rangle$
s_4	$\langle 3, 0.4\rangle$	$\langle 3, 0\rangle$	$\langle 3, 0.6\rangle$
$(\eta^{\tilde{t}}, \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	$\langle 3, 0.1\rangle$	$\langle 0, 0.5\rangle$	$\langle 0, 0.2\rangle$
s_2	$\langle 0, 0.2\rangle$	$\langle 0, 0.5\rangle$	$\langle 0, 0.7\rangle$
s_3	$\langle 0, 0.8\rangle$	$\langle 0, 0.3\rangle$	$\langle 0, 0.5\rangle$
s_4	$\langle 0, 0.6\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$

Table 8: Tabular form of F4-BS bottom weak complement of $(\mu, \eta, A, 4)$ in Example 1

$(\mu^{\tilde{b}}, A, 4)$	a_1	a_2	a_3
s_1	$\langle 0, 0.9\rangle$	$\langle 0, 0.5\rangle$	$\langle 0, 0.4\rangle$
s_2	$\langle 0, 0.5\rangle$	$\langle 0, 0.3\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 3, 0.1\rangle$	$\langle 0, 0.4\rangle$	$\langle 0, 0.5\rangle$
s_4	$\langle 0, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.6\rangle$
$(\eta^{\tilde{b}}, \neg A, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$
s_1	$\langle 3, 0.1\rangle$	$\langle 3, 0.5\rangle$	$\langle 3, 0.2\rangle$
s_2	$\langle 3, 0.2\rangle$	$\langle 3, 0.5\rangle$	$\langle 3, 0.7\rangle$
s_3	$\langle 0, 0.8\rangle$	$\langle 3, 0.3\rangle$	$\langle 3, 0.5\rangle$
s_4	$\langle 3, 0.6\rangle$	$\langle 3, 0\rangle$	$\langle 3, 0.1\rangle$

Definition 22. *An FN-BS set (μ^1, η^1, A^1, N) is considered a subset of (μ^2, η^2, A^2, N), denoted as $(\mu^1, \eta^1, A^1, N)\widetilde{\sqsubseteq}(\mu^2, \eta^2, A^2, N)$, if the following conditions are satisfied:*

1. $A^1 \subseteq A^2$.
2. *For every $a \in A^1$ and $s \in \mathcal{S}$, $g_a^1 \le g_a^2$ and $\mu^1_{g_a^1} \le \mu^2_{g_a^2}$, where $\langle g_a^1, \mu^1_{g_a^1}\rangle \in \mu^1(a)(s)$ and $\langle g_a^2, \mu^2_{g_a^2}\rangle \in \mu^2(a)(s)$.*
3. *For every $\neg a \in \neg A^1$ and $s \in \mathcal{S}$, $g^2_{\neg a} \le g^1_{\neg a}$ and $\eta^2_{g^2_{\neg a}} \le \eta^1_{g^2_{\neg a}}$, where $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}}\rangle \in \eta^1(\neg a)(s)$ and $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}}\rangle \in \eta^2(\neg a)(s)$.*

Definition 23. *Two FN-BS sets (μ^1, η^1, A^1, N) and (μ^2, η^2, A^2, N) over $\mathcal{S}$ are said to be FN-BS equal if the following conditions hold:*

1. $A^1 = A^2$.
2. *For every $a \in A^1 = A^2$ and $s \in \mathcal{S}$, $g_a^1 = g_a^2$ and $\mu^1_{g_a^1} = \mu^2_{g_a^2}$, where $\langle g_a^1, \mu^1_{g_a^1}\rangle \in \mu^1(a)(s)$ and $\langle g_a^2, \mu^2_{g_a^2}\rangle \in \mu^2(a)(s)$.*
3. *For every $\neg a \in \neg A^1 = \neg A^2$ and $s \in \mathcal{S}$, $g^1_{\neg a} = g^2_{\neg a}$ and $\eta^1_{g^1_{\neg a}} = \eta^2_{g^2_{\neg a}}$, where $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}}\rangle \in \eta^1(\neg a)(s)$ and $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}}\rangle \in \eta^2(\neg a)(s)$.*

Definition 24. *The FN-BS extended union of $(\mu^1, \eta^1, A^1, N^1)$ and $(\mu^2, \eta^2, A^2, N^2)$ is denoted and defined as $(\mu^1, \eta^1, A^1, N^1)\widetilde{\sqcup}_\varepsilon(\mu^2, \eta^2, A^2, N^2) = (\mu, \eta, A^1 \cup A^2, \max(N^1, N^2))$, where for all $a \in A^1 \cup A^2$ and $s \in \mathcal{S}$:*

$$\mu(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash A^2 \\ \mu^2(a)(s), & \text{if } a \in A^2 \backslash A^1 \\ \langle g_a, \mu_{g_a}\rangle, & \textit{such that } g_a = \max\{g_a^1, g_a^2\} \textit{ and } \mu_{g_a} = \max\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\} \\ & \textit{where } \langle g_a^1, \mu^1_{g_a^1}\rangle \in \mu^1(a)(s) \textit{ and } \langle g_a^2, \mu^2_{g_a^2}\rangle \in \mu^2(a)(s). \end{cases} \tag{11}$$

and for all $\neg a \in \neg A^1 \cup \neg A^2$ and $s \in \mathcal{S}$:

$$\eta(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \textit{if } \neg a \in \neg A^1 \backslash \neg A^2 \\ \eta^2(\neg a)(s), & \textit{if } \neg a \in \neg A^2 \backslash \neg A^1 \\ \langle g_{\neg a}, \eta_{g_{\neg a}}\rangle, & \textit{such that } g_{\neg a} = \min\{g^1_{\neg a}, g^2_{\neg a}\} \textit{ and } \eta_{g_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\} \\ & \textit{where } \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}}\rangle \in \eta^1(\neg a)(s) \textit{ and } \langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}}\rangle \in \eta^2(\neg a)(s). \end{cases} \tag{12}$$

Definition 25. *The FN-BS extended intersection of* $(\mu^1, \eta^1, A^1, N^1)$ *and* $(\mu^2, \eta^2, A^2, N^2)$ *is denoted and defined as* $(\mu^1, \eta^1, A^1, N^1)\widetilde{\sqcap}_{\varepsilon}(\mu^2, \eta^2, A^2, N^2) = (\mu, \eta, A^1 \cup A^2, \max(N^1, N^2))$, *where for all* $a \in A^1 \cup A^2$ *and* $s \in \mathcal{S}$:

$$\mu(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash A^2 \\ \mu^2(a)(s), & \text{if } a \in A^2 \backslash A^1 \\ \langle g_a, \mu_{g_a} \rangle, & \text{such that } g_a = \min\{g_a^1, g_a^2\} \text{ and } \mu_{g_a} = \min\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\} \\ & \text{where } \langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s) \text{ and } \langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s). \end{cases} \tag{13}$$

and for all $\neg a \in \neg A^1 \cup \neg A^2$ *and* $s \in \mathcal{S}$:

$$\eta(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \backslash \neg A^2 \\ \eta^2(\neg a)(s), & \text{if } \neg a \in \neg A^2 \backslash \neg A^1 \\ \langle g_{\neg a}, \eta_{g_{\neg a}} \rangle, & \text{such that } g_{\neg a} = \max\{g^1_{\neg a}, g^2_{\neg a}\} \text{ and } \eta_{g_{\neg a}} = \max\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\} \\ & \text{where } \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s). \end{cases} \tag{14}$$

Definition 26. *The FN-BS restricted union of* $(\mu^1, \eta^1, A^1, N^1)$ *and* $(\mu^2, \eta^2, A^2, N^2)$ *is denoted and defined as* $(\mu^1, \eta^1, A^1, N^1)\widetilde{\sqcup}_{\mathfrak{R}}(\mu^2, \eta^2, A^2, N^2) = (\mu, \eta, A^1 \cap A^2, \max(N^1, N^2))$, *where for all* $a \in A^1 \cap A^2 \neq \varnothing$ *and* $s \in \mathcal{S}$:

$$\mu(a)(s) = \langle g_a, \mu_{g_a} \rangle \tag{15}$$

where $g_a = \max\{g_a^1, g_a^2\}$ *and* $\mu_{g_a} = \max\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\}$, *with* $\langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s)$ *and* $\langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s)$, *and for all* $\neg a \in \neg A^1 \cap \neg A^2 \neq \varnothing$ *and* $s \in \mathcal{S}$:

$$\eta(\neg a)(s) = \langle g_{\neg a}, \eta_{g_{\neg a}} \rangle \tag{16}$$

where $g_{\neg a} = \min\{g^1_{\neg a}, g^2_{\neg a}\}$ *and* $\eta_{g_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\}$, *with* $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s)$ *and* $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s)$.

Definition 27. *The FN-BS restricted intersection of* $(\mu^1, \eta^1, A^1, N^1)$ *and* $(\mu^2, \eta^2, A^2, N^2)$ *is denoted and defined as* $(\mu^1, \eta^1, A^1, N^1)\widetilde{\sqcap}_{\mathfrak{R}}(\mu^2, \eta^2, A^2, N^2) = (\mu, \eta, A^1 \cap A^2, \max(N^1, N^2))$, *where for all* $a \in A^1 \cap A^2 \neq \varnothing$ *and* $s \in \mathcal{S}$:

$$\mu(a)(s) = \langle g_a, \mu_{g_a} \rangle \tag{17}$$

where $g_a = \min\{g_a^1, g_a^2\}$ *and* $\mu_{g_a} = \min\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\}$, *with* $\langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s)$ *and* $\langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s)$, *and for all* $\neg a \in \neg A^1 \cap \neg A^2 \neq \varnothing$ *and* $s \in \mathcal{S}$:

$$\eta(\neg a)(s) = \langle g_{\neg a}, \eta_{g_{\neg a}} \rangle \tag{18}$$

where $g_{\neg a} = \min\{g^1_{\neg a}, g^2_{\neg a}\}$ *and* $\eta_{g_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\}$, *with* $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s)$ *and* $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s)$.

