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Exact Solutions of the Cubic Duffing Equation by Leaf Functions under Free Vibration

Kazunori Shinohara[1, *]

Abstract: Exact solutions of the cubic Duffing equation with the initial conditions are presented. These exact solutions are expressed in terms of leaf functions and trigonometric functions. The leaf function $r=sleaf_n(t)$ or $r=cleaf_n(t)$ satisfies the ordinary differential equation $dx^2/dt^2=-nr^{2n-1}$. The second-order differential of the leaf function is equal to $-n$ times the function raised to the $(2n-1)$ power of the leaf function. By using the leaf functions, the exact solutions of the cubic Duffing equation can be derived under several conditions. These solutions are constructed using the integral functions of leaf functions $sleaf_2(t)$ and $cleaf_2(t)$ for the phase of a trigonometric function. Since the leaf function and the trigonometric function are used in combination, a highly accurate solution of the Duffing equation can be easily obtained based on the data of leaf functions. In this study, seven types of the exact solutions are derived from leaf functions; the derivation of the seven exact solutions is detailed in the paper. Finally, waves obtained by the exact solutions are graphically visualized with the numerical results.

Keywords: Duffing equation, nonlinear equations, ordinary differential equation, leaf functions.

1 Introduction

1.1 Leaf function

Trigonometric functions are generally used to mathematically describe a regular wave in terms of amplitude and period. The trigonometric functions $sin(\theta)$ and $cos(\theta)$ can be defined by using a unit circle. The coordinates ($cos(\theta)$, $sin(\theta)$) represent the intersection point between the unit circle and the straight line obtained by rotating the positive part of the x-axis counterclockwise around the origin through an angle θ. Therefore, by using trigonometric functions, solutions are obtained on the basis of the structure of the unit circle. This often makes it impossible to derive exact solutions that satisfy nonlinear equations. Therefore, approximate expressions obtained using trigonometric functions are generally applied to nonlinear equations.

The solution of some ordinary differential equations through numerical analysis may have the characteristics of waves. However, exact solutions representing these waves almost always cannot be derived using trigonometric functions.

To derive an exact solution of various ordinary differential equations, it is necessary to

[1] Daido University, 10-3 Takiharu-cho, Minami-ku, Nagoya 457-8530, Japan.

[*] Corresponding Author: Kazunori Shinohara. Email: shinohara@06.alumni.u-tokyo.ac.jp.

define a new base function instead of trigonometric functions. This study considers very simple ordinary differential equations: the second derivative of a function and the power exponent of a function [Shinohara (2015)].

$$\frac{d^2 r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \tag{1.1}$$

$$r(0) = 0 \tag{1.2}$$

$$\frac{dr(0)}{dl} = 1 \tag{1.3}$$

In this study, variable n is considered as the basis, which represents natural numbers ($n=1, 2, 3, \ldots$). Variable n in front of function $r(l)^{2n-1}$ in Eq. (1.1) represents a coefficient for normalizing the unit amplitude of the wave. Variable l represents the phase. It is different from angle θ except for $n=1$. As described later, l geometrically represents the length of a curve (for $n=1$, l is equal to θ.) Through numerical analysis, we can obtain a solution $r(l)$ that satisfies ordinary differential equations. We find that the ordinary differential equations (Eqs. (1.1)-(1.3)) produce a regular wave (solution $r(l)$) with a constant amplitude and period at all times. This wave is generated when the exponent of $r(l)$ is a positive odd number $2n-1$ but not when the exponent of $r(l)$ is an even number $2n$. For $n=1$ in Eq. (1.1), a function that satisfies the ordinary differential equation represents the trigonometric function $sin(l)$. For $n=2$ in Eq. (1.1), the lemniscate function $sl(l)$ is satisfied, while for $n \geq 3$, this function does not exist. As the basis n increases, a regular wave converges to the following function.

$$r(l) = (-1)^m (l - 2m) \quad (2m-1 \leq l \leq 2m+1)(m\text{: integer}) \tag{1.4}$$

Eq. (1.4) is therefore a discontinuous function. On the other hand, solution $r(l)$ for $n=1$, $n=2$, $n=5$, and $n=10$ is shown in the graph. As n increases, the curve converges to the wave obtained using Eq. (1.4), and the period of the wave reaches 4.

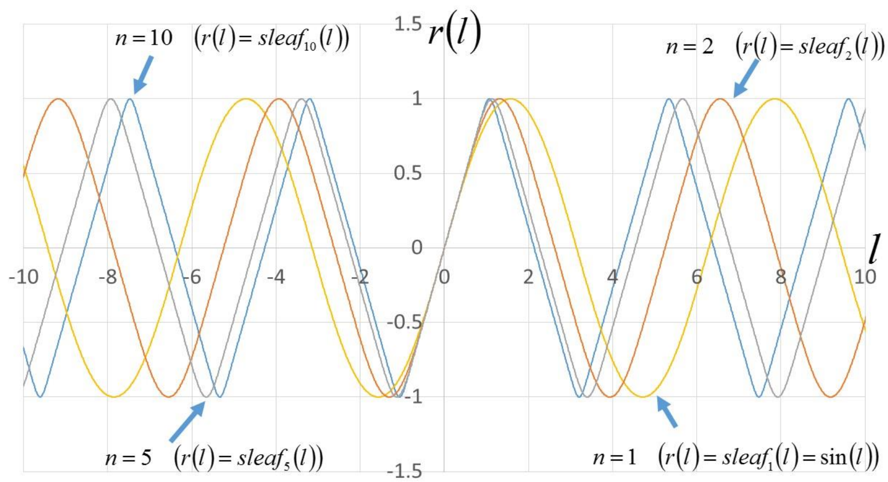

Figure 1: Curves obtained by the ordinary differential Eqs. (1.1)-(1.3) (basis: $n=1$, $n=2$, $n=5$, and $n=10$)

Although the curve shows concavity and convexity in Fig. 1, the curves obtained using solution *r(l)* are continuous. Such wave features obviously differ from those obtained using trigonometric functions. No conventional function satisfies the ordinary differential equations given by Eqs. (1.1)-(1.3). Therefore, the unknown function is defined as a leaf function as follows:

$$r = sleaf_n(l) \tag{1.5}$$

At this time, the relations among *x, y*, and *r* are defined as follows:

$$|r|^n = |\sin(n\theta)| \tag{1.6}$$

$$r^2 = x^2 + y^2 \tag{1.7}$$

$$x = r\cos\theta \tag{1.8}$$

$$y = r\sin\theta \tag{1.9}$$

The representation "*sleaf*" is combination of "*sin*" and "*leaf*." The subscript *n* represents the basis. As shown in Figs. 2-5, the numerical data (Tabs. 1 and 2) obtained using the function $sleaf_n(l)$ are plotted on a graph with variable *x* (Eq. (1.8)) along the horizontal axis and variable *y* (Eq. (1.9)) along the vertical axis. The curves obtained using leaf functions have a leaf-like shape, which is why the representation "*sleaf*" is used.

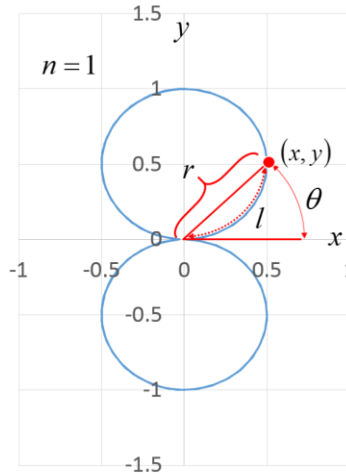

Figure 2: Geometrical relation among variables *l*, *θ*, and *r* for *n=1*

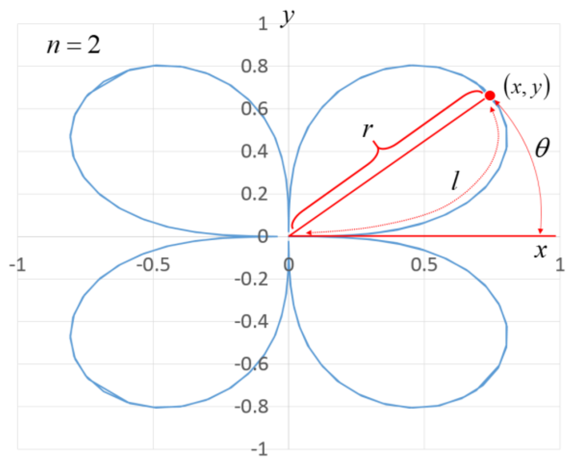

Figure 3: Geometrical relation among variables *l*, *θ*, and *r* for *n=2*

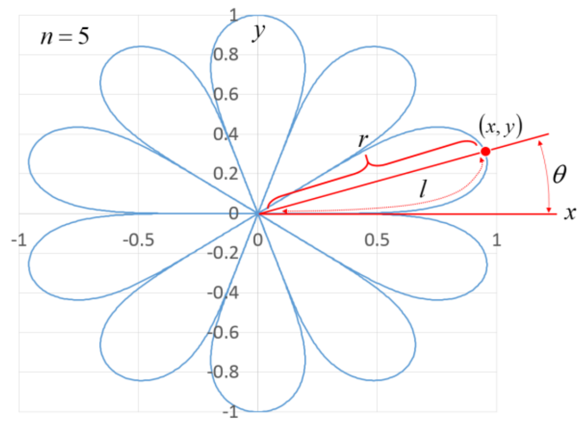

Figure 4: Geometrical relation among variables *l*, *θ*, and *r* for *n=5*

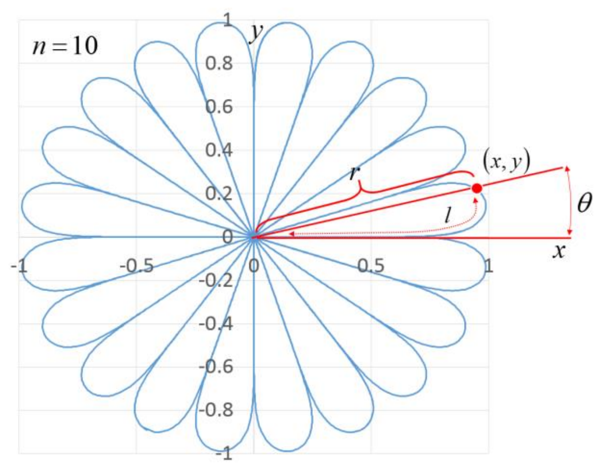

Figure 5: Geometrical relation among variables *l*, *θ*, and *r* for *n =10*

Table 1: Numerical data with respect to variable *l* (unit of parameter θ is °)

l	*n=1 (Fig. 2)* θ *(Eq. (1.6))*	r *(Eq. (1.5)*	*n=2 (Fig. 3)* θ *(Eq. (1.6))*	r *(Eq. (1.5)*	*n=5 (Fig. 4)* θ *(Eq. (1.6))*	r *(Eq. (1.5)*	*n=10 (Fig. 5)* θ *(Eq. (1.6))*	r *(Eq. (1.5)*
0.0	0.00000	0.000000	0.00000	0.000000	0.00000	0.000000	0.00000	0.000000
0.1	5.72957	0.099833	0.28647	0.099998	0.00011	0.100000	5.72958×10^{-10}	0.100000
0.2	11.4591	0.198669	1.14585	0.199967	0.00366	0.199999	5.86709×10^{-7}	0.200000
0.3	17.1887	0.295520	2.57761	0.299757	0.02784	0.299999	3.38326×10^{-5}	0.300000
0.4	22.9183	0.389418	4.57975	0.398978	0.11734	0.399998	0.00060	0.400000
0.5	28.6478	0.479425	7.14709	0.496891	0.35807	0.499977	0.00559	0.499999
0.6	34.3774	0.564642	10.2689	0.592307	0.89073	0.599835	0.03464	0.599999
0.7	40.1070	0.644217	13.9264	0.683522	1.92265	0.699103	0.16183	0.699986
0.8	45.8366	0.717356	18.0893	0.768313	3.73076	0.796139	0.61470	0.799780
0.9	51.5662	0.783326	22.7133	0.844009	6.62854	0.886245	1.98069	0.897430
1.0	57.2957	0.841470	27.7379	0.907683	10.8296	0.958858	5.36544	0.978597
1.1	63.0253	0.891207	33.0860	0.956432	16.1552	0.997400	7.17616	0.994858
1.2	68.7549	0.932039	38.6645	0.987748	21.8203	0.988736	15.1167	0.929666
1.3	74.4845	0.963558	44.3680	0.999878	26.8068	0.936130	17.0539	0.834802
1.4	80.2140	0.985449	50.0842	0.992115	30.5394	0.855664	17.7351	0.735307
1.5	85.9436	0.997494	55.6997	0.964914	33.0174	0.762260	17.9385	0.635343
1.6	91.6732	0.999573	61.1071	0.919815	34.5155	0.664110	17.9889	0.535344
1.7	97.4028	0.991664	66.2103	0.859192	35.3425	0.564533	17.9985	0.435344
1.8	103.132	0.973847	70.9285	0.785891	35.7519	0.464607	17.9998	0.335344
1.9	108.861	0.946300	75.1974	0.702864	35.9261	0.364617	17.9999	0.235344
2.0	114.591	0.909297	78.9694	0.612857	35.9851	0.264617	17.9999	0.135344
2.1	120.321	0.863209	82.2114	0.518203	35.9986	0.164617	18.0000	0.035344
2.2	126.050	0.808496	84.9022	0.420721	35.9999	0.064617	18.0000	-0.064655
2.3	131.780	0.745705	87.0296	0.321711	36.0000	-0.035382	18.0000	-0.164655
2.4	137.509	0.675463	88.5874	0.222003	36.0005	-0.135382	18.0000	-0.264655
2.5	143.239	0.598472	89.5732	0.122054	36.0082	-0.235382	18.0002	-0.364655
2.6	148.969	0.515501	89.9860	0.022057	36.0486	-0.335381	18.0026	-0.464655
2.7	154.698	0.427379	90.1740	-0.077942	36.1792	-0.435377	18.0188	-0.564654
2.8	160.428	0.334988	90.9070	-0.177924	36.5039	-0.535334	18.0964	-0.664650
2.9	166.157	0.239249	92.2126	-0.277776	37.1858	-0.635072	18.3914	-0.764570
3.0	171.887	0.141120	94.0892	-0.377172	38.4575	-0.733843	19.3331	-0.863538
3.1	177.616	0.041580	96.5326	-0.475459	40.6160	-0.829205	21.8669	-0.954060
3.2	183.346	-0.058374	99.5335	-0.571553	43.9633	-0.914704	26.8271	-0.999954
3.3	189.076	-0.157745	103.074	-0.663869	48.6133	-0.977299	31.9015	-0.958695
3.4	194.805	-0.255541	107.129	-0.750292	53.8236	-0.999976	34.5716	-0.869399
3.5	200.535	-0.350783	111.656	-0.828242	59.7059	-0.974439	35.5766	-0.770589
3.6	206.264	-0.442520	116.598	-0.894823	64.2806	-0.909942	35.8944	-0.670684
3.7	211.994	-0.529836	121.883	-0.947099	67.5481	-0.823534	35.9790	-0.570689
3.8	217.723	-0.611857	127.419	-0.982443	69.6429	-0.727832	35.9969	-0.470689
3.9	223.453	-0.687766	133.104	-0.998905	70.8702	-0.628957	35.9997	-0.370689
4.0	229.183	-0.756802	138.828	-0.995532	71.5242	-0.529194	35.9999	-0.270689
4.1	234.912	-0.818277	144.475	-0.972521	71.8330	-0.429231	35.9999	-0.170689

Table 2: Numerical data with respect to variable l (unit of parameter θ is °)

	n=1 (Fig. 2)		*n=2 (Fig. 3)*		*n=5 (Fig. 4)*		*n=10 (Fig. 5)*	
l	θ (Eq. (1.6))	r (Eq. (1.5)	θ (Eq. (1.6))	r (Eq. (1.5)	θ (Eq. (1.6))	r (Eq. (1.5)	θ (Eq. (1.6))	r (Eq. (1.5)
4.2	240.642	-0.871575	149.937	-0.931190	71.9556	-0.329235	36.0000	-0.070689
4.3	246.371	-0.916165	155.115	-0.873757	71.9927	-0.229235	36.0000	0.029310
4.4	252.101	-0.951602	159.924	-0.802997	71.9995	-0.129235	36.0000	0.129310
4.5	257.831	-0.977530	164.297	-0.721869	71.9999	-0.029235	36.0000	0.229310
4.6	263.560	-0.993691	168.182	-0.633184	72.0000	0.070764	36.0000	0.329310
4.7	269.290	-0.999923	171.543	-0.539380	72.0016	0.170764	36.0012	0.429310
4.8	275.019	-0.996164	174.356	-0.442393	72.0166	0.270763	36.0098	0.529310
4.9	280.749	-0.982452	176.609	-0.343633	72.0802	0.370763	36.0558	0.629308
5.0	286.478	-0.958924	178.293	-0.244028	72.2649	0.470752	36.2438	0.729278
5.1	292.208	-0.925814	179.405	-0.144108	72.6939	0.570668	36.8801	0.828843
5.2	297.938	-0.883454	179.944	-0.044115	73.5542	0.670202	38.7084	0.924335
5.3	303.667	-0.832267	179.910	0.055884	75.1032	0.768191	42.8490	0.992807
5.4	309.397	-0.772764	180.696	0.155875	77.6516	0.861102	48.3453	0.982050
5.5	315.126	-0.705540	181.875	0.255775	81.4660	0.940349	51.8849	0.903084
5.6	320.856	-0.631266	183.626	0.355314	86.5174	0.990662	53.3374	0.805778
5.7	326.585	-0.550685	185.945	0.453922	87.8074	0.996323	53.8236	0.706018
5.8	332.315	-0.464602	188.824	0.550618	97.4657	0.955194	53.9617	0.606034
5.9	338.045	-0.373876	192.248	0.643931	101.586	0.881067	53.9936	0.506034
6.0	343.774	-0.279415	196.193	0.731861	104.408	0.790304	53.9993	0.406034
6.1	349.504	-0.182162	200.620	0.811921	106.160	0.693040	53.9999	0.306034
6.2	355.233	-0.083089	205.476	0.881266	107.153	0.593706	53.9999	0.206034
6.3	360.963	0.016813	210.692	0.936940	107.663	0.493834	54.0000	0.106034
6.4	366.692	0.116549	216.181	0.976215	107.891	0.393852	54.0000	0.006034
6.5	372.422	0.215119	221.843	0.996963	107.974	0.293853	54.0000	-0.093965
6.6	378.152	0.311541	222.431	0.997989	107.996	0.193853	53.9999	-0.193965
6.7	383.881	0.404849	233.241	0.979234	107.999	0.093854	54.0000	-0.293965
6.8	389.611	0.494113	238.753	0.941780	108.000	-0.006145	54.0005	-0.393965
6.9	395.340	0.578439	244.002	0.887674	108.000	-0.106145	54.0049	-0.493965
7.0	401.070	0.656986	248.899	0.819600	108.004	-0.206145	54.0313	-0.593964
7.1	406.800	0.728969	253.373	0.740507	108.030	-0.306145	54.1484	-0.693954
7.2	412.529	0.793667	257.369	0.653264	108.126	-0.406143	54.5699	-0.793778
7.3	418.259	0.850436	260.848	0.560404	108.380	-0.506120	55.8538	-0.891730
7.4	423.988	0.898708	263.784	0.463979	108.937	-0.605961	59.0855	-0.974903
7.5	429.718	0.937999	266.161	0.365515	110.008	-0.705159	61.5089	-0.996580
7.6	435.447	0.967919	267.970	0.266039	111.874	-0.801949	68.9327	-0.934905
7.7	441.177	0.988168	269.208	0.166160	114.848	-0.891358	70.9835	-0.840748
7.8	446.907	0.998543	269.874	0.066172	119.129	-0.962392	71.7126	-0.741335
7.9	452.636	0.998941	269.967	-0.033827	124.504	-0.998292	71.9325	-0.641377
8.0	458.366	0.989358	270.513	-0.133822	130.155	-0.986640	71.9876	-0.541379
8.1	464.095	0.969889	271.566	-0.233757	135.074	-0.931807	71.9983	-0.441379
8.2	469.825	0.940730	273.191	-0.333412	138.725	-0.850180	71.9998	-0.341379
8.3	475.554	0.902171	275.385	-0.432294	141.134	-0.756314	71.9999	-0.241379

Variables x and y obtained using Eqs. (1.8) and (1.9) always pass through the origin. Variable r represents the distance between the origin and the point (x, y). Variable θ is the angle between the x-axis and the straight line connecting the point (x, y) and the origin. Variable l represents the length of the curve. As n increases, the number of leaves increases.

Leaf function $sleaf_n(l)$ is supposed to be an extension of the trigonometric function $sin(\theta)$. It is assumed that a similar system can be extended using trigonometric function $cos(\theta)$. We define the ordinary differential equations that satisfy another leaf function as follows:

$$\frac{d^2 r(l)}{dl^2} = -n \cdot r(l)^{2n-1} \tag{1.10}$$

$$r(0) = 1 \tag{1.11}$$

$$\frac{dr(0)}{dl} = 0 \tag{1.12}$$

The function satisfying the ordinary differential equations is represented as $cleaf_n(l)$. For $n=1$, $cleaf_1(l)$ represents trigonometric function $cos(l)$. For $n=2$, $cleaf_2(l)$ represents lemniscate function $cl(l)$. For $n=1$, 2, and 3, by defining functions *sleaf* and *cleaf*, the relational expressions between *sleaf* and *cleaf*, *sleaf* and *sin*, and *cleaf* and *cos* are derived [Shinohara (2015); Shinohara (2017)]. We can also derive the addition theorem of *sleaf* and *cleaf*. Leaf functions *sleaf* and *cleaf* can be flexibly transformed through various relational expressions and addition theorems to fit the Duffing equation, a nonlinear second-order ordinary differential equation.

This study provides seven types of exact solutions of the Duffing equation using $sleaf_2(l)$ and $cleaf_2(l)$. To verify that these exact solutions satisfy the ordinary differential equations of the Duffing equation, the waveform types and numerical data of each type are shown through graphs and numerical analysis.

1.2 Relation among leaf functions, trigonometric functions, and Lemniscatic elliptic function

The motivation for studying leaf functions is the ordinary differential Eq. (1.1) with initial conditions (1.2) and (1.3). Eq. (1.1) is very simple, but when it is solved numerically, we notice the following. When point (l, r) is plotted on the graph with the horizontal axis as t and the vertical axis as r, a curve that has the characteristics of a wave is obtained; it has a regular periodicity for any basis parameter n. In a solution satisfying an ordinary differential equation, we generally imagine a solution indicated by a trigonometric function. However, in Eq. (1.1), if $n=2$ or more, it is obvious that the solution is not a trigonometric function. Therefore, in this study, to define an exact solution of ordinary differential Eq. (1.1), leaf functions are created artificially. In the case of $n=1$, the relation between trigonometric functions and leaf functions is as follows:

$$sleaf_1(l) = \sin(l) \tag{1.13}$$

$$cleaf_1(t) = \cos(l) \tag{1.14}$$

In addition, the leaf function for *n=2* is essentially equivalent to the lemniscate function [Ranjan (2017)]. Leaf functions $sleaf_2(l)$ and $cleaf_2(l)$ have the same meaning as lemniscate functions *sl(l)* and *cl(l)*.

$$sleaf_2(l) = \mathrm{s}l(l) \tag{1.15}$$

$$cleaf_2(t) = cl(l) \tag{1.16}$$

The lemniscate functions are further extended to the Jacobi elliptic function. Given the historical background of these functions, the Fagnano's doubling theorem is the beginning [Fagnano (1750)]. Based on Fagnano's study, Euler derived the general solution of the following differential equation [Euler (1911)].

$$\frac{du}{\sqrt{1-u^4}} = \frac{dv}{\sqrt{1-v^4}} \tag{1.17}$$

The general solution of this equation is derived as follows:

$$u^2 + v^2 = c^2 + 2uv\sqrt{1-c^4} - c^2u^2v^2 \tag{1.18}$$

Solving Eq. (1.18) for *u* yields the following equation.

$$u = \frac{v\sqrt{1-c^4} \pm c\sqrt{1-v^4}}{1+c^2v^2} \tag{1.19}$$

The addition theorem of the lemniscate function is derived from Eq. (1.19). Incidentally, we also consider the following ordinary differential equations.

$$\frac{du}{\sqrt{1-u^2}} = \frac{dv}{\sqrt{1-v^2}} \tag{1.20}$$

The general solution of this equation is derived as follows.

$$u^2 + v^2 = c^2 + 2uv\sqrt{1-c^2} \tag{1.21}$$

Solving Eq. (1.21) for *u* yields the following equation.

$$u = u\sqrt{1-c^2} \pm c\sqrt{1-u^2} \tag{1.22}$$

The addition theorem of the trigonometric function is derived from Eq. (1.22). However, historically, further higher orders (*n=3* or more) has not been considered as described below:

$$\frac{du}{\sqrt{1-u^{2n}}} = \frac{dv}{\sqrt{1-v^{2n}}} \tag{1.23}$$

The search for a general solution that satisfies the above equation is essentially the same as deriving the addition theorem of the leaf function of base *n*. In this study, the addition theorem of only the leaf function of *n=3* can be derived [Shinohara (2017)]. In fact, as of 2018, the addition theorem of the leaf function of *n=4* or more is not known.