Example 3. *Consider F4-BS set* $(\mu, \eta, A, 4)$ *in Example 1 and the F3-BS set* $(\mu^1, \eta^1, A^1, 3)$ *described in tabular form by Table 9. The FN-BS extended union (intersection) and the FN-BS restricted union (intersection) are given in Tables 10–13.*

Table 9: Tabular form of F3-BS set $(\mu^1, \eta^1, A^1, 3)$ in Example 3

$(\mu^1, A^1, 3)$	a_1	a_2	a_4
s_1	$\langle 1, 0.4\rangle$	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$
s_2	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.7\rangle$
s_3	$\langle 1, 0.3\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.7\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 2, 1\rangle$	$\langle 0, 0\rangle$
$(\eta^1, \neg A^1, 3)$	$\neg a_1$	$\neg a_2$	$\neg a_4$
s_1	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.6\rangle$
s_2	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$

Table 10: Tabular form of $(\mu, \eta, A, 4)\widetilde{\sqcup}_\varepsilon(\mu^1, \eta^1, A^1, 3) = (\mu^2, \eta^2, A \cup A^1, 4)$ in Example 3

$(\mu^2, A \cup A^1, 4)$	a_1	a_2	a_3	a_4
s_1	$\langle 3, 0.9\rangle$	$\langle 2, 0.6\rangle$	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$
s_2	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.7\rangle$
s_3	$\langle 1, 0.3\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 2, 1\rangle$	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$
$(\eta^2, \neg A \cup \neg A^1, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	$\langle 0, 0.1\rangle$	$\langle 0, 0.1\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.6\rangle$
s_2	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$	$\langle 0, 0.1\rangle$

Table 11: Tabular form of $(\mu, \eta, A, 4)\widetilde{\sqcap}_\varepsilon(\mu^1, \eta^1, A^1, 3) = (\mu^3, \eta^3, A \cup A^1, 4)$ in Example 3

$(\mu^3, A \cup A^1, 4)$	a_1	a_2	a_3	a_4
s_1	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$
s_2	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$	$\langle 0, 0.1\rangle$	$\langle 2, 0.7\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$
$(\eta^3, \neg A \cup \neg A^1, 4)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.6\rangle$
s_2	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.7\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 3, 0.8\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
s_4	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$	$\langle 0, 0.1\rangle$	$\langle 0, 0.1\rangle$

Table 12: Tabular form of $(\mu, \eta, A, 4)\widetilde{\sqcup}_{\Re}(\mu^1, \eta^1, A^1, 3) = (\mu^4, \eta^4, A \cap A^1, 4)$ in Example 3

$(\mu^4, A \cap A^1, 4)$	a_1	a_2
s_1	$\langle 3, 0.9\rangle$	$\langle 2, 0.6\rangle$
s_2	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$
s_3	$\langle 1, 0.3\rangle$	$\langle 1, 0.4\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 2, 1\rangle$
$(\eta^4, \neg A \cap \neg A^1, 4)$	$\neg a_1$	$\neg a_2$
s_1	$\langle 0, 0.1\rangle$	$\langle 0, 0.1\rangle$
s_2	$\langle 0, 0\rangle$	$\langle 2, 0.5\rangle$
s_3	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$

Table 13: Tabular form of $(\mu, \eta, A, 4)\widetilde{\sqcap}_{\Re}(\mu^1, \eta^1, A^1, 3) = (\mu^5, \eta^5, A \cap A^1, 4)$ in Example 3

$(\mu^5, A \cap A^1, 4)$	a_1	a_2
s_1	$\langle 1, 0.4\rangle$	$\langle 2, 0.5\rangle$
s_2	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 1, 0.4\rangle$
s_4	$\langle 1, 0.4\rangle$	$\langle 0, 0\rangle$
$(\eta^5, \neg A \cap \neg A^1, 4)$	$\neg a_1$	$\neg a_2$
s_1	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$
s_2	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$
s_3	$\langle 3, 0.8\rangle$	$\langle 1, 0.3\rangle$
s_4	$\langle 2, 0.6\rangle$	$\langle 0, 0\rangle$

3.3 Algebraic Properties of Operations on Fuzzy N-Bipolar Soft Sets

The algebraic properties associated with operations on FN-BS sets are explored in this subsection. These include distributive, associative, and commutative properties, among others. The relationships between these properties are highlighted, and their implications for the structure of FN-BS sets are discussed.

Proposition 1. *Let (μ^1, η^1, A, N), (μ^2, η^2, A, N), and (μ^3, η^3, A, N) be three FN-BS sets. Then*

1. $(\mu^1, \eta^1, A, N) \mathrel{\widetilde{\sqsubseteq}} (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A, N)$.
2. $(\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A, N) \mathrel{\widetilde{\sqsubseteq}} (\mu^1, \eta^1, A, N)$.
3. *If* $(\mu^1, \eta^1, A, N) \mathrel{\widetilde{\sqsubseteq}} (\mu^2, \eta^2, A, N)$ *and* $(\mu^2, \eta^2, A, N) \mathrel{\widetilde{\sqsubseteq}} (\mu^3, \eta^3, A, N)$, *then* $(\mu^1, \eta^1, A, N) \mathrel{\widetilde{\sqsubseteq}} (\mu^3, \eta^3, A, N)$.

Proof. Straightforward. ■

Proposition 2. *Let (μ^1, η^1, A^1, N) and (μ^2, η^2, A^2, N) be two FN-BS sets. Then*

1. $(\mu^1, \eta^1, A^1, N) \mathrel{\widetilde{\sqcup}_{\varepsilon}} (\mu^2, \eta^2, A^2, N)$ *is the smallest FN-BS set which contains both* (μ^1, η^1, A^1, N) *and* (μ^2, η^2, A^2, N).
2. $(\mu^1, \eta^1, A^1, N) \mathrel{\widetilde{\sqcap}_{\Re}} (\mu^2, \eta^2, A^2, N)$ *is the largest FN-BS set which is contained in both* (μ^1, η^1, A^1, N) *and* (μ^2, η^2, A^2, N).

Proof. Straightforward. ■

Proposition 3. *Let* (μ^1, η^1, A, N) *and* (μ^2, η^2, A, N) *be two FN-BS sets. Then*

1. $(\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A, N)^{\tilde{c}} = (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A, N)$.
2. $(\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A, N)^{\tilde{c}} = (\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A, N)$.
3. $((\mu^1, \eta^1, A, N)^{\tilde{c}})^{\tilde{c}} = (\mu^1, \eta^1, A, N)$.
4. *If* $(\mu^1, \eta^1, A, N)\widetilde{\sqsubseteq}(\mu^2, \eta^2, A, N)$, *then* $(\mu^2, \eta^2, A, N)^{\tilde{c}}\widetilde{\sqsubseteq}(\mu^1, \eta^1, A, N)^{\tilde{c}}$.
5. $(\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A, N) \;\widetilde{\sqsubseteq}\; (\mu^1, \eta^1, A, N) \;\widetilde{\sqcap}_{\mathfrak{R}}\; (\mu^1, \eta^1, A, N)^{\tilde{c}} \;\widetilde{\sqsubseteq}\; (\mu^1, \eta^1, A, N) \;\widetilde{\sqcup}_{\mathfrak{R}}\; (\mu^1, \eta^1, A, N)^{\tilde{c}} \;\widetilde{\sqsubseteq}\; (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A, N)$.
6. *If* $(\mu^1, \eta^1, A, N)\widetilde{\sqsubseteq}(\mu^2, \eta^2, A, N)$, *then* $(\mu^1, \eta^1, A, N)\widetilde{\sqcap}_{\mathfrak{R}}(\mu^2, \eta^2, A, N) = (\mu^1, \eta^1, A, N)$.
7. *If* $(\mu^1, \eta^1, A, N)\widetilde{\sqsubseteq}(\mu^2, \eta^2, A, N)$, *then* $(\mu^1, \eta^1, A, N)\widetilde{\sqcup}_{\mathfrak{R}}(\mu^2, \eta^2, A, N) = (\mu^2, \eta^2, A, N)$.

Proof. Straightforward. ■

Proposition 4. *Let* (μ^1, η^1, A^1, N) *and* (μ^2, η^2, A^2, N) *be two FN-BS sets. Then*

1. $\left((\mu^1, \eta^1, A^1, N)\,\widetilde{\sqcup}_{\varepsilon}\,(\mu^2, \eta^2, A^2, N)\right)^{\tilde{c}} = (\mu^1, \eta^1, A^1, N)^{\tilde{c}}\,\widetilde{\sqcap}_{\varepsilon}\,(\mu^2, \eta^2, A^2, N)^{\tilde{c}}$.
2. $\left((\mu^1, \eta^1, A^1, N)\,\widetilde{\sqcap}_{\varepsilon}\,(\mu^2, \eta^2, A^2, N)\right)^{\tilde{c}} = (\mu^1, \eta^1, A^1, N)^{\tilde{c}}\,\widetilde{\sqcup}_{\varepsilon}\,(\mu^2, \eta^2, A^2, N)^{\tilde{c}}$.
3. $\left((\mu^1, \eta^1, A^1, N)\,\widetilde{\sqcup}_{\mathfrak{R}}\,(\mu^2, \eta^2, A^2, N)\right)^{\tilde{c}} = (\mu^1, \eta^1, A^1, N)^{\tilde{c}}\,\widetilde{\sqcap}_{\mathfrak{R}}\,(\mu^2, \eta^2, A^2, N)^{\tilde{c}}$.
4. $\left((\mu^1, \eta^1, A^1, N)\,\widetilde{\sqcap}_{\mathfrak{R}}\,(\mu^2, \eta^2, A^2, N)\right)^{\tilde{c}} = (\mu^1, \eta^1, A^1, N)^{\tilde{c}}\,\widetilde{\sqcup}_{\mathfrak{R}}\,(\mu^2, \eta^2, A^2, N)^{\tilde{c}}$.