1.3 Solving the Duffing equation

The Duffing equation is an ordinary differential equation that was originally proposed by Georg Duffing [Kovacic and Brennan (2011); Cveticanin (2013)]. In the literature, the Duffing equation is generally solved by approximate solutions using computer analysis.

Harmonic Balance Method is applied to determine approximate analytic solutions for strongly nonlinear duffing oscillators [Hosen and Chowdhury (2016)]. A new reliable analytical technique based on the Harmonic Balance Method (HBM) [Chowdhury, Hosen and Ahmad et al. (2017)] and the improved constrained optimization [Liao (2014)] has been established to derive approximate periodic solutions for the nonlinear Duffing oscillations. The iterative method proposed by Temimi and Ansari namely (TAM) has been presented to solve the Duffing equation [Al-Jawary and Al-Razaq (2016)]. The first-order approximation of the iteration perturbation method (IPM) is used to approximate the behavior of the cubic-quintic Duffing oscillators [Ganji, Barari, Karimpour et al. (2012)]. The modified perturbation technique has been applied to solve nonlinear fifth-order duffing oscillators [El-Naggar and Ismail (2016)]. Additionally, the homotopy analysis method (HAM) has been used to obtain the analytical solution for nonlinear cubic-quantic duffing oscillators [Sayevand, Baleanu and Fardi (2014)]. This technique represents a blending of the Chebyshev Pseudo-spectral method and the homotopy perturbation method (HPM). The method is tested by solving nonlinear Duffing equation for undamped oscillators [Sibanda and Khidir (2011)]. The dynamic behavior of SBB with the effect of a random parameter has been investigated by applying global analysis. The Chebyshev orthogonal polynominal approximation method has been applied to reduce RP-DS [Zhang, Du, Yue et al. (2015)]. The simple collocation method has been applied to determine the harmonic period solutions to the duffing equation [Dai (2012)]. The complexity of a nonlinear Duffing oscillator has been revealed by using a method that leverages sign function [Liu (2014)].

On the other hand, the Duffing equation can be solved by using exact solutions that leverage Jacobi elliptic functions. The current study aims to derive the exact solution of the Duffing equation by using the leaf function. In the literature [Elías-Zúñiga (2013); Beléndez, Beléndez, Martínez et al. (2016)], an exact solution for a cubic-quintic Duffing oscillator has been derived by using the Jacobi elliptic functions. The present study differs from the literature in that an exact solution of the Duffing equation is constructed using the integral functions of leaf functions $sleaf_2(t)$ and $cleaf_2(t)$ for the phase of a trigonometric function. Since only the leaf function and the trigonometric function are used in combination, a highly accurate solution of the Duffing equation can be easily obtained without using computer analysis if already we have obtained the data via the leaf function. In the literature, the analytical solution of a damped cubic-quintic Duffing oscillator was derived by using the Jacobi elliptic function [Elías-Zúñiga (2014)]; When using this method, to determine the coefficients of the exact solution, we need to find the roots of a sextic equation by using software such as Mathematica.

It is possible to derive an exact solution to the Duffing eqution, including the damping term, simply by using the leaf function; this method, not described in this paper, does not require the use of Mathematica software.

2 Numerical data of leaf functions

The periodicity of functions $sleaf_n(l)$ and $cleaf_n(l)$ depends on parameter n. The constant values of the periodicity are defined as follows:

$$\frac{\pi_n}{2}=\int_0^1 \frac{1}{\sqrt{1-u^{2n}}}du \quad (n=1,2,3,\cdots) \tag{2.1}$$

The constant values $2\pi_n$ represent one periodicity with respect to the arbitrary parameter n. The numerical results of π_n (for $n=1, 2, 3\ldots$) are summarized in Tab. 3.

Table 3: Values of constant π_n

n	π_n
1	π_1=3.141…
2	π_2=2.622…
3	π_3=2.429…
…	...

The inverse leaf function for $n=2$ is as follows:

$$arcsleaf_2(r)=\int_0^r \frac{1}{\sqrt{1-u^4}}du=l \tag{2.2}$$

$$arccleaf_2(r)=\int_r^1 \frac{1}{\sqrt{1-u^4}}du=l \tag{2.3}$$

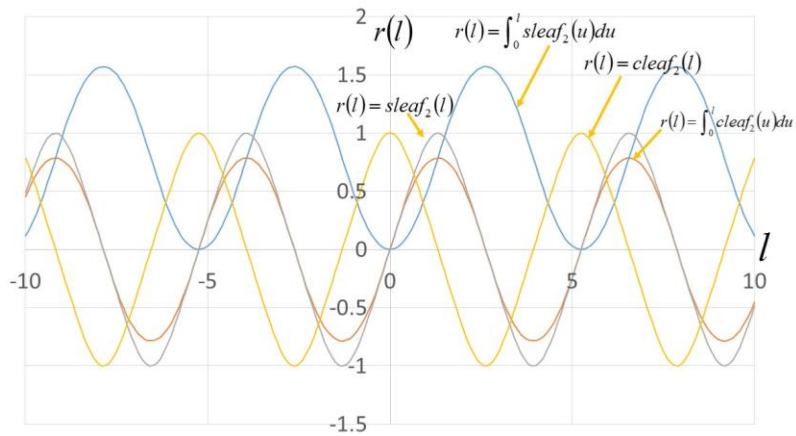

Figure 6: Waves of $sleaf_2(l)$, $cleaf_2(l)$, $\int_0^l sleaf_2(u)du$, and $\int_0^l cleaf_2(u)du$

Using Eqs. (2.2) and (2.3), the numerical data between parameters r and l can be obtained by numerical analyses and are summarized in Tab. 4. The curves of leaf functions $sleaf_2(l)$ and $cleaf_2(l)$ and integral leaf functions $\int_0^r sleaf_2(u)du$ and $\int_0^r cleaf_2(u)du$ are shown in Fig. 6. The values of the integral functions can be obtained by numerical integration. The periodicity of leaf functions $sleaf_2(l)$ and $cleaf_2(l)$ is $2\pi_2$. The mathematical description is as follows:

$$sleaf_2(l+2\pi_2)=sleaf_2(l) \tag{2.4}$$

$$\mathrm{cleaf}_2(l+2\pi_2)=\mathrm{cleaf}_2(l) \tag{2.5}$$

$$\mathrm{sleaf}_2(m\pi_2)=0 \quad (m=0,\pm1,\pm2,\pm3\cdots) \tag{2.6}$$

$$\mathrm{sleaf}_2\left(\frac{\pi_2}{2}(4m-3)\right)=1 \quad (m=0,\ \pm1,\pm2,\pm3\cdots) \tag{2.7}$$

$$\mathrm{sleaf}_2\left(\frac{\pi_2}{2}(4m-1)\right)=-1 \quad (m=0,\ \pm1,\pm2,\pm3\cdots) \tag{2.8}$$

$$\mathrm{cleaf}_2\left(\frac{\pi_2}{2}(2m-1)\right)=0 \quad (m=0,\pm1,\pm2,\pm3\cdots) \tag{2.9}$$

$$\mathrm{cleaf}_2(2m\pi_2)=1 \quad (m=0,\pm1,\pm2,\pm3\cdots) \tag{2.10}$$

$$\mathrm{cleaf}_2(\pi_2(2m-1))=-1 \quad (m=0,\pm1,\pm2,\pm3\cdots) \tag{2.11}$$

In this paper, using leaf functions $sleaf_2(l)$ and $cleaf_2(l)$ for $n=2$, seven types of the exact solutions are presented for the cubic Duffing equation. In each case, the mathematical derivations and the numerical results of the seven types are shown in detail. Thereafter, the features of the waveforms are discussed.

Table 4: Numerical data of $sleaf_2(t)$, $cleaf_2(l)$, $\int_0^l sleaf_2(u)du$, and $\int_0^l cleaf_2(u)du$

(All results have been rounded to no more than six significant figures)

l	$sleaf_2(l)$	$cleaf_2(l)$	$\int_0^l sleaf_2(u)du$	$\int_0^l cleaf_2(u)du$
-10.0	0.48547	0.78647	0.11895	0.45195
-9.0	0.96908	-0.17718	0.96076	0.76969
-8.0	0.13382	-0.98225	1.56184	0.13303
-7.0	-0.81960	-0.44312	1.20252	-0.68658
-6.0	-0.73186	0.54991	0.28262	-0.63179
-5.0	0.24402	0.94212	0.02979	0.23935
-4.0	0.99553	0.06691	0.71858	0.78315
-3.0	0.37717	-0.86655	1.49942	0.36067
-2.0	-0.61286	-0.67373	1.37828	-0.54982
-1.0	-0.90768	0.31073	0.48411	-0.73704
0.0	0.00000	1.00000	0.00000	0.00000
1.0	0.90768	0.31073	0.48411	0.73704
2.0	0.61285	-0.67373	1.37828	0.54982
3.0	-0.37717	-0.86655	1.49942	-0.36067
4.0	-0.99553	0.06691	0.71858	-0.78316
5.0	-0.24403	0.94212	0.02979	-0.23935
6.0	0.73186	0.54991	0.28262	0.63179
7.0	0.81960	-0.44312	1.20252	0.68657

8.0	*-0.13382*	*-0.98225*	*1.56184*	*-0.13303*
9.0	*-0.96908*	*-0.17718*	*0.96076*	*-0.76970*
10.0	*-0.48547*	*0.78647*	*0.11895*	*-0.45196*

3 Exact solutions of cubic Duffing equation using leaf functions

We try to apply the leaf function to the Duffing equation, given as follows:

$$\frac{d^2x(t)}{dt^2}+\alpha x(t)+\beta x(t)^3=0 \tag{3.1}$$

For mechanical vibration, the above equation represents the free vibration by a nonlinear spring. Variable *x(t)* represents the unknown function and depends on parameter *t*. Differential operators *dx(t)/dt* and $d^2x(t)/dt^2$ represent the first- and second-order differentials, respectively. Symbols *α* and *β* represent coefficients that do not depend on time *t*. In mechanical engineering fields, Eq. (3.1) is regarded as the mathematical model for the nonlinear vibration. In the left side of the equation, the first, second, and third terms represent inertia, stiffness, and nonlinear stiffness, respectively.

By using the leaf functions, seven types of exact solutions can be set and then the equation that the solution satisfies can be derived. In this paper, types (I)-(VII) are defined for the exact solutions, and the ordinary difference equations and the initial conditions are given as follows:

・Type (I) (See Appendix I for details)

Exact solution:

$$x(t)=A\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\left(=A\cos\left(\int_0^{\omega t+\phi}cl(u)du\right)\right) \tag{3.2}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}-3\omega^2x(t)+4\left(\frac{\omega}{A}\right)^2x(t)^3=0 \tag{3.3}$$

Initial position:

$$x(0)=A\cos\left(\int_0^{\phi}cleaf_2(u)du\right) \tag{3.4}$$

Initial velocity:

$$\frac{dx(0)}{dt}=-A\cdot\omega\cdot cleaf_2(\phi)\cdot\sin\left(\int_0^{\phi}cleaf_2(u)du\right) \tag{3.5}$$

・Type (II) (See Appendix II for details)

Exact solution:

$$x(t)=A\sin\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\left(=A\sin\left(\int_0^{\omega t+\phi}cl(u)du\right)\right) \tag{3.6}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}+3\omega^2x(t)-4\left(\frac{\omega}{A}\right)^2x(t)^3=0 \tag{3.7}$$

Initial position:

$$x(0)=A\sin\left(\int_0^{\phi}cleaf_2(u)du\right) \tag{3.8}$$

Initial velocity:

$$\frac{dx(0)}{dt}=A\cdot\omega\cdot cleaf_2(\phi)\cdot\cos\left(\int_0^{\phi}cleaf_2(u)du\right) \tag{3.9}$$

・Type (III) (See Appendix III for details)

Exact solution:

$$\begin{aligned}x(t)&=A\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+A\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\\&\left(=A\cos\left(\int_0^{\omega t+\phi}sl(u)du\right)+A\sin\left(\int_0^{\omega t+\phi}sl(u)du\right)\right)\end{aligned} \tag{3.10}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}-3\omega^2x(t)+2\left(\frac{\omega}{A}\right)^2x(t)^3=0 \tag{3.11}$$

Initial position:

$$x(0)=\sqrt{2}A\cos\left(\int_0^{\phi}sleaf_2(u)du-\frac{\pi}{4}\right) \tag{3.12}$$

Initial velocity:

$$\frac{dx(0)}{dt}=\sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^{\phi}sleaf_2(u)du+\frac{\pi}{4}\right) \tag{3.13}$$

・Type (IV) (See Appendix IV for details)

Exact solution:

$$\begin{aligned}x(t)&=A\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)-A\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\\&=A\cos\left(\int_0^{\omega t+\phi}sl(u)du\right)-A\sin\left(\int_0^{\omega t+\phi}sl(u)du\right)\end{aligned} \tag{3.14}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}+3\omega^2x(t)-2\left(\frac{\omega}{A}\right)^2x(t)^3=0 \tag{3.15}$$

Initial position:

$$x(0)=\sqrt{2}A\cos\left(\int_0^{\phi} sleaf_2(u)du+\frac{\pi}{4}\right) \tag{3.16}$$

Initial velocity:

$$\frac{dx(0)}{dt}=\sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^{\phi} sleaf_2(u)du-\frac{\pi}{4}\right) \tag{3.17}$$

・Type (V) (See Appendix V for details)

Exact solution:

$$\begin{gathered} x(t)=A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \\ \left(=A\cos\left(\int_0^{\omega t+\phi} sl(u)du\right)+A\sin\left(\int_0^{\omega t+\phi} sl(u)du\right)+\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cl(u)du\right)\right) \end{gathered} \tag{3.18}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}-3\omega^2\left(1+2\sqrt{2}\right)x(t)+2\frac{\omega^2}{A^2}x(t)^3=0 \tag{3.19}$$

Initial position:

$$x(0)=\sqrt{2}A\left\{\sin\left(\int_0^{\phi} sleaf_2(u)du+\frac{\pi}{4}\right)+\cos\left(\int_0^{\phi} cleaf_2(u)du\right)\right\} \tag{3.20}$$

Initial velocity:

$$\frac{dx(0)}{dt}=\sqrt{2}A\cdot\omega\cdot\left\{\cos\left(\int_0^{\phi} sleaf_2(u)du+\frac{\pi}{4}\right)\cdot sleaf_2(\phi)-\sin\left(\int_0^{\phi} cleaf_2(u)du\right)\cdot cleaf_2(\phi)\right\} \tag{3.21}$$

・Type (VI) (See Appendix VI for details)

Exact solution:

$$\begin{gathered} x(t)=A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)-\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \\ \left(=A\cos\left(\int_0^{\omega t+\phi} sl(u)du\right)+A\sin\left(\int_0^{\omega t+\phi} sl(u)du\right)-\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cl(u)du\right)\right) \end{gathered} \tag{3.22}$$

Ordinary differential equation:

$$\frac{d^2x(t)}{dt^2}+3\omega^2\left(2\sqrt{2}-1\right)x(t)+2\frac{\omega^2}{A^2}x(t)^3=0 \tag{3.23}$$

Initial position:

$$x(0)=\sqrt{2}A\left\{\sin\left(\int_0^{\phi} sleaf_2(u)du+\frac{\pi}{4}\right)-\cos\left(\int_0^{\phi} cleaf_2(u)du\right)\right\} \tag{3.24}$$

Initial velocity:

$$\frac{dx(0)}{dt}=\sqrt{2}A\cdot\omega\cdot\left\{\cos\left(\int_0^{\phi} sleaf_2(u)du+\frac{\pi}{4}\right)\cdot sleaf_2(\phi)+\sin\left(\int_0^{\phi} cleaf_2(u)du\right)\cdot cleaf_2(\phi)\right\} \tag{3.25}$$

・Type (VII) (See Appendix VII for details)

Exact solution:

$$x(t) = A \cdot sleaf_2(\omega \cdot t + \phi) \cdot cleaf_2(\omega \cdot t + \phi)$$
$$(= A \cdot sl(\omega \cdot t + \phi) \cdot cl(\omega \cdot t + \phi)) \quad (3.26)$$

Ordinary differential equation:

$$\frac{d^2 x(t)}{dt^2} + 6\omega^2 x(t) - 2\left(\frac{\omega}{A}\right)^2 x(t)^3 = 0 \quad (3.27)$$

Initial position:

$$x(0) = A \cdot sleaf_2(\phi) \cdot cleaf_2(\phi) \quad (3.28)$$

Initial velocity:

$$\frac{dx(0)}{dt} = A\omega\left\{(cleaf_2(\phi))^2 - (sleaf_2(\phi))^2\right\} \quad (3.29)$$

Variables *x(t), A, t, ω, u,* and *Φ* represent displacement, amplitude, time, angular frequency, dummy variable, and initial phase, respectively. The exact solutions of types (I)-(VII) satisfy the cubic Duffing equation (Eq. (3.1)), verification of which is summarized in the appendix.

4 Numerical results of exact solutions

4.1 Numerical results of exact solution of type (I)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.2), the wave of the exact solution of type (I) is compared to those of functions $cleaf_2(t)$ and $\int_0^t cleaf_2(u)du$. These waves are shown in Fig. 7. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. In function $cleaf_2(t)$, the amplitude and period become 1.0 and $2\pi_2$, respectively (see Tab. 3). The center of the displacement is $x(t)=0$. In function $\int_0^t cleaf_2(u)du$, the amplitude is as follows (see Appendix A):

$$\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du = \pm\frac{\pi}{4} (\cong \pm 0.785398) \ (m\text{: integer}) \quad (4.1)$$

The period becomes constant π_2 (see Tab. 3). The center of the displacement is $x(t)=0$. Next, the wave obtained by the type (I) exact solution is discussed. The minimum of variable $x(t)$ is obtained as follows:

$$x\left(\frac{\pi_2}{2}(2m-1)\right) = \cos\left(\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du\right) = \cos\left(\pm\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \ (m\text{: integer}) \quad (4.2)$$

The first-order differential of the type (I) exact solution is obtained as follows:

$$\frac{dx(t)}{dt} = -\sin\left(\int_0^t cleaf_2(u)du\right)\cdot cleaf_2(t) \text{ (}m\text{: integer)} \tag{4.3}$$

By substituting $t = \frac{\pi_2}{2}(2m-1)$ into Eq. (4.3), Eq. (4.3) is satisfied as follows:

$$\begin{aligned} \frac{d}{dt}x\left(\frac{\pi_2}{2}(2m-1)\right) &= -\sin\left(\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du\right)\cdot cleaf_2\left(\frac{\pi_2}{2}(2m-1)\right) \\ &= -\sin\left(\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du\right)\cdot 0 = 0 \end{aligned} \tag{4.4}$$

Then, Eq. (2.9) is applied to the above equation. Next, the maximum variable *x(t)* is obtained as follows:

$$x\left(\frac{\pi_2}{2}(2m)\right) = \cos\left(\int_0^{\frac{\pi_2}{2}(2m)} cleaf_2(u)du\right) = \cos(0) = 1.0 \tag{4.5}$$

By substituting $t = \frac{\pi_2}{2}(2m)$ into Eq. (4.3), Eq. (4.3) is satisfied as follows:

$$\frac{d}{dt}x\left(\frac{\pi_2}{2}(2m)\right) = -\sin\left(\int_0^{\frac{\pi_2}{2}(2m)} cleaf_2(u)du\right)\cdot cleaf_2\left(\frac{\pi_2}{2}(2m)\right) = -\sin(0)\cdot(\pm 1) = 0 \tag{4.6}$$

In the type (I) exact solution, the range of *x(t)* is $\frac{1}{\sqrt{2}} \le x(t) \le 1$. The centers of the displacement and amplitude are $\frac{2+\sqrt{2}}{4}$ and $\frac{2-\sqrt{2}}{4}$, respectively.

In the exact solution of type (I), integration function $\int_0^{m\pi_2} cleaf_2(u)du$ represents the phase of the cosine function. As shown in Fig. 6, the range of this function is the inequality $-\frac{\pi}{4} \le \int_0^{m\pi_2} cleaf_2(t)dt \le \frac{\pi}{4}$ (see Appendix A). The periodicity of integration function $\int_0^{m\pi_2} cleaf_2(u)du$ is constant $2\pi_2$ as shown in Tab. 3. Therefore, for $\omega=1$, $A=1$, and $\Phi=1$, the type (I) exact solution is as follows:

$$x(t) = \cos\left(\int_0^t cleaf_2(u)du\right) \tag{4.7}$$

The periodicity of the above solution becomes constant π_2 because the cosine function satisfies the following equation:

$$\cos\left(\int_0^t cleaf_2(u)du\right) = \cos\left(-\int_0^t cleaf_2(u)du\right) \tag{4.8}$$

The ordinary difference equation is as follows:

$$\frac{d^2x(t)}{dt^2} - 3x(t) + 4x(t)^3 = 0 \tag{4.9}$$

The second derivative $d^2x(t)/dt^2$ is as follows (see Appendix I):

$$\frac{d^2x(t)}{dt^2} = -\cos\left(\int_0^t cleaf_2(u)du\right)\cdot\left(cleaf_2(t)\right)^2 - \sin\left(\int_0^t cleaf_2(u)du\right)\cdot\frac{d}{dt}cleaf_2(t) \tag{4.10}$$

where the derivative of $cleaf_2(t)$ with respect to parameter t is as follows:

$$\frac{d}{dt}cleaf_2(t) = -\sqrt{1-\left(cleaf_2(t)\right)^4} \quad (2m-2)\pi_2 \le t \le (2m-1)\pi_2 \tag{4.11}$$

$$\frac{d}{dt}cleaf_2(t) = \sqrt{1-\left(cleaf_2(t)\right)^4} \quad (2m-1)\pi_2 \le t \le 2m\pi_2 \tag{4.12}$$

Table 5: Numerical data of the type (I) exact solution
(All results have been rounded to no more than six significant figures)

t	$x(t)$ (by the Eq.(4.7))	$x(t)^3$ (by the Eq.(4.7))	$d^2x(t)/dt^2$ (by the Eq. (4.10))
-10.0	0.89959	0.72801	-0.21327
-9.0	0.71812	0.37033	0.67303
-8.0	0.99116	0.97372	-0.92141
-7.0	0.77341	0.46264	0.46969
-6.0	0.80697	0.52550	0.31890
-5.0	0.97149	0.91689	-0.75308
-4.0	0.70868	0.35593	0.70234
-3.0	0.93565	0.81913	-0.46954
-2.0	0.85261	0.61981	0.07858
-1.0	0.74045	0.40597	0.59746
0.0	1.00000	1.00000	-1.00000
1.0	0.74045	0.40597	0.59746
2.0	0.85261	0.61981	0.07858
3.0	0.93565	0.81913	-0.46954
4.0	0.70868	0.35593	0.70234
5.0	0.97149	0.91689	-0.75308
6.0	0.80697	0.52550	0.31890
7.0	0.77341	0.46264	0.46969
8.0	0.99116	0.97372	-0.92141
9.0	0.71812	0.37033	0.67303
10.0	0.89959	0.72801	-0.21327

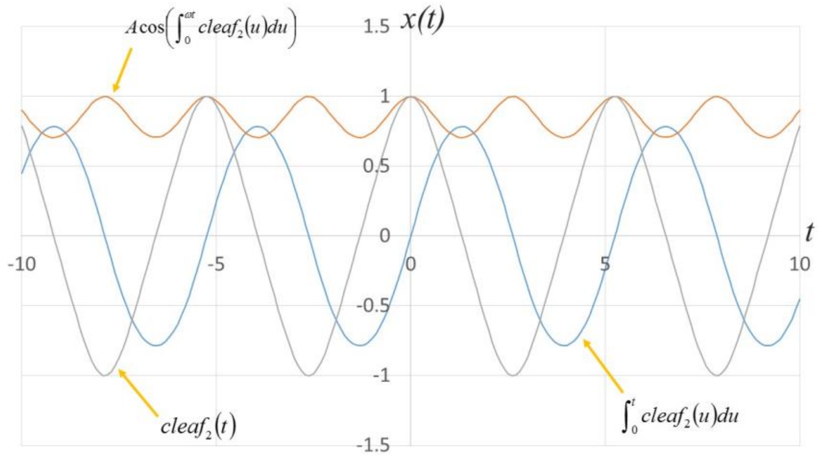

Figure 7: Waves obtained by the type (I) exact solution; leaf function $cleaf_2(t)$ and function $\int_0^t cleaf_2(u)du$