Proof. (1) Let $(\mu^1, \eta^1, A^1, N)\ \widetilde{\sqcup}_{\varepsilon}\ (\mu^2, \eta^2, A^2, N) = (\mu^3, \eta^3, A^1 \cup A^2, N)$. Then, $\left((\mu^1, \eta^1, A^1, N)\ \widetilde{\sqcup}_{\varepsilon}\ (\mu^2, \eta^2, A^2, N)\right)^{\tilde{c}} = (\mu^3, \eta^3, A^1 \cup A^2, N)^{\tilde{c}} = (\mu^{3^{\tilde{c}}}, \eta^{3^{\tilde{c}}}, A^1 \cup A^2, N)$. By definition, for all $a \in A^1 \cup A^2$ and $s \in \mathcal{S}$:

$$\mu^3(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash A^2 \\ \mu^2(a)(s), & \text{if } a \in A^2 \backslash A^1 \\ \langle g_a^3, \mu^3_{g_a^3} \rangle, & \text{such that } g_a^3 = \max\{g_a^1, g_a^2\} \text{ and } \mu^3_{g_a^3} = \max\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\} \\ & \text{where } \langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s) \text{ and } \langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s). \end{cases} \tag{19}$$

and for all $\neg a \in \neg A^1 \cup \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^3(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \backslash \neg A^2 \\ \eta^2(\neg a)(s), & \text{if } \neg a \in \neg A^2 \backslash \neg A^1 \\ \langle g_{\neg a}^3, \eta^3_{g_{\neg a}^3} \rangle, & \text{such that } g_{\neg a}^3 = \min\{g_{\neg a}^1, g_{\neg a}^2\} \text{ and } \eta^3_{g_{\neg a}^3} = \min\{\eta^1_{g_{\neg a}^1}, \eta^2_{g_{\neg a}^2}\} \\ & \text{where } \langle g_{\neg a}^1, \eta^1_{g_{\neg a}^1} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g_{\neg a}^2, \eta^2_{g_{\neg a}^2} \rangle \in \eta^2(\neg a)(s). \end{cases} \tag{20}$$

Then, for all $a \in A^1 \cup A^2$ and $s \in \mathcal{S}$:

$$\mu^{3^{\tilde{c}}}(a)(s) = \eta^3(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \text{if } a \in A^1 \backslash A^2 \\ \eta^2(\neg a)(s), & \text{if } a \in A^2 \backslash A^1 \\ \langle g_{\neg a}^3, \eta^3_{g_{\neg a}^3} \rangle, & \text{such that } g_{\neg a}^3 = \min\{g_{\neg a}^1, g_{\neg a}^2\} \text{ and } \eta^3_{g_{\neg a}^3} = \min\{\eta^1_{g_{\neg a}^1}, \eta^2_{g_{\neg a}^2}\} \\ & \text{where } \langle g_{\neg a}^1, \eta^1_{g_{\neg a}^1} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g_{\neg a}^2, \eta^2_{g_{\neg a}^2} \rangle \in \eta^2(\neg a)(s). \end{cases} \tag{21}$$

and for all $\neg a \in \neg A^1 \cup \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^{3^{\tilde{c}}}(\neg a)(s) = \mu^3(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } \neg a \in \neg A^1 \backslash \neg A^2 \\ \mu^2(a)(s), & \text{if } \neg a \in \neg A^2 \backslash \neg A^1 \\ \langle g_a^3, \mu^3_{g_a^3} \rangle, & \text{such that } g_a^3 = \max\{g_a^1, g_a^2\} \text{ and } \mu^3_{g_a^3} = \max\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\} \\ & \text{where } \langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s) \text{ and } \langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s). \end{cases} \tag{22}$$

On the other hand, let $(\mu^1, \eta^1, A^1, N)^{\tilde{c}} \mathbin{\widetilde{\sqcap}_\varepsilon} (\mu^2, \eta^2, A^2, N)^{\tilde{c}} = (\mu^4, \eta^4, A^1 \cup A^2, N)$. By definition, for all $a \in A^1 \cup A^2$ and $s \in \mathcal{S}$:

$$\mu^4(a)(s) = \begin{cases} \mu^{1^{\tilde{c}}}(a)(s) = \eta^1(\neg a)(s), & \text{if } a \in A^1 \backslash A^2 \\ \mu^{2^{\tilde{c}}}(a)(s) = \eta^2(\neg a)(s), & \text{if } a \in A^2 \backslash A^1 \\ \langle g_a^{4^{\tilde{c}}}, {\mu^{4^{\tilde{c}}}}_{g_a^4} \rangle = \langle g_{\neg a}^4, \eta^4_{g_{\neg a}^4} \rangle, & \text{such that } g_{\neg a}^4 = \min\{g_{\neg a}^1, g_{\neg a}^2\} \text{ and } \eta^4_{g_{\neg a}^4} = \min\{\eta^1_{g_{\neg a}^1}, \eta^2_{g_{\neg a}^2}\} \\ & \text{where } \langle g_{\neg a}^1, \eta^1_{g_{\neg a}^1} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g_{\neg a}^2, \eta^2_{g_{\neg a}^2} \rangle \in \eta^2(\neg a)(s). \end{cases} \tag{23}$$

and for all $\neg a \in \neg A^1 \cup \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^4(\neg a)(s) = \begin{cases} \eta^{1^{\tilde{c}}}(\neg a)(s) = \mu^1(a)(s), & \text{if } \neg a \in \neg A^1 \backslash \neg A^2 \\ \eta^{2^{\tilde{c}}}(\neg a)(s) = \mu^2(a)(s), & \text{if } \neg a \in \neg A^2 \backslash \neg A^1 \\ \langle {g^{4^{\tilde{c}}}}_{\neg a}, {\eta^{4^{\tilde{c}}}}_{g_{\neg a}^4} \rangle = \langle g_a^4, \mu^4_{g_a^4} \rangle, & \text{such that } g_a^4 = \max\{g_a^1, g_a^2\} \text{ and } \mu^4_{g_a^4} = \max\{\mu^1_{g_a^1}, \mu^2_{g_a^2}\} \\ & \text{where } \langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s) \text{ and } \langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s). \end{cases} \tag{24}$$

Since $(\mu^3, \eta^3, A^1 \cup A^2, N)^{\tilde{c}}$ and $(\mu^4, \eta^4, A^1 \cup A^2, N)$ are equivalent for all $a \in A^1 \cup A^2$ and $s \in \mathcal{S}$, the proof is concluded.

The other parts can be proven in a similar manner. ■

Proposition 5. *Let (μ^1, η^1, A^1, N) and (μ^2, η^2, A^1, N) be two FN-BS sets. Then*

1. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mu^2, \eta^2, A^1, N) = (\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mu^2, \eta^2, A^1, N)$.
2. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\varepsilon} (\mu^2, \eta^2, A^1, N) = (\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mu^2, \eta^2, A^1, N)$.
3. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mu^1, \eta^1, A^1, N) = (\mu^1, \eta^1, A^1, N)$ *and* $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mu^1, \eta^1, A^1, N) = (\mu^1, \eta^1, A^1, N)$.
4. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A^1, N) = (\mu^1, \eta^1, A^1, N)$ *and* $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A^1, N) = (\mu^{\mathbb{O}}, \eta^{\mathbb{O}}, A^1, N)$.
5. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A^1, N) = (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A^1, N)$ *and* $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mu^{\mathbb{W}}, \eta^{\mathbb{W}}, A^1, N) = (\mu^1, \eta^1, A^1, N)$.

Proof. Straightforward. ■

Proposition 6. *Let (μ^1, η^1, A^1, N) and (μ^2, η^2, A^2, N) be two FN-BS sets. Then*

1. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\varepsilon} \big((\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} (\mu^2, \eta^2, A^2, N)\big) = (\mu^1, \eta^1, A^1, N)$.
2. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\varepsilon} \big((\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} (\mu^2, \eta^2, A^2, N)\big) = (\mu^1, \eta^1, A^1, N)$.
3. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\mathfrak{R}} \big((\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\varepsilon} (\mu^2, \eta^2, A^2, N)\big) = (\mu^1, \eta^1, A^1, N)$.
4. $(\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcap}_\mathfrak{R}} \big((\mu^1, \eta^1, A^1, N) \mathbin{\widetilde{\sqcup}_\varepsilon} (\mu^2, \eta^2, A^2, N)\big) = (\mu^1, \eta^1, A^1, N)$.

Proof. (1) Suppose that $(\mu^1, \eta^1, A^1, N) \,\widetilde{\sqcap}_{\mathfrak{R}}\, (\mu^2, \eta^2, A^2, N) = (\mu^3, \eta^3, A^1 \cap A^2, N)$. Then, for all $a \in A^1 \cap A^2$ and $s \in \mathcal{S}$:

$$\mu^3(a)(s) = \langle g^3_a, \mu^3_{g^3_a} \rangle \tag{25}$$

where $g^3_a = \min\{g^1_a, g^2_a\}$ and $\mu^3_{g^3_a} = \min\{\mu^1_{g^1_a}, \mu^2_{g^2_a}\}$ with $\langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s)$ and $\langle g^2_a, \mu^2_{g^2_a} \rangle \in \mu^2(a)(s)$ and for all $\neg a \in \neg A^1 \cap \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^3(\neg a)(s) = \langle g^3_{\neg a}, \eta^3_{g^3_{\neg a}} \rangle \tag{26}$$

where $g^3_{\neg a} = \max\{g^1_{\neg a}, g^2_{\neg a}\}$ and $\eta^3_{g^3_{\neg a}} = \max\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\}$ with $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s)$ and $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s)$.