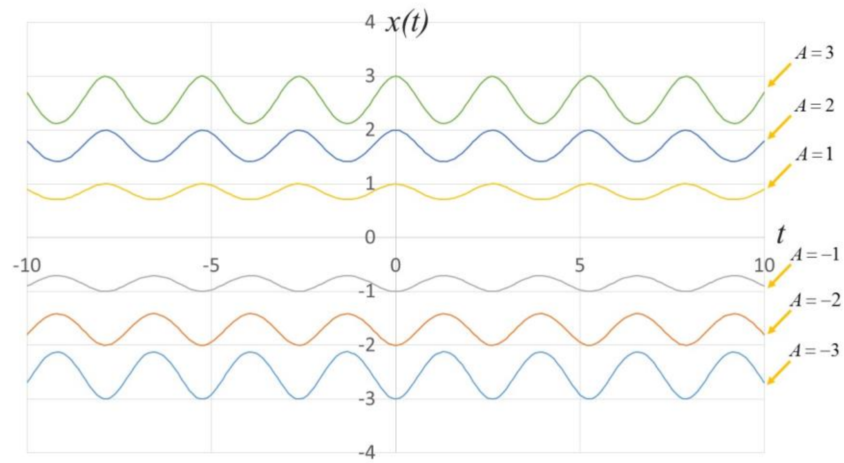

Figure 8: Wave obtained by the type (I) exact solution at varying amplitude *A*

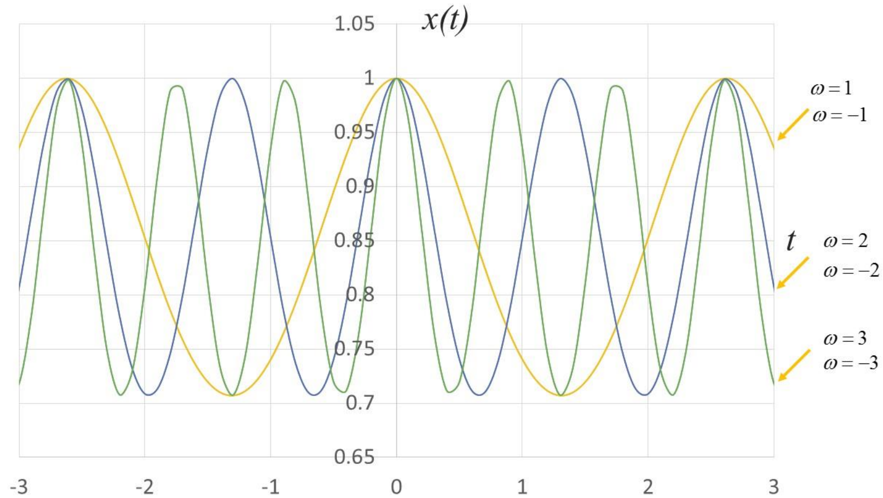

Figure 9: Wave obtained by the type (I) exact solution at varying angular frequency ω

Note that the sign of the derivation $dcleaf_2(t)/dt$ depends on the range of parameter t. The three terms $d^2x(t)/dt^2$, $x(t)$, and $x(t)^3$ in Eq. (4.9) are obtained using the data in Tab. 4 and are summarized in Tab. 5. The value of $d^2x(t)/dt^2-3x(t)+4\,x(t)^3$ is zero, as shown in Tab. 5. The type (I) exact solution satisfies Eq. (4.9) using the numerical data given in Tab. 5. The variables $\Phi=0$ and $\omega=1$ are fixed. whereas amplitude A is varied. Under these conditions, the waves obtained by the type (I) exact solution are shown in Fig. 8. As amplitude A varies, the initial position (in Eq. (3.4)) also varies. The range of displacement $x(t)$ can be obtained by the following inequality:

$$\frac{1}{\sqrt{2}}A \le x(t) \le A \quad (A \ge 0) \tag{4.13}$$

$$A \le x(t) \le \frac{1}{\sqrt{2}}A \quad (A < 0) \tag{4.14}$$

The center of displacement $x(t)$ is obtained as follows:

$$(Center\ of\ displacement) = \frac{1+\frac{1}{\sqrt{2}}}{2}A = \frac{2+\sqrt{2}}{4}A \tag{4.15}$$

Amplitude A is obtained as follows:

$$(Amplitude) = A - \frac{2+\sqrt{2}}{4}A = \frac{2-\sqrt{2}}{4}A \tag{4.16}$$

Next, the variables $\Phi=0$ and $A=1$ are set, whereas angular frequency ω is varied. The waves obtained by the type (I) exact solution are shown in Fig. 9. The period of the waves varies according to the absolute value $|\omega|$. As $|\omega|$ increases, period T decreases. For $\omega=\pm 1$, period T is constant π_2, for $\omega=\pm 2$, it is $\pi_2/2$, and for $\omega=\pm 3$, it is $\pi_2/3$. By using

ω, period T is obtained as follows:

$$T = \frac{\pi_2}{|\omega|} \tag{4.17}$$

4.2 Numerical results of exact solution of type (II)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.6), the exact solution of type (II), the second derivative, and the ordinary difference equation are, respectively, as follows:

$$x(t) = \sin\left(\int_0^t cleaf_2(u)du\right) \tag{4.18}$$

$$\frac{d^2x(t)}{dt^2} = -\sin\left(\int_0^t cleaf_2(u)du\right)\cdot\left(cleaf_2(t)\right)^2 + \cos\left(\int_0^t cleaf_2(u)du\right)\cdot\frac{d}{dt}cleaf_2(t) \tag{4.19}$$

$$\frac{d^2x(t)}{dt^2} + 3x(t) - 4x(t)^3 = 0 \tag{4.20}$$

The data of the terms $d^2x(t)/dt^2$, x(t), and $x(t)^3$ in Eq. (4.20) are summarized in Tab. 6. The type (II) solution satisfies Eq. (4.20), as shown in Tab. 6. In this case, the wave of the exact solution of type (II) is compared to those of functions $cleaf_2(t)$ and $\int_0^t cleaf_2(u)du$. These waves are shown in Fig. 10. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. The maximum $x(t)$ of the type (II) exact solution is obtained as follows:

$$\sin\left(\int_0^{\frac{\pi_2}{2}(4m+1)} cleaf_2(u)du\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \ (m\text{: integer}) \tag{4.21}$$

The minimum $x(t)$ of the type (II) exact solution is obtained as follows:

$$\sin\left(\int_0^{\frac{\pi_2}{2}(4m-1)} cleaf_2(u)du\right) = \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \ (m\text{: integer}) \tag{4.22}$$

The range of $x(t)$ is $-\frac{1}{\sqrt{2}} \le x(t) \le \frac{1}{\sqrt{2}}$. The centers of the displacement and amplitude are *0.0* and $\frac{1}{\sqrt{2}}$, respectively. The period of the type (II) exact solution is $2\pi_2$.

Next, the variables $\Phi=0$ and $\omega=1$ are set, whereas amplitude A is varied. Under these conditions, the waves obtained by the type (II) exact solution are shown in Figs. 11 and 12. The center of displacement $x(t)$ is 0.0. The range of displacement $x(t)$ can be obtained by the following inequality:

$$-\frac{1}{\sqrt{2}}|A| \le x(t) \le \frac{1}{\sqrt{2}}|A| \tag{4.23}$$

The amplitude is obtained as follows:

$$(Amplitude) = \frac{1}{\sqrt{2}}|A| \tag{4.24}$$

Next, the variables $\Phi=0$ and $A=1$ are set, whereas angular frequency ω is varied. The waves obtained by the type (II) exact solution are shown in Figs. 13 and 14. For $\omega=\pm 1$, the period is constant $2\pi_2$, for $\omega=\pm 2$, it becomes $2\pi_2/2$, and for $\omega=\pm 3$, it becomes $2\pi_2/3$. By using parameter ω, period T is obtained as follows:

$$T = \frac{2\pi_2}{|\omega|} \tag{4.25}$$

Table 6: Numerical data of the type (II) exact solution
(All results have been rounded to no more than six significant figures)

t	$x(t)$ (by the Eq. (4.18))	$x(t)^3$ (by the Eq. (4.18))	$d^2x(t)/dt^2$ (by the Eq. (4.19))
-10.0	0.43672	0.08329	-0.97699
-9.0	0.69591	0.33703	-0.73961
-8.0	0.13263	0.00233	-0.38858
-7.0	-0.63389	-0.25471	0.88283
-6.0	-0.59059	-0.20599	0.94778
-5.0	0.23707	0.01332	-0.65791
-4.0	0.70552	0.35118	-0.71184
-3.0	0.35290	0.04395	-0.88290
-2.0	-0.52253	-0.14267	0.99690
-1.0	-0.67210	-0.30360	0.80189
0.0	0.00000	0.00000	0.00000
1.0	0.67210	0.30360	-0.80189
2.0	0.52253	0.14267	-0.99690
3.0	-0.35290	-0.04395	0.88290
4.0	-0.70552	-0.35118	0.71184
5.0	-0.23707	-0.01332	0.65791
6.0	0.59059	0.20599	-0.94778
7.0	0.63389	0.25471	-0.88283
8.0	-0.13263	-0.00233	0.38858
9.0	-0.69591	-0.33703	0.73961
10.0	-0.43672	-0.08329	0.97699

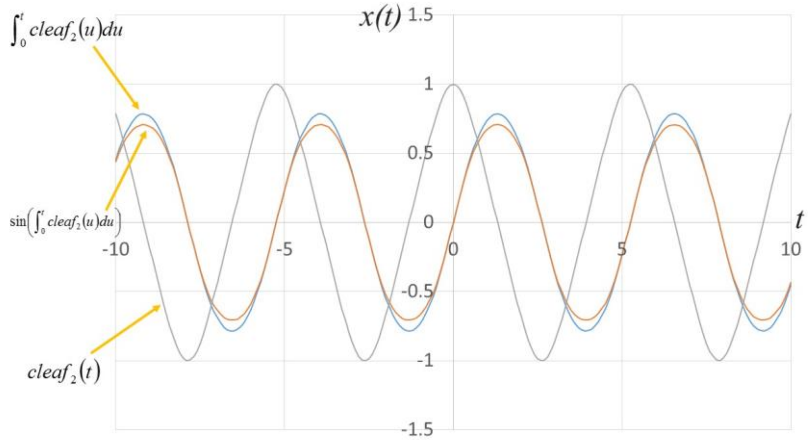

Figure 10: Waves obtained by the type (II) exact solution; leaf function $cleaf_2(t)$ and function $\int_0^t cleaf_2(u)du$

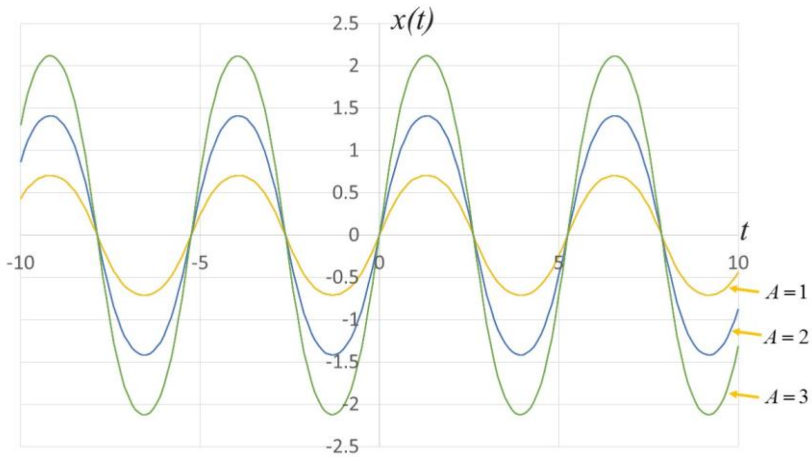

Figure 11: Waves obtained by the type(II) exact solution at varying amplitude A (*A=1, 2, 3*)