Now, let $(\mu^1, \eta^1, A^1, N) \,\widetilde{\sqcup}_{\varepsilon}\, (\mu^3, \eta^3, A^1 \cap A^2, N) = (\mu^4, \eta^4, A^1 \cup (A^1 \cap A^2), N) = (\mu^4, \eta^4, A^1, N)$. Then, for all $a \in A^1 \cup (A^1 \cap A^2)$ and $s \in \mathcal{S}$:

$$\mu^4(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash (A^1 \cap A^2) \\ \mu^3(a)(s), & \text{if } a \in (A^1 \cap A^2) \backslash A^1 = \varnothing \\ \langle g^4_a, \mu^4_{g^4_a} \rangle, & \text{such that } g^4_a = \max\{g^1_a, g^3_a\} \text{ and } \mu^4_{g^4_a} = \max\{\mu^1_{g^1_a}, \mu^3_{g^3_a}\} \\ & \text{where } \langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s) \text{ and } \langle g^3_a, \mu^3_{g^3_a} \rangle \in \mu^3(a)(s). \end{cases} \tag{27}$$

$$= \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash (A^1 \cap A^2) \\ \langle g^4_a, \mu^4_{g^4_a} \rangle, & \text{such that } g^4_a = \max\{g^1_a, \min\{g^1_a, g^2_a\}\} \text{ and } \mu^4_{g^4_a} = \max\{\mu^1_{g^1_a}, \min\{\mu^1_{g^1_a}, \mu^2_{g^2_a}\} \\ & \text{where } \langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s) \text{ and } \langle g^2_a, \mu^2_{g^2_a} \rangle \in \mu^2(a)(s). \end{cases} \tag{28}$$

and for all $\neg a \in \neg A^1 \cup (\neg A^1 \cap \neg A^2)$ and $s \in \mathcal{S}$:

$$\eta^4(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \backslash (\neg A^1 \cap \neg A^2) \\ \eta^3(\neg a)(s), & \text{if } \neg a \in (\neg A^1 \cap \neg A^2) \backslash \neg A^1 = \varnothing \\ \langle g^4_{\neg a}, \eta^4_{g^4_{\neg a}} \rangle, & \text{such that } g^4_{\neg a} = \max\{g^1_{\neg a}, g^3_{\neg a}\} \text{ and } \eta^4_{g^4_{\neg a}} = \max\{\eta^1_{g^1_{\neg a}}, \eta^3_{g^3_{\neg a}}\} \\ & \text{where } \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g^3_{\neg a}, \eta^3_{g^3_{\neg a}} \rangle \in \eta^3(\neg a)(s). \end{cases} \tag{29}$$

$$= \begin{cases} \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \backslash (\neg A^1 \cap \neg A^2) \\ \langle g^4_{\neg a}, \eta^4_{g^4_{\neg a}} \rangle, & \text{such that } g^4_{\neg a} = \max\{g^1_{\neg a}, \min\{g^1_{\neg a}, g^2_{\neg a}\}\} \text{ and } \eta^4_{g^4_{\neg a}} = \max\{\eta^1_{g^1_{\neg a}}, \min\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\} \\ & \text{where } \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s) \text{ and } \langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s). \end{cases} \tag{30}$$

Hence,

$$\mu^4(a)(s) = \begin{cases} \mu^1(a)(s), & \text{if } a \in A^1 \backslash (A^1 \cap A^2) \\ \mu^1(a)(s), & \text{if } a \in A^1 \cap A^2 \end{cases} \tag{31}$$

and

$$\eta^4(\neg a)(s) = \begin{cases} \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \backslash (\neg A^1 \cap \neg A^2) \\ \eta^1(\neg a)(s), & \text{if } \neg a \in \neg A^1 \cap \neg A^2 \end{cases} \tag{32}$$

Therefore, $(\mu^1, \eta^1, A^1, N) \,\widetilde{\sqcup}_{\varepsilon}\, \left((\mu^1, \eta^1, A^1, N) \,\widetilde{\sqcap}_{\mathfrak{R}}\, (\mu^2, \eta^2, A^2, N)\right) = (\mu^1, \eta^1, A^1, N)$.

The other parts can be proven in a similar manner. ■

Proposition 7. *Let* $(\mu^1, \eta^1, A^1, N^1)$, $(\mu^2, \eta^2, A^2, N^2)$, *and* $(\mu^3, \eta^3, A^3, N^3)$ *be three FN-BS sets and let* $\odot \in \{\widetilde{\sqcup}_\varepsilon, \widetilde{\sqcap}_\varepsilon, \widetilde{\sqcup}_\mathfrak{R}, \widetilde{\sqcap}_\mathfrak{R}\}$. *Then*

1. $(\mu^1, \eta^1, A^1, N^1) \odot (\mu^2, \eta^2, A^2, N^2) = (\mu^2, \eta^2, A^2, N^2) \odot (\mu^1, \eta^1, A^1, N^1)$.
2. $(\mu^1, \eta^1, A^1, N^1) \odot \big((\mu^2, \eta^2, A^2, N^2) \odot (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \odot (\mu^2, \eta^2, A^2, N^2)\big) \odot (\mu^3, \eta^3, A^3, N^3)$.

Proof. Straightforward. ■

Proposition 8. *Let* $(\mu^1, \eta^1, A^1, N^1)$, $(\mu^2, \eta^2, A^2, N^2)$, *and* $(\mu^3, \eta^3, A^3, N^3)$ *be three FN-BS sets. Then*

1. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\varepsilon\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\varepsilon\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcap}_\mathfrak{R}\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\varepsilon\, (\mu^3, \eta^3, A^3, N^3)\big)$.
2. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcup}_\varepsilon\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcup}_\varepsilon\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big)$.
3. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\varepsilon\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\varepsilon\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcup}_\mathfrak{R}\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\varepsilon\, (\mu^3, \eta^3, A^3, N^3)\big)$.
4. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcap}_\varepsilon\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcap}_\varepsilon\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big)$.
5. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcap}_\mathfrak{R}\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big)$.
6. $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, \big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big) = \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^2, \eta^2, A^2, N^2)\big) \,\widetilde{\sqcup}_\mathfrak{R}\, \big((\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcap}_\mathfrak{R}\, (\mu^3, \eta^3, A^3, N^3)\big)$.

Proof. (4) Suppose that $\big((\mu^2, \eta^2, A^2, N^2) \,\widetilde{\sqcap}_\varepsilon\, (\mu^3, \eta^3, A^3, N^3)\big) = (\mu^4, \eta^4, A^2 \cup A^3, \max(N^2, N^3))$, then for all $a \in A^2 \cup A^3$ and $s \in \mathcal{S}$:

$$\mu^4(a)(s) = \begin{cases} \mu^2(a)(s), & \text{if } a \in A^2 \backslash A^3 \\ \mu^3(a)(s), & \text{if } a \in A^3 \backslash A^2 \\ \langle g_a^4, \mu^4_{g_a^4} \rangle, & \text{such that } g_a^4 = \min\{g_a^2, g_a^3\} \text{ and } \mu^4_{g_a^4} = \min\{\mu^2_{g_a^2}, \mu^3_{g_a^3}\} \\ & \text{where } \langle g_a^2, \mu^2_{g_a^2} \rangle \in \mu^2(a)(s) \text{ and } \langle g_a^3, \mu^3_{g_a^3} \rangle \in \mu^3(a)(s). \end{cases} \tag{33}$$

and for all $\neg a \in \neg A^2 \cup \neg A^3$ and $s \in \mathcal{S}$:

$$\eta^4(\neg a)(s) = \begin{cases} \eta^2(\neg a)(s), & \text{if } \neg a \in \neg A^2 \backslash \neg A^3 \\ \eta^3(\neg a)(s), & \text{if } \neg a \in \neg A^3 \backslash \neg A^2 \\ \langle g_{\neg a}^4, \eta^4_{g_{\neg a}^4} \rangle, & \text{such that } g_{\neg a}^4 = \max\{g_{\neg a}^2, g_{\neg a}^3\} \text{ and } \eta^4_{g_{\neg a}^4} = \max\{\eta^2_{g_{\neg a}^2}, \eta^3_{g_{\neg a}^3}\} \\ & \text{where } \langle g_{\neg a}^2, \eta^2_{g_{\neg a}^2} \rangle \in \eta^2(\neg a)(s) \text{ and } \langle g_{\neg a}^3, \eta^3_{g_{\neg a}^3} \rangle \in \eta^3(\neg a)(s). \end{cases} \tag{34}$$

Let $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_\mathfrak{R}\, (\mu^4, \eta^4, A^2 \cup A^3, \max(N^2, N^3)) = (\mu^5, \eta^5, A^1 \cap (A^2 \cup A^3), \max(N^1, \max(N^2, N^3))) = (\mu^5, \eta^5, O \cup P, \max(N^1, N^2, N^3))$ where $O = A^1 \cap A^2$ and $P = A^1 \cap A^3$, then for all $a \in O \cup P$ and $s \in \mathcal{S}$:

$$\mu^5(a)(s) = \langle g_a^5, \mu^5_{g_a^5} \rangle \tag{35}$$

where $g_a^5 = \max\{g_a^1, g_a^4\}$ and $\mu^5_{g_a^5} = \max\{\mu^1_{g_a^1}, \mu^4_{g_a^4}\}$ with $\langle g_a^1, \mu^1_{g_a^1} \rangle \in \mu^1(a)(s)$ and $\langle g_a^4, \mu^4_{g_a^4} \rangle \in \mu^4(a)(s)$.

and for all $\neg a \in \neg O \cup \neg P$ and $s \in \mathcal{S}$:

$$\eta^5(\neg a)(s) = \langle g^5_{\neg a}, \eta^5_{g^5_{\neg a}} \rangle \tag{36}$$

where $g^5_{\neg a} = \min\{g^1_{\neg a}, g^4_{\neg a}\}$ and $\eta^5_{g^5_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^4_{g^4_{\neg a}}\}$ with $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s)$ and $\langle g^4_{\neg a}, \eta^4_{g^4_{\neg a}} \rangle \in \eta^4(\neg a)(s)$. Hence, for all $a \in O \cup P$ and $s \in \mathcal{S}$:

$$\mu^5(a)(s) = \begin{cases} \max\{\mu^1(a)(s), \mu^2(a)(s)\}, & \text{if } a \in O \backslash P \\ \max\{\mu^1(a)(s), \mu^3(a)(s)\}, & \text{if } a \in P \backslash O \\ \langle g^5_a, \mu^5_{g^5_a} \rangle, & \text{such that } g^5_a = \max\{g^1_a, \min\{g^2_a, g^3_a\} \text{ and} \\ & \mu^5_{g^5_a} = \max\{\mu^1_{g^1_a}, \min\{\mu^2_{g^2_a}, \mu^3_{g^3_a}\} \text{ where} \\ & \langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s), \langle g^2_a, \mu^2_{g^2_a} \rangle \in \mu^2(a)(s), \\ & \text{and } \langle g^3_a, \mu^3_{g^3_a} \rangle \in \mu^3(a)(s).\} \end{cases} \tag{37}$$

and for all $\neg a \in \neg O \cup \neg P$ and $s \in \mathcal{S}$:

$$\eta^5(\neg a)(s) = \begin{cases} \min\{\eta^1(\neg a)(s), \eta^2(\neg a)(s)\}, & \text{if } \neg a \in \neg O \backslash \neg P \\ \min\{\eta^1(\neg a)(s), \eta^3(\neg a)(s)\}, & \text{if } \neg a \in \neg P \backslash \neg O \\ \langle g^5_{\neg a}, \eta^5_{g^5_{\neg a}} \rangle, & \text{such that } g^5_{\neg a} = \min\{g^1_{\neg a}, \max\{g^2_{\neg a}, g^3_{\neg a}\}\} \text{ and} \\ & \eta^5_{g^5_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \max\{\eta^2_{g^2_{\neg a}}, \eta^3_{g^3_{\neg a}}\} \text{ where} \\ & \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s), \langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s), \\ & \text{and } \langle g^3_{\neg a}, \eta^3_{g^3_{\neg a}} \rangle \in \eta^3(\neg a)(s). \end{cases} \tag{38}$$

On the other hand, let $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_{\mathfrak{R}}\, (\mu^2, \eta^2, A^2, N^2) = (\mu^6, \eta^6, A^1 \cap A^2, \max(N^1, N^2))$, then for all $a \in A^1 \cap A^2$ and $s \in \mathcal{S}$:

$$\mu^6(a)(s) = \langle g^6_a, \mu^6_{g^6_a} \rangle \tag{39}$$

where $g^6_a = \max\{g^1_a, g^2_a\}$ and $\mu^6_{g^6_a} = \max\{\mu^1_{g^1_a}, \mu^2_{g^2_a}\}$ with $\langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s)$ and $\langle g^2_a, \mu^2_{g^2_a} \rangle \in \mu^2(a)(s)$.
and for all $\neg a \in \neg A^1 \cap \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^6(\neg a)(s) = \langle g^6_{\neg a}, \eta^6_{g^6_{\neg a}} \rangle \tag{40}$$

where $g^6_{\neg a} = \min\{g^1_{\neg a}, g^2_{\neg a}\}$ and $\eta^6_{g^6_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^2_{g^2_{\neg a}}\}$ with $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}} \rangle \in \eta^1(\neg a)(s)$ and $\langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}} \rangle \in \eta^2(\neg a)(s)$. Next, let $(\mu^1, \eta^1, A^1, N^1) \,\widetilde{\sqcup}_{\mathfrak{R}}\, (\mu^3, \eta^3, A^3, N^3) = (\mu^7, \eta^7, A^1 \cap A^2, \max(N^1, N^2))$, then for all $a \in A^1 \cap A^3$ and $s \in \mathcal{S}$:

$$\mu^7(a)(s) = \langle g^7_a, \mu^7_{g^7_a} \rangle \tag{41}$$

where $g^7_a = \max\{g^1_a, g^3_a\}$ and $\mu^7_{g^7_a} = \max\{\mu^1_{g^1_a}, \mu^3_{g^3_a}\}$ with $\langle g^1_a, \mu^1_{g^1_a} \rangle \in \mu^1(a)(s)$ and $\langle g^3_a, \mu^3_{g^3_a} \rangle \in \mu^3(a)(s)$.
and for all $\neg a \in \neg A^1 \cap \neg A^2$ and $s \in \mathcal{S}$:

$$\eta^7(\neg a)(s) = \langle g^7_{\neg a}, \eta^7_{g^7_{\neg a}} \rangle \tag{42}$$

where $g^7_{\neg a} = \min\{g^1_{\neg a}, g^3_{\neg a}\}$ and $\eta^7_{g^7_{\neg a}} = \min\{\eta^1_{g^1_{\neg a}}, \eta^3_{g^3_{\neg a}}\}$ with $\langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}}\rangle \in \eta^1(\neg a)(s)$ and $\langle g^3_{\neg a}, \eta^3_{g^3_{\neg a}}\rangle \in \eta^3(\neg a)(s)$. Now, suppose that $(\mu^6, \eta^6, A^1 \cap A^2, \max(N^1, N^2))\ \widetilde{\sqcap}_\varepsilon\ (\mu^7, \eta^7, A^1 \cap A^3, \max(N^1, N^3)) = (\mu^8, \eta^8, O \cup P), \max(N^1, N^2, N^3))$ where $O = A^1 \cap A^2$ and $P = A^1 \cap A^3$, then for all $a \in O \cup P$ and $s \in \mathcal{S}$:

$$\mu^8(a)(s) = \begin{cases} \mu^6(a)(s), & \text{if } a \in O\backslash P \\ \mu^7(a)(s), & \text{if } a \in P\backslash O \\ \langle g^8_a, \mu^8_{g^8_a}\rangle, & \text{such that } g^8_a = \min\{g^6_a, g^7_a\} \text{ and } \mu^8_{g^8_a} = \min\{\mu^6_{g^6_a}, \mu^7_{g^7_a}\} \\ & \text{where } \langle g^6_a, \mu^6_{g^6_a}\rangle \in \mu^6(a)(s) \text{ and } \langle g^7_a, \mu^7_{g^7_a}\rangle \in \mu^7(a)(s). \end{cases} \tag{43}$$

$$= \begin{cases} \max\{\mu^1(a)(s), \mu^2(a)(s)\}, & \text{if } a \in O\backslash P \\ \max\{\mu^1(a)(s), \mu^3(a)(s)\}, & \text{if } a \in P\backslash O \\ \langle g^8_a, \mu^8_{g^8_a}\rangle, & \text{such that } g^8_a = \min\{\max\{g^1_a, g^2_a\}, \max\{g^1_a, g^3_a\}\} \text{ and} \\ & \mu^8_{g^8_a} = \min\{\max\{\mu^1_{g^1_a}, \mu^2_{g^2_a}\}, \max\{\mu^1_{g^1_a}, \mu^3_{g^3_a}\}\} \text{ where} \\ & \langle g^1_a, \mu^1_{g^1_a}\rangle \in \mu^1(a)(s), \langle g^2_a, \mu^2_{g^2_a}\rangle \in \mu^2(a)(s), \text{and } \langle g^3_a, \mu^3_{g^3_a}\rangle \in \mu^3(a)(s). \end{cases} \tag{44}$$

and for all $\neg a \in \neg O \cup \neg P$ and $s \in \mathcal{S}$:

$$\eta^8(\neg a)(s) = \begin{cases} \eta^6(\neg a)(s), & \text{if } \neg a \in \neg O\backslash\neg P \\ \eta^7(\neg a)(s), & \text{if } \neg a \in \neg P\backslash\neg O \\ \langle g^8_{\neg a}, \eta^8_{g^8_{\neg a}}\rangle, & \text{such that } g^8_{\neg a} = \max\{g^6_{\neg a}, g^7_{\neg a}\} \text{ and } \eta^8_{g^8_{\neg a}} = \max\{\eta^6_{g^6_{\neg a}}, \eta^7_{g^7_{\neg a}}\} \\ & \text{where } \langle g^6_{\neg a}, \eta^6_{g^6_{\neg a}}\rangle \in \eta^6(\neg a)(s) \text{ and } \langle g^7_{\neg a}, \eta^7_{g^7_{\neg a}}\rangle \in \eta^7(\neg a)(s). \end{cases} \tag{45}$$

$$= \begin{cases} \min\{\eta^1(\neg a)(s), \eta^2(\neg a)(s)\}, & \text{if } \neg a \in \neg O\backslash\neg P \\ \min\{\eta^1(\neg a)(s), \eta^3(\neg a)(s)\}, & \text{if } \neg a \in \neg P\backslash\neg O \\ \langle g^8_{\neg a}, \eta^8_{g^8_{\neg a}}\rangle, & \text{such that } g^8_{\neg a} = \max\{\min\{g^1_{\neg a}, g^2_{\neg a}\}\} \text{ and} \\ & \eta^8_{g^8_{\neg a}} = \max\{\min\{\eta^1_{g^1_{\neg a}}, \eta^3_{g^3_{\neg a}}\}\} \text{ where} \\ & \langle g^1_{\neg a}, \eta^1_{g^1_{\neg a}}\rangle \in \eta^1(\neg a)(s), \langle g^2_{\neg a}, \eta^2_{g^2_{\neg a}}\rangle \in \eta^2(\neg a)(s), \\ & \text{and } \langle g^3_{\neg a}, \eta^3_{g^3_{\neg a}}\rangle \in \eta^3(\neg a)(s). \end{cases} \tag{46}$$

Since $(\mu^5, \eta^5, O \cup P, \max(N^1, N^2, N^3))$ and $(\mu^8, \eta^8, O \cup P, \max(N^1, N^2, N^3))$ are equivalent for all $a \in O \cup P$ and $s \in \mathcal{S}$, the proof is concluded.