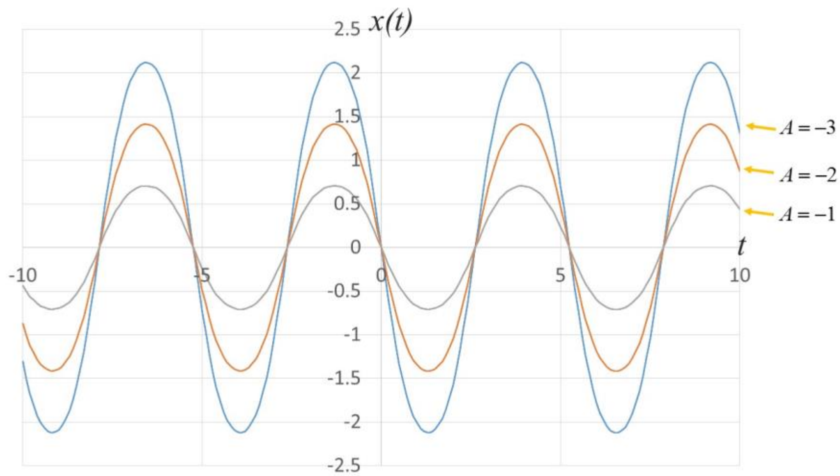

Figure 12: Waves obtained by the type(II) exact solution at varying amplitude A (*A=-1, -2, -3)*

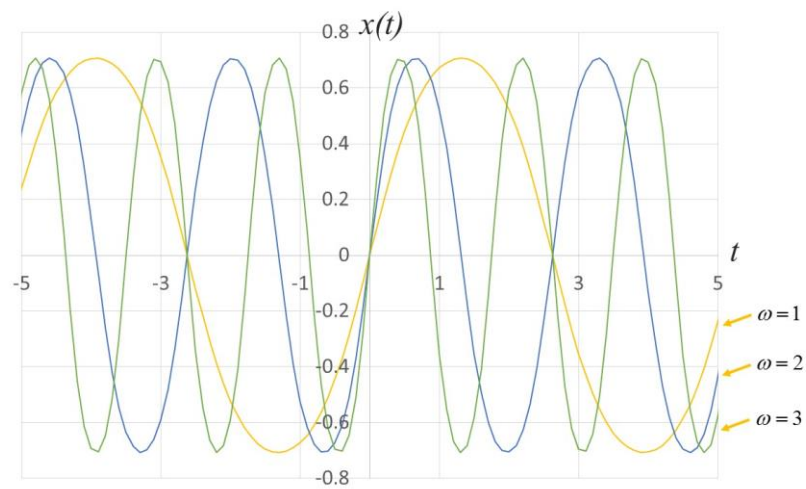

Figure 13: Waves obtained by the type (II) exact solution at varying angular frequency *ω* (*ω=1, 2, 3*)

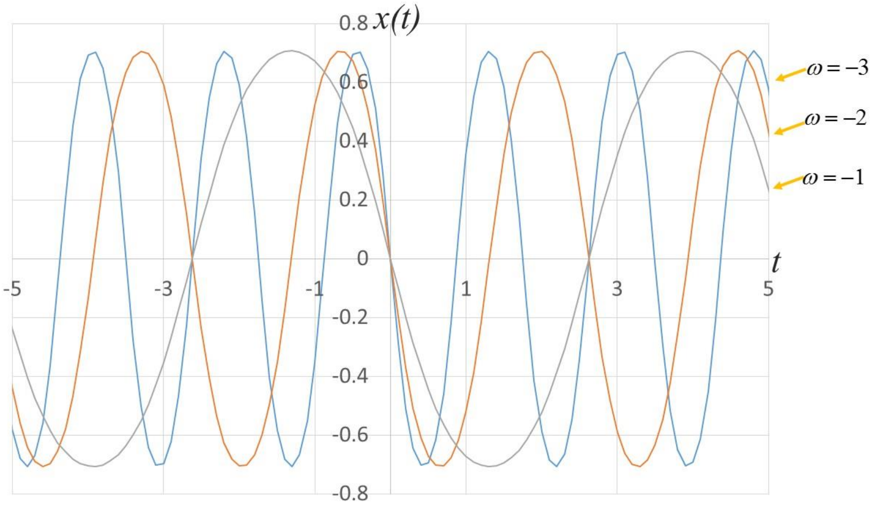

Figure 14: Waves obtained by the type (II) exact solution at varying angular frequency ω (ω=-1, -2, -3)

4.3 Numerical results of exact solution of type (III)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.10), the exact solution of type (III), the second derivative, and the ordinary difference equation are, respectively, as follows:

$$x(t)=\cos\left(\int_0^t sleaf_2(u)du\right)+\sin\left(\int_0^t sleaf_2(u)du\right) \tag{4.26}$$

$$\begin{aligned}\frac{d^2x(t)}{dt^2}&=4\left\{\cos\left(\int_0^t sleaf_2(u)du\right)\right\}^3+4\left\{\sin\left(\int_0^t sleaf_2(u)du\right)\right\}^3\\&-3\cos\left(\int_0^t sleaf_2(u)du\right)-3\sin\left(\int_0^t sleaf_2(u)du\right)\end{aligned} \tag{4.27}$$

$$\frac{d^2x(t)}{dt^2}-3x(t)+2x(t)^3=0 \tag{4.28}$$

The data of the terms $d^2x(t)/dt^2$, $x(t)$, and $x(t)^3$ in Eq. (4.28) are summarized in Tab. 7. The type (III) exact solution satisfies Eq. (4.28) from the numerical data given in Tab. 7. For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.10), the wave of the exact solution of type (III) is compared with those of functions $sleaf_2(t)$ and $\int_0^t sleaf_2(u)du$. These waves are shown in Fig. 15. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. In function $sleaf_2(t)$, the amplitude and the period are 1.0 and $2\pi_2$($\cong$5.244) [Shinohara (2015)], respectively. The center of the displacement becomes $x(t)$=0. In function $\int_0^t sleaf_2(u)du$, the minimum value of the displacement is 0.0. The maximum value of the displacement is given as follows:

$$\int_0^{\pi_2} sleaf_2(u)du = \frac{\pi}{2}(\cong 1.5708) \tag{4.29}$$

The center of displacement is obtained as follows:

$$(Center\, of\ displacement) = \frac{0+\frac{\pi}{2}}{2} = \frac{\pi}{4}(\cong 0.7854\cdots) \tag{4.30}$$

The amplitude of function $\int_0^t sleaf_2(u)du$ is *π/4*. Next, the wave obtained by the type (III) exact solution is discussed. By using the addition theorem, the type (III) exact solution can be transformed as follows:

$$x(t) = \sqrt{2}\sin\left(\int_0^t sleaf_2(u)du + \frac{\pi}{4}\right) \tag{4.31}$$

Table 7: Numerical data of the type (III) exact solution
(All results have been rounded to no more than six significant figures)

t	$\cos\left(\int_0^t sleaf_2(u)du\right)$	$\sin\left(\int_0^t sleaf_2(u)du\right)$	*x(t) (by the Eq. (4.26))*	*$x(t)^3$ (by the Eq. (4.26))*	*$d^2x(t)/dt^2$ (by the Eq. (4.27))*
-10.0	0.99293	0.11867	1.11161	1.37359	0.58764
-9.0	0.57289	0.81962	1.39252	2.70027	-1.22297
-8.0	0.00895	0.99995	1.00891	1.02698	0.97277
-7.0	0.36000	0.93294	1.29295	2.16148	-0.44410
-6.0	0.96032	0.27887	1.23920	1.90294	-0.08828
-5.0	0.99955	0.02978	1.02934	1.09064	0.90674
-4.0	0.75273	0.65831	1.41105	2.80953	-1.38589
-3.0	0.07131	0.99745	1.06876	1.22082	0.76466
-2.0	0.19132	0.98152	1.17285	1.61336	0.29183
-1.0	0.88508	0.46542	1.35051	2.46318	-0.87483
0.0	1.00000	0.00000	1.00000	1.00000	1.00000
1.0	0.88508	0.46542	1.35051	2.46318	-0.87483
2.0	0.19132	0.98152	1.17285	1.61336	0.29183
3.0	0.07131	0.99745	1.06876	1.22082	0.76466
4.0	0.75273	0.65831	1.41105	2.80953	-1.38589
5.0	0.99955	0.02978	1.02934	1.09064	0.90674
6.0	0.96032	0.27887	1.23920	1.90294	-0.08828
7.0	0.36000	0.93294	1.29295	2.16148	-0.44410
8.0	0.00895	0.99995	1.00891	1.02698	0.97277
9.0	0.57289	0.81962	1.39252	2.70027	-1.22297
10.0	0.99293	0.11867	1.11161	1.37359	0.58764

The minimum *x(t)* is obtained as follows:

$$x(2m\pi_2)=\sqrt{2}\sin\left(\int_0^{2m\pi_2} sleaf_2(u)du+\frac{\pi}{4}\right)=\sqrt{2}\sin\left(0+\frac{\pi}{4}\right)=1.0\,(m:\ integer) \quad (4.32)$$

or

$$x((2m-1)\pi_2)=\sqrt{2}\sin\left(\int_0^{(2m-1)\pi_2} sleaf_2(u)du+\frac{\pi}{4}\right)=\sqrt{2}\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=1.0\ (m:\ integer) \quad (4.33)$$

In contrast, the maximum *x(t)* is obtained as follows:

$$\sqrt{2}\sin\left(\int_0^{(2m-1)\frac{\pi_2}{2}} sleaf_2(u)du+\frac{\pi}{4}\right)=\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\sqrt{2} \quad (m:\ integer) \quad (4.34)$$

The range of *x(t)* is $1.0\le x(t)\le\sqrt{2}$. The centers of displacement and amplitude are $x(t)=\frac{1+\sqrt{2}}{2}$ and $\frac{-1+\sqrt{2}}{2}$, respectively. The variables $\Phi=0$ and $\omega=1$ are set, whereas amplitude A is varied. Under these conditions, the waves obtained by the type (III) exact solution are shown in Fig. 16. As amplitude A is varied, the initial position (in Eq. (3.12)) also varies. The range of displacement *x(t)* can be obtained by the following inequality:

$$A\le x(t)\le\sqrt{2}A \quad (A\ge 0) \quad (4.35)$$

$$\sqrt{2}A\le x(t)\le A \quad (A<0) \quad (4.36)$$

The center of displacement *x(t)* is obtained as follows:

$$(Center\ of\ displacement)=\frac{1+\sqrt{2}}{2}A \quad (4.37)$$

The amplitude is obtained as follows:

$$(Amplitude)=\frac{-1+\sqrt{2}}{2}|A| \quad (4.38)$$

Next, the variables $\Phi=0$ and $A=1$ are set, whereas angular frequency ω is varied. The waves obtained by the type (III) exact solution are shown in Fig. 17. For $\omega=\pm 1$, the period is constant π_2, for $\omega=\pm 2$, it is constant $\pi_2/2$, and for $\omega=\pm 3$, it is $\pi_2/3$. By using parameter ω, period T is obtained as follows:

$$T=\frac{\pi_2}{|\omega|} \quad (4.39)$$

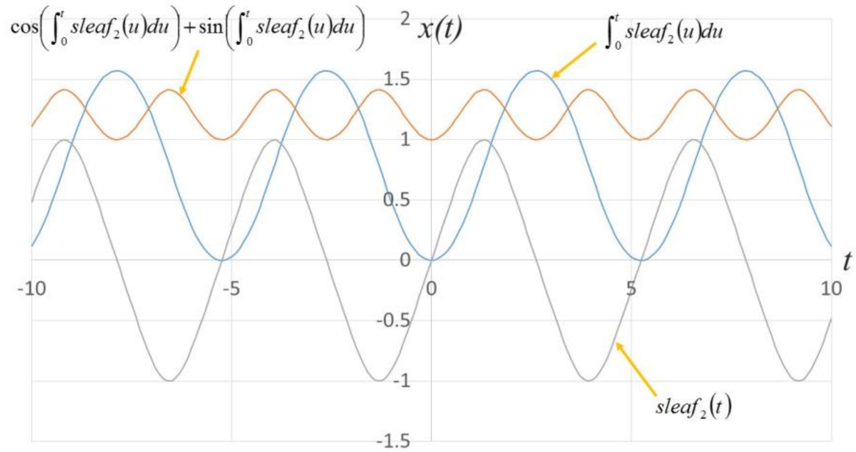

Figure 15: Waves obtained by type (III) exact solution; leaf function $sleaf_2(t)$ and function $\int_0^t sleaf_2(u)du$