The other parts can be proven in a similar manner. ■

4 Application to MCDM Using Fuzzy N-Bipolar Soft Sets

In this section, we present two algorithms based on the proposed FN-BS set model for evaluating and selecting optimal choices in DM scenarios. Subsequently, we demonstrate the application of these algorithms to a practical case study: selecting the best vaccination program plan across multiple countries.

We can summarize these algorithms through flowcharts that demonstrate their respective steps. The first flowchart in Fig. 1 illustrates the process for determining choice values using the FN-BS set, as described in Algorithm 1. The second flowchart in Fig. 2 represents the procedure for computing choice values with thresholds, as detailed in Algorithm 2.

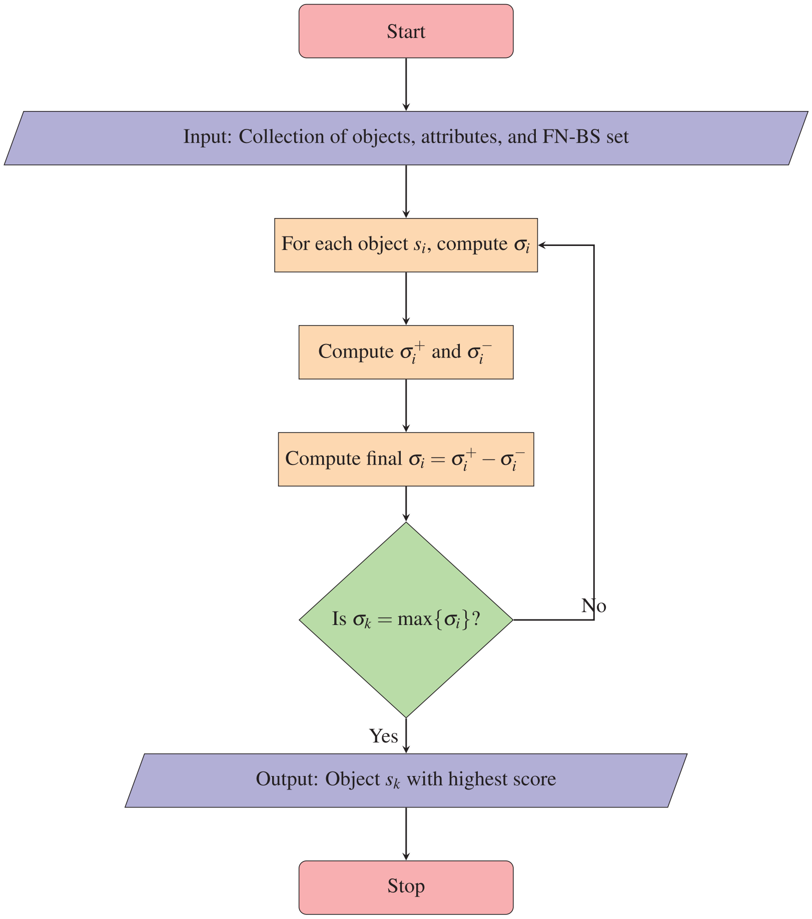

Figure 1: Flowchart for determining choice values using FN-BS set (Algorithm 1)

Algorithm 1: Algorithm for determining choice values using FN-BS set

- **Input:**
  1. A collection of objects $\mathcal{S} = \{s_i : i = 1, 2, \ldots, m\}$.
  2. A collection of attributes $A = \{a_j : j = 1, 2, \ldots, n\}$.
  3. The FN-BS set (μ, η, A, N).
- **Procedure**:
  1. For each object s_i, compute the aggregate score σ_i as:

$$\sigma_i = \sigma_i^+ - \sigma_i^-, \tag{47}$$

where:

$$\sigma_i^+ = \left\langle \sum_{j=1}^{n} g_{ij}^+, \sum_{j=1}^{n} \mu_{ij} \right\rangle, \tag{48}$$

$$\sigma_i^- = \left\langle \sum_{j=1}^{n} g_{ij}^-, \sum_{j=1}^{n} \eta_{ij} \right\rangle, \tag{49}$$

(Continued)

Algorithm 1 (continued)

and subtraction is performed component-wise as:

$$\sigma_i = \left\langle \sum_{j=1}^{n} g_{ij}^{+} - \sum_{j=1}^{n} g_{ij}^{-}, \sum_{j=1}^{n} \mu_{ij} - \sum_{j=1}^{n} \eta_{ij} \right\rangle. \quad (50)$$

2. Identify the object index k such that:

$$\sigma_k = \max\{\sigma_i : i = 1, 2, \ldots, m\}. \quad (51)$$

- **Output:** The result is the object s_k that satisfies the condition defined in Step 2 of the procedure.

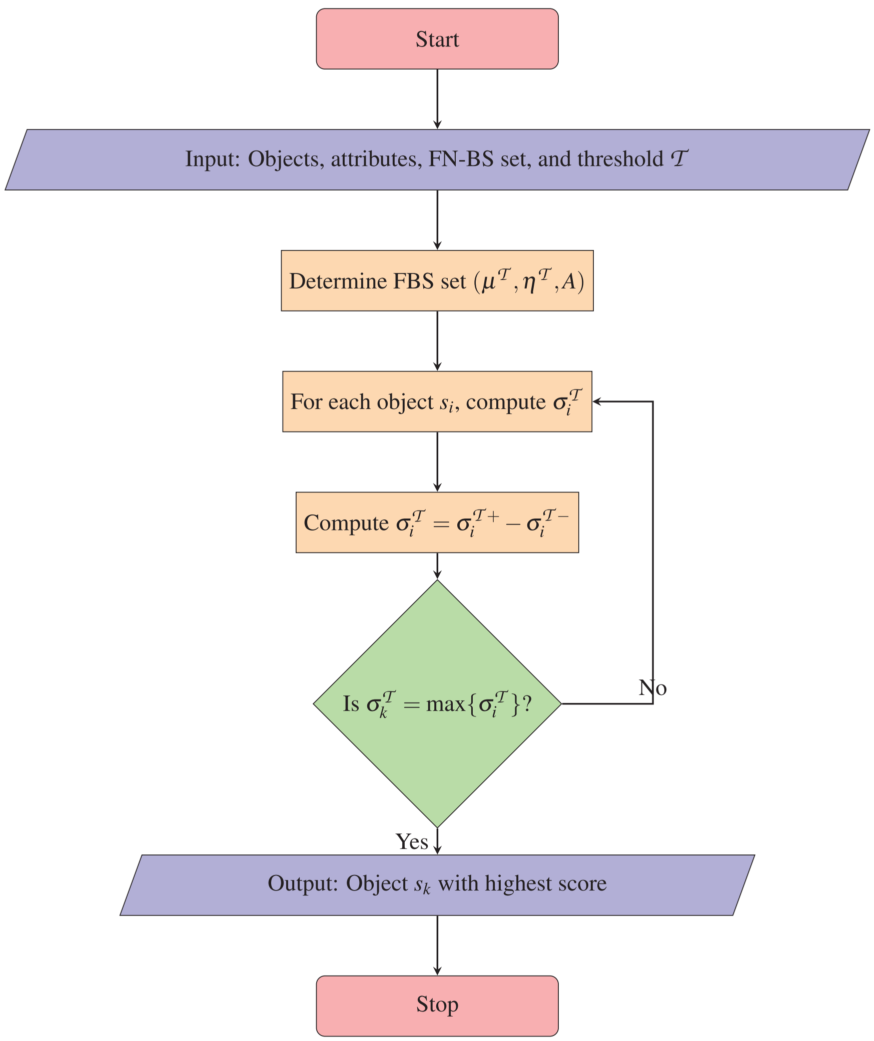

Figure 2: Flowchart for computing choice values using FN-BS set with thresholds (Algorithm 2)

Algorithm 2: Algorithm for computing choice values using FN-BS set with thresholds

- **Input:**
  1. A collection of objects $\mathcal{S} = \{s_i : i = 1, 2, \ldots, m\}$.
  2. A collection of attributes $A = \{a_j : j = 1, 2, \ldots, n\}$.
  3. The FN-BS set (μ, η, A, N).
  4. A predefined threshold value $\mathcal{T}$.
- **Procedure:**
  1. Determine the FBS set $(\mu^{\mathcal{T}}, \eta^{\mathcal{T}}, A)$ corresponding to the FN-BS set and the threshold $\mathcal{T}$, as outlined in Definition 15.
  2. For each object s_i, compute the aggregate score $\sigma_i^{\mathcal{T}}$ as:

$$\sigma_i^{\mathcal{T}} = \sigma_i^{\mathcal{T}^+} - \sigma_i^{\mathcal{T}^-}, \tag{52}$$

where:

$$\sigma_i^{\mathcal{T}^+} = \left\langle \sum_{j=1}^{n} \mu_{ij}^{\mathcal{T}} \right\rangle, \tag{53}$$

and

$$\sigma_i^{\mathcal{T}^-} = \left\langle \sum_{j=1}^{n} \eta_{ij}^{\mathcal{T}} \right\rangle. \tag{54}$$

3. Identify the object index k such that:

$$\sigma_k^{\mathcal{T}} = \max\{\sigma_i^{\mathcal{T}} : i = 1, 2, \ldots, m\}. \tag{55}$$

- **Output:** The result is the object s_k that satisfies the condition defined in Step 2 of the procedure.