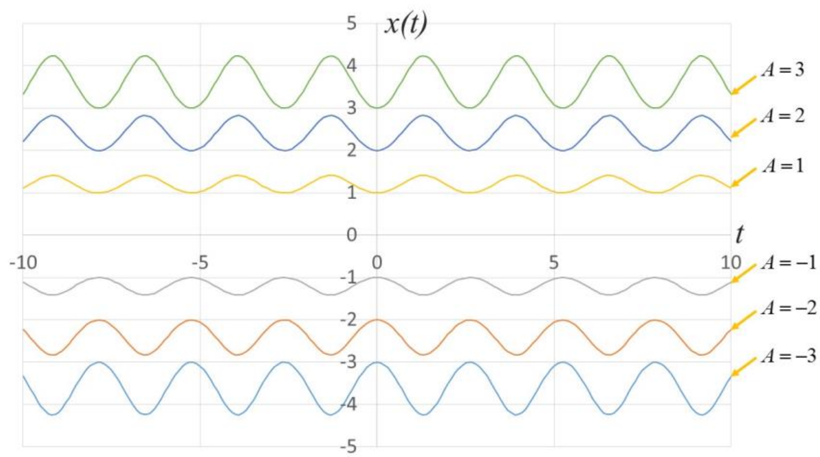

Figure 16: Wave obtained by the type(III) exact solution at varying amplitude *A*

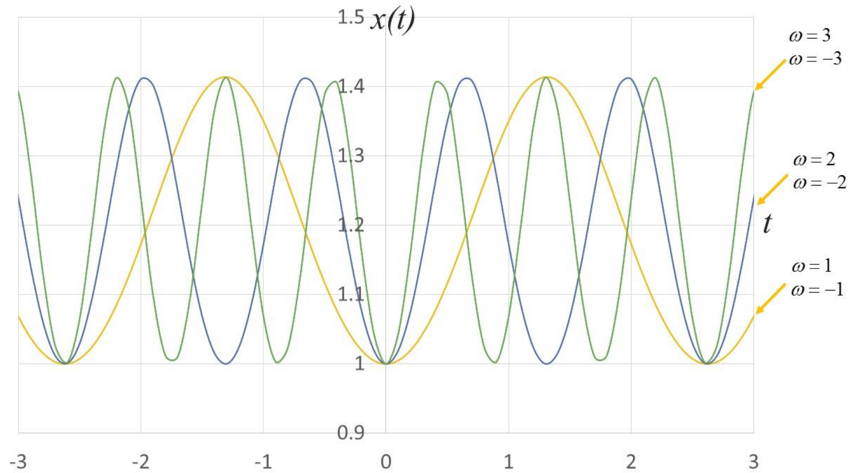

Figure 17: Wave obtained by the type (III) exact solution at varying angular frequency ω

4.4 Numerical results of exact solution of type (IV)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.14), the waves of the exact solution of type (IV), function $sleaf_2(t)$, and function $\int_0^t sleaf_2(u)du$ are shown in Fig. 18. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. The numerical data of the type (IV) exact solution can be obtained by using data in Tab. 7. By using the addition theorem, the type (IV) exact solution can be transformed as follows:

$$x(t) = -\sqrt{2}\sin\left(\int_0^t sleaf_2(u)du - \frac{\pi}{4}\right) \tag{4.40}$$

The maximum value of displacement is given as follows:

$$x(2m\pi_2) = -\sqrt{2}\sin\left(\int_0^{2m\pi_2} sleaf_2(u)du - \frac{\pi}{4}\right) = -\sqrt{2}\sin\left(0 - \frac{\pi}{4}\right) = 1.0 \quad (m{:}integer) \tag{4.41}$$

In contrast, the minimum value of displacement is given as follows:

$$x((2m-1)\pi_2) = -\sqrt{2}\sin\left(\int_0^{(2m-1)\pi_2} sleaf_2(u)du - \frac{\pi}{4}\right) = -\sqrt{2}\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = -1.0 \quad (m{:}integer) \tag{4.42}$$

The range of $x(t)$ is $-1.0 \leqq x(t) \leqq 1.0$. The centers of displacement and amplitude are $x(t)=0$ and 1.0, respectively.

Next, the variables $\Phi=0$ and $\omega=1$ are set, whereas amplitude A is varied. Under these conditions, the waves obtained by the type (IV) exact solution are shown in Figs. 19 and 20. The range of displacement $x(t)$ can be obtained by the following inequality:

$$-|A| \le x(t) \le |A| \tag{4.43}$$

The centers of displacement and amplitude are $x(t)=0$ and $|A|$, respectively. Next, the variables $\Phi=0$ and $A=1$ are set, whereas angular frequency ω is varied. The waves obtained by the type (IV) exact solution are shown in Fig. 21. The period of the waves varies according to the absolute value ω; as ω increases, the period decreases For $\omega=\pm 1$, the period is constant $2\pi_2$, for $\omega=\pm 2$, it is $2\pi_2/2$, and for $\omega=\pm 3$, it is $2\pi_2/3$. By using ω, period T is obtained as follows:

$$T = \frac{2\pi_2}{|\omega|} \tag{4.44}$$

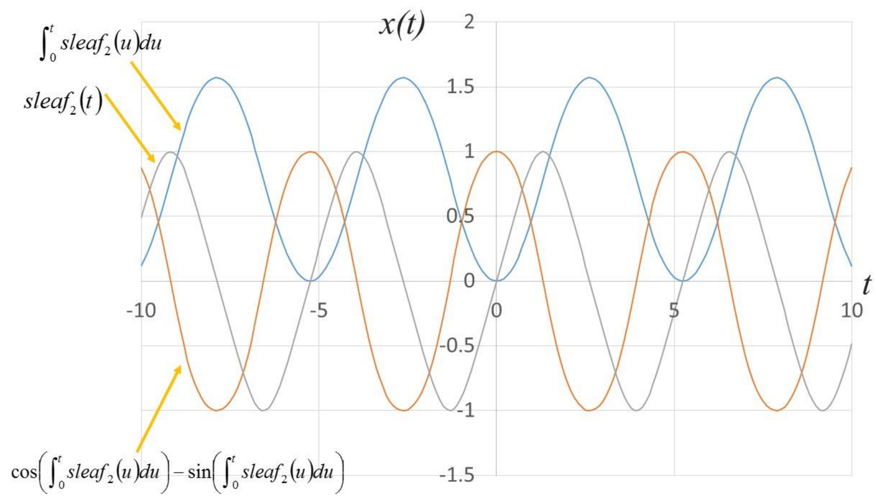

Figure 18: Waves obtained by the type (IV) exact solution; leaf function $sleaf_2(t)$ and function $\int_0^t sleaf_2(u)du$

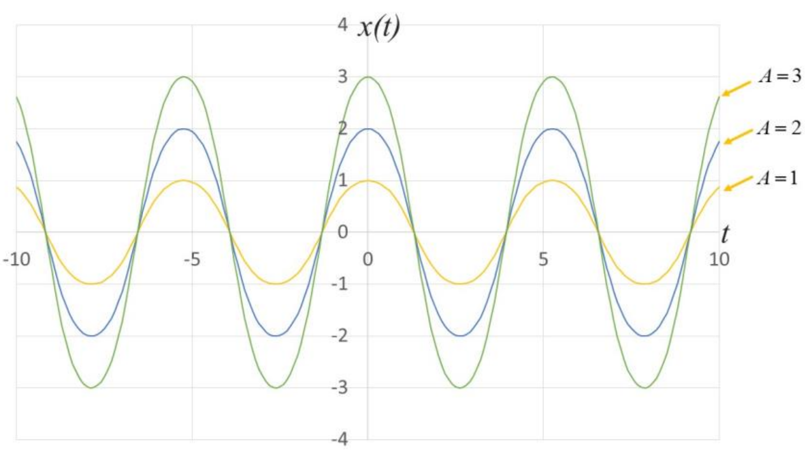

Figure 19: Wave obtained by the type(IV) exact solution at varying amplitude A ($A=1, 2, 3$)

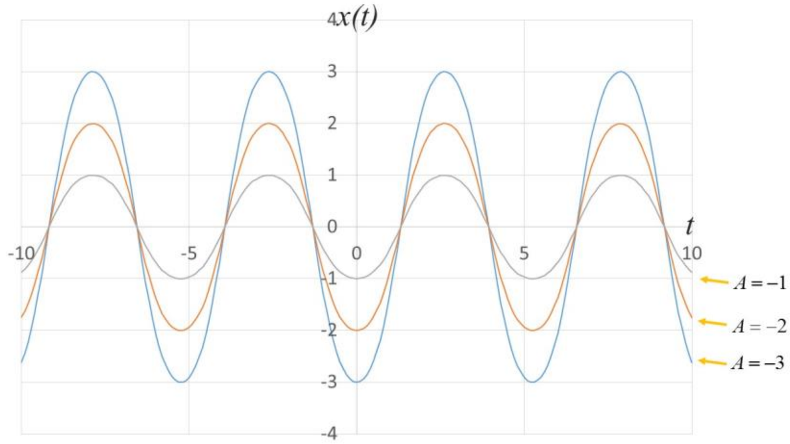

Figure 20: Wave obtained by the type(IV) exact solution at varying amplitude A (A=-1, -2, -3)

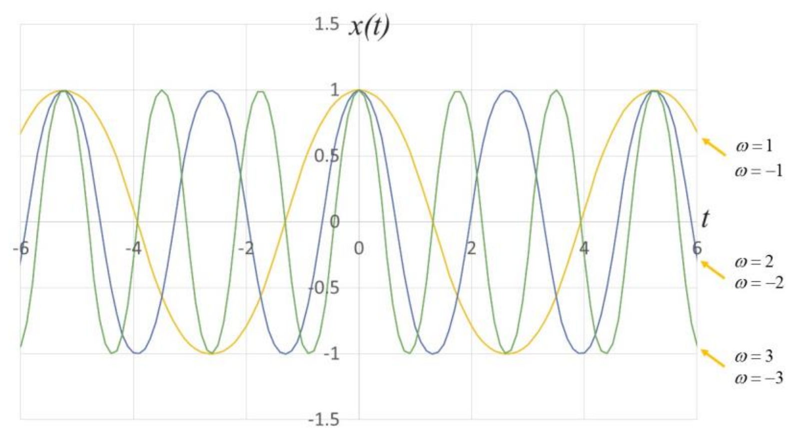

Figure 21: Wave obtained by the type (IV) exact solution at varying angular frequency ω

4.5 Numerical results of exact solution of type (V)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.18), the waves of the exact solution of type (V), leaf functions $sleaf_2(t)$ and $cleaf_2(t)$, and functions $\int_0^t sleaf_2(u)du$ and $\int_0^t cleaf_2(u)du$ are shown in Fig. 22. To show the wave of the type (V) exact solution, Fig. 23 shows an enlarged view of Fig. 22. The horizontal and vertical axes represent time t and displacement $x(t)$, respectively. The numerical data of the type (V) exact solution can be obtained by using data given in Tabs. 5 and 7. To discuss the range of $x(t)$in the type (V) exact solution, the following equation is transformed:

$$x_1(t) = \cos\left(\int_0^t sleaf_2(u)du\right) + \sin\left(\int_0^t sleaf_2(u)du\right) \tag{4.45}$$

The following equation is obtained by squaring both sides of the above equation.

$$\begin{aligned}\{x_1(t)\}^2 &= \left\{\cos\left(\int_0^t sleaf_2(u)du\right)\right\}^2 + \left\{\sin\left(\int_0^t sleaf_2(u)du\right)\right\}^2 \\ &+ 2\sin\left(\int_0^t sleaf_2(u)du\right)\cos\left(\int_0^t sleaf_2(u)du\right) \\ &= 1+\sin\left(2\int_0^t sleaf_2(u)du\right) = 1+(sleaf_2(t))^2\end{aligned} \tag{4.46}$$

Eq. (III.4) in Appendix III is applied to the above equation. As shown in Fig. 15, the inequality $x_1(t)>0$ is obvious. The following equation is obtained.

$$x_1(t) = \sqrt{1+(sleaf_2(t))^2} \tag{4.47}$$

Next, the following equation is transformed:

$$x_2(t) = \sqrt{2}\cos\left(\int_0^t cleaf_2(u)du\right) \tag{4.48}$$

Eq. (4.48) is squared on both sides to obtain

$$\begin{aligned}\{x_2(t)\}^2 &= 2\left\{\cos\left(\int_0^t cleaf_2(u)du\right)\right\}^2 = 2\frac{1+\cos\left(2\int_0^t cleaf_2(u)du\right)}{2} \\ &= 1+\cos\left(2\int_0^t cleaf_2(u)du\right) = 1+(cleaf_2(t))^2\end{aligned} \tag{4.49}$$

As shown in Fig. 7, the inequality $x_2(t)>0$ is obvious. The following equation is obtained.

$$x_2(t) = \sqrt{1+(cleaf_2(t))^2} \tag{4.50}$$

Therefore, the type (V) exact solution is obtained as follows:

$$x(t) = x_1(t)+x_2(t) = \sqrt{1+(sleaf_2(t))^2} + \sqrt{1+(cleaf_2(t))^2} \tag{4.51}$$

The sign of the first-order differential with respect to leaf functions $sleaf_2(t)$ and $cleaf_2(t)$ depends on domain t of $x(t)$ (see Shinohara [Shinohara (2015)] or Eqs. (4.11) and (4.12)). The first-order differential of Eq. (4.51) is discussed by dividing the domain as shown in Fig. 24.