Numerical Example: Evaluation of Vaccination Programs in Preventing Disease Outbreaks

Vaccination programs are critical in preventing the spread of infectious diseases and maintaining public health. The success of such programs depends on several factors that ensure effective immunization coverage and disease control.

Let us consider a set of countries $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$, where each s_i represents a country. The set of attributes $A = \{a_1 =$ Vaccination coverage rate, $a_2 =$ Availability of vaccines, $a_3 =$ Trained healthcare workers for vaccine delivery, $a_4 =$ Public awareness about vaccination benefits$\}$ are the key factors to evaluate the effectiveness of the vaccination programs. Additionally, we define the challenges in $\neg A = \{\neg a_1 =$ Low vaccination coverage, $\neg a_2 =$ Shortage of vaccines, $\neg a_3 =$ Lack of trained staff, $\neg a_4 =$ Misinformation about vaccines$\}$, which represent obstacles that may negatively impact the vaccination effort. A 5-BS set can be obtained from Table 14, where

- One circle "$\circ$" represents not recommended.
- One star "$\star$" represents below average.
- Two stars "$\star\star$" represent average.
- Three stars "$\star\star\star$" represent strongly recommended.
- Four stars "$\star\star\star\star$" represent beyond excellent.

Table 14: Information extracted from related data in vaccination program

$S\backslash A$	a_1	a_2	a_3	a_4
s_1	$\star\star\star\star$	$\star\star$	$\star$	$\star\star$
s_2	$\star\star\star$	$\star$	$\circ$	$\star\star\star\star$
s_3	$\circ$	$\star\star$	$\star\star$	$\star\star$
s_4	$\star$	$\circ$	$\star\star\star$	$\star\star\star$
s_5	$\circ$	$\star\star$	$\star$	$\star$
$S\backslash\neg A$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	$\circ$	$\star\star$	$\star\star\star$	$\star$
s_2	$\circ$	$\star\star\star$	$\star\star$	$\circ$
s_3	$\star\star\star\star$	$\star$	$\star\star$	$\circ$
s_4	$\star\star\star$	$\circ$	$\star$	$\star$
s_5	$\star\star\star$	$\star$	$\star\star\star$	$\star$

This graded evaluation by symbols can easily be identified with numbers as in $G = \{0, 1, 2, 3, 4\}$ where

- 0 corresponds to $\circ$.
- 1 corresponds to $\star$.
- 2 corresponds to $\star\star$.
- 3 corresponds to $\star\star\star$.
- 4 corresponds to $\star\star\star\star$.

The tabular representation of the 5-BS set $(\mu', \eta', A, 5)$ is shown in Table 15.

Table 15: Tabular form of 5-BS set $(\mu', \eta', A, 5)$ for the vaccination program

$(\mu', A, 5)$	a_1	a_2	a_3	a_4
s_1	4	2	1	2
s_2	3	1	0	4
s_3	0	2	2	2
s_4	1	0	3	3
s_5	0	2	1	1
$(\eta', \neg A, 5)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	0	2	3	1
s_2	0	3	2	0
s_3	4	1	2	0
s_4	3	0	1	1
s_5	3	1	3	1

Membership values are assigned based on the evaluation grades of the vaccination program attributes as follows:

For positive membership values (μ_{g_a}):

$$\begin{array}{lll} 0 \le \mu_{g_a} < 0.2 & \text{when} & g_a = 0, \\ 0.2 \le \mu_{g_a} < 0.4 & \text{when} & g_a = 1, \\ 0.4 \le \mu_{g_a} < 0.6 & \text{when} & g_a = 2, \\ 0.6 \le \mu_{g_a} < 0.8 & \text{when} & g_a = 3, \\ 0.8 \le \mu_{g_a} \le 1 & \text{when} & g_a = 4. \end{array} \tag{56}$$

For negative membership values ($\eta_{g_{\neg a}}$):

$$\begin{array}{lll} 0 \le \eta_{g_{\neg a}} < 0.2 & \text{when} & g_{\neg a} = 0, \\ 0.2 \le \eta_{g_{\neg a}} < 0.4 & \text{when} & g_{\neg a} = 1, \\ 0.4 \le \eta_{g_{\neg a}} < 0.6 & \text{when} & g_{\neg a} = 2, \\ 0.6 \le \eta_{g_{\neg a}} < 0.8 & \text{when} & g_{\neg a} = 3, \\ 0.8 \le \eta_{g_{\neg a}} \le 1 & \text{when} & g_{\neg a} = 4. \end{array} \tag{57}$$

The F5-BS set $(\mu, \eta, A, 5)$, derived from these information, is presented in Table 16.

Table 16: Tabular form of F5-BS set $(\mu, \eta, A, 5)$ for the vaccination program

$(\mu, A, 5)$	a_1	a_2	a_3	a_4
s_1	$\langle 4, 0.8\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$
s_2	$\langle 3, 0.6\rangle$	$\langle 1, 0.2\rangle$	$\langle 0, 0.1\rangle$	$\langle 4, 0.9\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 2, 0.5\rangle$
s_4	$\langle 1, 0.2\rangle$	$\langle 0, 0.1\rangle$	$\langle 3, 0.7\rangle$	$\langle 3, 0.7\rangle$
s_5	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$
$(\eta, \neg A, 5)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$
s_2	$\langle 0, 0.1\rangle$	$\langle 3, 0.7\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
s_3	$\langle 4, 0.9\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$
s_4	$\langle 3, 0.6\rangle$	$\langle 0, 0.1\rangle$	$\langle 1, 0.2\rangle$	$\langle 1, 0.3\rangle$
s_5	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.3\rangle$

4.1 Approach 1: Score Aggregation for Vaccination Program Evaluation Using Algorithm 1

In this approach, we apply Algorithm 1 to compute the aggregate score σ_i for each country s_i based on the vaccination program attributes and challenges outlined previously. The detailed steps are as follows:

Step: 1 For each object s_i, compute σ_i^+ and σ_i^- using Eqs. (48) and (49), respectively. The data required for these computations are provided in Table 16. The computed values for each country are shown in Table 17.

Step: 2 For each object s_i, compute the overall score σ_i as shown in Eq. (50). The results of this computation are displayed in Table 18.

Step: 3 Based on the aggregate scores σ_i shown in Table 18, the country with the highest total score is identified as the optimal choice for the best vaccination program. From the results, it is evident that country s_1 has the highest score, indicating that it offers the most effective vaccination program among the considered countries.

Table 17: Vaccination program attribute and challenge evaluation (Approach 1)

$(\mu, A, 5)$	a_1	a_2	a_3	a_4	σ_i^+
s_1	$\langle 4, 0.8\rangle$	$\langle 2, 0.4\rangle$	$\langle 1, 0.3\rangle$	$\langle 2, 0.5\rangle$	$\langle 9, 2\rangle$
s_2	$\langle 3, 0.6\rangle$	$\langle 1, 0.3\rangle$	$\langle 0, 0\rangle$	$\langle 4, 0.9\rangle$	$\langle 8, 1.8\rangle$
s_3	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.4\rangle$	$\langle 2, 0.5\rangle$	$\langle 6, 1.4\rangle$
s_4	$\langle 1, 0.3\rangle$	$\langle 0, 0.1\rangle$	$\langle 3, 0.7\rangle$	$\langle 3, 0.6\rangle$	$\langle 7, 1.7\rangle$
s_5	$\langle 0, 0.1\rangle$	$\langle 2, 0.5\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 4, 1.2\rangle$
$(\eta, \neg A, 5)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$	σ_i^-
s_1	$\langle 0, 0.1\rangle$	$\langle 2, 0.4\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.2\rangle$	$\langle 6, 1.4\rangle$
s_2	$\langle 0, 0\rangle$	$\langle 3, 0.7\rangle$	$\langle 2, 0.5\rangle$	$\langle 0, 0.1\rangle$	$\langle 5, 1.3\rangle$
s_3	$\langle 4, 0.9\rangle$	$\langle 1, 0.2\rangle$	$\langle 2, 0.4\rangle$	$\langle 0, 0.1\rangle$	$\langle 7, 1.6\rangle$
s_4	$\langle 3, 0.7\rangle$	$\langle 0, 0.1\rangle$	$\langle 1, 0.3\rangle$	$\langle 1, 0.3\rangle$	$\langle 5, 1.4\rangle$
s_5	$\langle 3, 0.6\rangle$	$\langle 1, 0.3\rangle$	$\langle 3, 0.7\rangle$	$\langle 1, 0.2\rangle$	$\langle 8, 1.8\rangle$

Table 18: Overall evaluation of vaccination program effectiveness (Approach 1)

$\mathcal{S}$	$\sigma_i^+ - \sigma_i^-$	$\sigma_i = \sigma_i^+ - \sigma_i^-$
s_1	$\langle 9-6, 2-1.4\rangle$	$\langle 3, 0.6\rangle$
s_2	$\langle 8-5, 1.8-1.3\rangle$	$\langle 3, 0.5\rangle$
s_3	$\langle 6-7, 1.4-1.6\rangle$	$\langle -1, -0.2\rangle$
s_4	$\langle 7-5, 1.7-1.4\rangle$	$\langle 2, 0.3\rangle$
s_5	$\langle 4-8, 1.2-1.8\rangle$	$\langle -4, -0.6\rangle$

4.2 Approach 2: Score Aggregation for Vaccination Program Evaluation Using Algorithm 2

In this approach, we utilize Algorithm 2 to calculate the aggregate score σ_i^2 for each country s_i. The detailed steps are as follows:

Step: 1 Determine the FBS set (μ^2, η^2, A), which corresponds to the F5-BS set as presented in Table 16. The FBS set (μ^2, η^2, A) is shown in Table 19.

Step: 2 For each object s_i, calculate σ_i^{2+} and σ_i^{2-} using Eqs. (53) and (54), respectively. The computed values for each country are displayed in Table 20.

Step: 3 For each object s_i, calculate the overall score. The results are presented in Table 21.

Step: 4 Based on the aggregate scores σ_i^2 shown in Table 21, the optimal choice is s_1, which has the highest score of 1.3.

Table 19: Tabular form of FBS set $(\mu^{\mathcal{T}=2}, \eta^{\mathcal{T}=2}, A)$ for the vaccination scenario

$(\mu^{\mathcal{T}=2}, A)$	a_1	a_2	a_3	a_4
s_1	0.8	0.4	0.3	0.5
s_2	0.6	0	0	0.9
s_3	0	0.4	0.4	0.5
s_4	0	0	0.7	0.6

(Continued)

Table 19 (continued)

$(\mu^{\mathcal{T}=2}, A)$	a_1	a_2	a_3	a_4
s_5	0	0.5	0	0
$(\eta^{\mathcal{T}=2}, \neg A)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$
s_1	0	0	0.7	0
s_2	0	0.7	0	0
s_3	0.9	0	0	0
s_4	0.7	0	0	0
s_5	0.6	0	0.7	0

Table 20: Vaccination program attribute and challenge evaluation (Approach 2)

$(\mu^{\mathcal{T}=2}, A)$	a_1	a_2	a_3	a_4	$\sigma_i^{2^+}$
s_1	0.8	0.4	0.3	0.5	2
s_2	0.6	0	0	0.9	1.5
s_3	0	0.4	0.4	0.5	1.3
s_4	0	0	0.7	0.6	1.3
s_5	0	0.5	0	0	0.5
$(\eta^{\mathcal{T}=2}, \neg A)$	$\neg a_1$	$\neg a_2$	$\neg a_3$	$\neg a_4$	$\sigma_i^{2^-}$
s_1	0	0	0.7	0	0.7
s_2	0	0.7	0	0	0.7
s_3	0.9	0	0	0	0.9
s_4	0.7	0	0	0	0.7
s_5	0.6	0	0.7	0	1.3

Table 21: Overall evaluation of vaccination program effectiveness (Approach 2)

$\mathcal{S}$	$\sigma_i^{2^+} - \sigma_i^{2^-}$	$\sigma_i^2 = \sigma_i^{2^+} - \sigma_i^{2^-}$
s_1	2 − 0.7	1.3
s_2	1.5 − 0.7	0.8
s_3	1.3 − 0.9	0.4
s_4	1.3 − 0.7	0.6
s_5	0.5 − 1.3	−0.8

5 Comparisons and Discussions

In this section, Approaches 4.1 and 4.2 are applied to assess and compare the results generated by the two algorithms. By examining the outcomes produced by both algorithms (as shown in Table 22), it is observed that the rankings among the objects are identical. This striking similarity in the results suggests that the algorithms provide consistent rankings under the given threshold settings. However, it is crucial to note that the decision rankings are sensitive to the choice of threshold values. A small change in the threshold could lead to a different ranking of the objects, highlighting the model's inherent sensitivity to these values.

Table 22: Comparison of FN-BS algorithms for choice value determination and computation

Algorithm no.	Algorithm name	Decision ranking
Algorithm 1	FN-BS choice value determination	$s_1 > s_2 > s_4 > s_3 > s_5$
Algorithm 2	FN-BS choice value computation with thresholds	$s_1 > s_2 > s_4 > s_3 > s_5$

This sensitivity to threshold values is a critical aspect of the model's performance. In real-world DM scenarios, especially in healthcare applications where the threshold might represent different levels of risk or patient severity, small variations in threshold values can significantly alter the outcomes. For example, in a clinical decision support system, shifting the threshold for a medical condition might change the prioritization of treatment protocols, which could have a significant impact on patient outcomes. Therefore, decision-makers must be aware of this sensitivity and consider the potential consequences of changing threshold values.

To mitigate this, it is suggested that a more dynamic threshold adjustment mechanism could be integrated into the model. This could allow the model to adapt to varying conditions such as changes in available data, user preferences, or external factors, providing more stable and reliable results. Despite this, the algorithms demonstrate strong adaptability and versatility in handling the DM process, as evidenced by their ability to produce consistent rankings in controlled settings.

6 Comparative Analysis

The FN-BS model has been designed to address the complexities of DM with multiple conflicting criteria. In this section, we compare the FN-BS model with existing DM approaches, highlighting its advantages, limitations, and potential defects, while also identifying ways to improve its performance.

6.1 Advantages of the Model

The FN-BS model offers several advantages over traditional models:

1. **Dual Uncertainty Representation:** FN-BS captures both positive and negative aspects of uncertainty, providing a more comprehensive model that allows decision-makers to better represent conflicting perspectives. This makes the model particularly useful in fields like healthcare, where both benefits and risks need to be assessed in a balanced way.
2. **Flexibility:** FN-BS supports multinary evaluations, which enhances its application in complex DM scenarios involving more than just binary choices. This flexibility makes the model applicable to a wider range of real-world problems, such as in financial DM or resource allocation, where multiple factors must be considered simultaneously.
3. **Real-world Applicability:** FN-BS is particularly well-suited for MCDM problems with conflicting criteria, offering more reliable decision support for practical decision-makers in various sectors like healthcare, urban planning, and industrial engineering.

6.2 Limitations of the Model

While the FN-BS model is robust, it does have some limitations. These include:

- **Reliance on Expert Knowledge:** The model's dependency on expert input introduces a degree of subjectivity, which may affect its accuracy, especially in domains with less expert availability or in more subjective DM contexts.

- **Sensitivity to Threshold Values:** As mentioned earlier, the model's sensitivity to threshold values can lead to different decision outcomes. This becomes a significant concern in critical DM environments where small changes in thresholds could result in drastically different outcomes.
- **Assumption of Equal Attribute Significance:** The model assumes that all attributes contribute equally to the final decision, which may not always reflect real-world situations. In many applications, some attributes may have a higher importance than others.

6.3 Defects and Mitigation Strategies

A key defect of the FN-BS model is its computational complexity, especially when handling large datasets. Future improvements could focus on optimizing computation through heuristic or approximation methods to reduce the time complexity. Additionally, the use of threshold values can lead to unreliable outcomes without clear thresholds. A dynamic adjustment mechanism could address this issue by automatically adjusting the threshold based on the context of the DM problem. Lastly, incorporating a weighting system for attributes could better reflect their importance, which would improve the model's accuracy and adaptiveness.

6.4 Comparison with Existing Models

Table 23 compares the FN-BS model with other relevant DM frameworks based on key features such as basic DM support (BDS), multi-attribute handling (MAH), dual-perspective assessment (DPA), and flexible membership representation (FMR). The FN-BS model excels in all these aspects, demonstrating its adaptability and efficiency.

Table 23: Feature-wise comparison of FN-BS model with existing approaches

Authors/Model	BDS	MAH	DPA	FMR
Molodtsov [5] (S-sets)	Supported	Not supported	Not supported	Not supported
Maji et al. [6] (FS sets)	Not supported	Not supported	Not supported	Supported
Fatimah et al. [25] (N-S sets)	Supported	Supported	Not supported	Not supported
Akram et al. [26] (FN-S sets)	Supported	Supported	Not supported	Supported
Shabir et al. [13] (BS sets)	Supported	Not supported	Supported	Not supported
Naz et al. [14] (FBS sets)	Supported	Not supported	Supported	Supported
Shabir et al. [35] (N-BS sets)	Supported	Supported	Supported	Not supported
Proposed Model (FN-BS sets)	Supported	Supported	Supported	Supported

The comparison confirms that the FN-BS model integrates all the key features, positioning it as a superior solution in handling multi-attribute, multi-perspective decision problems compared to other models. This highlights the flexibility and comprehensiveness of the FN-BS framework for diverse applications. However, the sensitivity of the results to threshold values should not be overlooked. This reinforces the need for further refinement, particularly in making the model more stable across varying thresholds.

7 Conclusions

In this paper, we introduced the FN-BS framework to address the complexities of MCDM in uncertain environments. By combining fuzzy logic, bipolar evaluations, and non-binary attributes, FN-BS sets offer a comprehensive approach for evaluating decision objects across various criteria. We presented its formal definition, explored set-theoretic operations, and examined algebraic properties. Additionally, we developed and applied two FN-BS-based algorithms to a case study of vaccination program evaluation, demonstrating the practical utility of the framework. The comparative analysis showed that FN-BS outperforms existing DM models in flexibility, adaptability, and its ability to handle diverse evaluation scenarios.

Future work could involve extending the FN-BS model to advanced variants, such as intuitionistic, complex, picture, Pythagorean, and q-rung orthopair FN-BS sets, to tackle more sophisticated DM challenges. Integrating FN-BS sets with emerging technologies like machine learning and artificial intelligence could further enhance decision support systems.

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