Domain (1): $2m\pi_2 \le t < \frac{\pi_2}{2} + 2m\pi_2$ *(m: integer)*

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{d}{dt}\left(1+(sleaf_2(t))^2\right)^{\frac{1}{2}} + \frac{d}{dt}\left(1+(cleaf_2(t))^2\right)^{\frac{1}{2}} \\ &= \frac{1}{2}\left(1+(sleaf_2(t))^2\right)^{\frac{1}{2}-1}\cdot 2(sleaf_2(t))\cdot\sqrt{1-(sleaf_2(t))^4} \\ &+ \frac{1}{2}\left(1+(cleaf_2(t))^2\right)^{\frac{1}{2}-1}\cdot 2(cleaf_2(t))\cdot\left(-\sqrt{1-(cleaf_2(t))^4}\right) \\ &= \frac{(sleaf_2(t))\sqrt{\left(1+(sleaf_2(t))^2\right)\left(1-(sleaf_2(t))^2\right)}}{\sqrt{1+(sleaf_2(t))^2}} - \frac{(cleaf_2(t))\sqrt{\left(1+(cleaf_2(t))^2\right)\left(1-(cleaf_2(t))^2\right)}}{\sqrt{1+(cleaf_2(t))^2}} \\ &= sleaf_2(t)\sqrt{1-(sleaf_2(t))^2} - cleaf_2(t)\sqrt{1-(cleaf_2(t))^2}\end{aligned} \tag{4.52}$$

Domain (2): $\frac{\pi_2}{2}+2m\pi_2 \le t < \pi_2 + 2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt}=\frac{d}{dt}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}}+\frac{d}{dt}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(sleaf_2(t)\right)\cdot\left(-\sqrt{1-\left(sleaf_2(t)\right)^4}\right)$$
$$+\frac{1}{2}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(cleaf_2(t)\right)\cdot\left(-\sqrt{1-\left(cleaf_2(t)\right)^4}\right)$$
$$=-sleaf_2(t)\sqrt{1-\left(sleaf_2(t)\right)^2}-cleaf_2(t)\sqrt{1-\left(cleaf_2(t)\right)^2} \quad (4.53)$$

Domain (3): $\pi_2+2m\pi_2 \le t < \frac{3\pi_2}{2}+2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt}=\frac{d}{dt}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}}+\frac{d}{dt}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(sleaf_2(t)\right)\cdot\left(-\sqrt{1-\left(sleaf_2(t)\right)^4}\right)$$
$$+\frac{1}{2}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(cleaf_2(t)\right)\cdot\left(\sqrt{1-\left(cleaf_2(t)\right)^4}\right)$$
$$=-sleaf_2(t)\sqrt{1-\left(sleaf_2(t)\right)^2}+cleaf_2(t)\sqrt{1-\left(cleaf_2(t)\right)^2} \quad (4.54)$$

Domain (4): $\frac{3\pi_2}{2}+2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt}=\frac{d}{dt}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}}+\frac{d}{dt}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}}$$
$$=\frac{1}{2}\left(1+\left(sleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(sleaf_2(t)\right)\cdot\left(\sqrt{1-\left(sleaf_2(t)\right)^4}\right)$$
$$+\frac{1}{2}\left(1+\left(cleaf_2(t)\right)^2\right)^{\frac{1}{2}-1}\cdot 2\left(cleaf_2(t)\right)\cdot\left(\sqrt{1-\left(cleaf_2(t)\right)^4}\right)$$
$$=sleaf_2(t)\sqrt{1-\left(sleaf_2(t)\right)^2}+cleaf_2(t)\sqrt{1-\left(cleaf_2(t)\right)^2} \quad (4.55)$$

The extreme value of the type (V) exact solution is obtained by *dx(t)/dt=0* with Eqs. (4.52)-(4.55), as shown below.

$$\left(sleaf_2(t)-cleaf_2(t)\right)\cdot\left(sleaf_2(t)+cleaf_2(t)\right)\cdot\left\{1-\left(sleaf_2(t)\right)^2-\left(cleaf_2(t)\right)^2\right\}=0 \quad (4.56)$$

For the above equation to satisfy *dx(t)/dt=0*, one of the following conditions (i)-(iii) must be satisfied.

$$(i)\quad sleaf_2(t)-cleaf_2(t)=0 \quad (4.57)$$

$$(ii)\quad sleaf_2(t)+cleaf_2(t)=0 \quad (4.58)$$

$$(iii)\quad 1-\left(sleaf_2(t)\right)^2-\left(cleaf_2(t)\right)^2=0 \quad (4.59)$$

As shown in Fig. 24, solutions *t* of condition (i) (Eq. (4.57)) are obtained as follows:

$$t=(4m+1)\frac{\pi_2}{4} \quad \text{(Domain (1) and Domain (3) in Fig. 24)} \quad (4.60)$$

Using Eqs. (4.57) and (B.1), the values of the leaf function are obtained. For *m=2k* (*m* and *k*: integer), the leaf function value is as follows:

$$sleaf_2\left((8k+1)\frac{\pi_2}{4}\right)=cleaf_2\left((8k+1)\frac{\pi_2}{4}\right)=\sqrt{\sqrt{2}-1}\quad \text{(Domain (1) in Fig. 24)} \tag{4.61}$$

For *m=2k+1* (*m* and *k*: integer), the value is as follows:

$$sleaf_2\left((8k+5)\frac{\pi_2}{4}\right)=cleaf_2\left((8k+5)\frac{\pi_2}{4}\right)=-\sqrt{\sqrt{2}-1}\quad \text{(Domain (3) in Fig. 24)} \tag{4.62}$$

As shown in Fig. 24, solutions *t* of condition (ii) are obtained as follows:

$$t=(4m+3)\frac{\pi_2}{4}\quad \text{(Domain (2) and Domain (4) in Fig. 24)} \tag{4.63}$$

Using Eqs. (4.58) and (B.1), the values of the leaf function are obtained. For *m=2k* (*m* and *k*: integer), the leaf function value is as follows:

$$sleaf_2\left((8k+3)\frac{\pi_2}{4}\right)=cleaf_2\left((8k+3)\frac{\pi_2}{4}\right)=\sqrt{\sqrt{2}-1}\quad \text{(Domain (2) in Fig. 24)} \tag{4.64}$$

For *m=2k+1* (*m* and *k*: integer), the value is as follows:

$$sleaf_2\left((8k+7)\frac{\pi_2}{4}\right)=cleaf_2\left((8k+7)\frac{\pi_2}{4}\right)=-\sqrt{\sqrt{2}-1}\quad \text{(Domain (4) in Fig. 24)} \tag{4.65}$$

Using Eqs. (4.59) and (B.1), the following equation is obtained.

$$(sleaf_2(t))^2(cleaf_2(t))^2=0 \tag{4.66}$$

For $sleaf_2(t)=0$, solutions *t* of the condition given in Eq. (4.59) are obtained as follows:

$$t=m\pi_2\quad \text{(Domain (1) and Domain (3) in Fig. 24)} \tag{4.67}$$

For $cleaf_2(t)=0$, solutions *t* of the conditions given in Eq. (4.59) are obtained as follows:

$$t=(2m+1)\frac{\pi_2}{2}\quad \text{(Domain (2) and Domain (4) in Fig. 24)} \tag{4.68}$$

Domain (1): Here, we consider the case where condition (i) is satisfied in Domain (1). Using Eq. (4.61), the minimum value of Eq. (4.51) is obtained as follows:

$$x\left((8k+1)\frac{\pi_2}{4}\right)=\sqrt{1+\left(\sqrt{\sqrt{2}-1}\right)^2}+\sqrt{1+\left(\sqrt{\sqrt{2}-1}\right)^2}=2^{\frac{5}{4}}(=2.378\cdots) \tag{4.69}$$

Next, we consider the case where condition (ii) is satisfied in Domain (1): $2m\pi_2\le t<\frac{\pi_2}{2}+2m\pi_2$. As shown in Fig. 24, the curve of function $sleaf_2(t)$ does not intersect with the curve of function -c$leaf_2(t)$.

Therefore, *t* does not satisfy condition (ii).

Next, we consider the case where condition (iii) is satisfied in Domain (1): $2m\pi_2\le t<\frac{\pi_2}{2}+2m\pi_2$. Using Eq. (4.67), the maximum value of Eq. (4.51) is obtained as follows:

$$x(m\pi_2)=\sqrt{1+(0)^2}+\sqrt{1+(\pm1)^2}=1+\sqrt{2}==2.414214\cdots \tag{4.70}$$

Domain (2): We consider the case where condition (i) is satisfied in Domain (2): $\frac{\pi_2}{2}+2m\pi_2 \le t < \pi_2+2m\pi_2$. As shown in Fig. 24, the curve of function *sleaf*$_2$*(t)* does not intersect with the curve of function -c*leaf*$_2$*(t)*. Therefore, *t* does not satisfy condition (i).

Next, we consider the case where condition (ii) is satisfied in Domain (2): $\frac{\pi_2}{2}+2m\pi_2 \le t < \pi_2+2m\pi_2$. Using Eq. (4.64), the minimum value of Eq. (4.51) is obtained as follows:

$$x\left((8k+3)\frac{\pi_2}{4}\right)=\sqrt{1+\left(\sqrt{\sqrt{2}-1}\right)^2}+\sqrt{1+\left(\sqrt{\sqrt{2}-1}\right)^2}=2^{\frac{5}{4}}(=2.378\cdots) \tag{4.71}$$

Next, we consider the case where condition (iii) is satisfied in Domain (2): $\frac{\pi_2}{2}+2m\pi_2 \le t < \pi_2+2m\pi_2$. Using Eq. (4.68), the maximum value of Eq. (4.51) is obtained as follows:

$$x\left((2m+1)\frac{\pi_2}{2}\right)=\sqrt{1+(\pm 1)^2}+\sqrt{1+(0)^2}=1+\sqrt{2}==2.414214\cdots \tag{4.72}$$

Domain (3): Next, we consider the case where condition (i) is satisfied in Domain (3): $\pi_2+2m\pi_2 \le t < \frac{3\pi_2}{2}+2m\pi_2$. Using Eq. (4.62), the minimum value of Eq. (4.51) is obtained as follows:

$$x\left((8k+5)\frac{\pi_2}{4}\right)=\sqrt{1+\left(-\sqrt{\sqrt{2}-1}\right)^2}+\sqrt{1+\left(-\sqrt{\sqrt{2}-1}\right)^2}=2^{\frac{5}{4}} \tag{4.73}$$

Next, we consider the case where condition (ii) is satisfied in Domain (3): $\pi_2+2m\pi_2 \le t < \frac{3\pi_2}{2}+2m\pi_2$. As shown in Fig. 24, the curve of function *sleaf*$_2$*(t)* does not intersect with the curve of function -c*leaf*$_2$*(t)*. Therefore, *t* does not satisfy condition (i).

Next, we consider the case where condition (iii) is satisfied in Domain (3): $\pi_2+2m\pi_2 \le t < \frac{3\pi_2}{2}+2m\pi_2$. Using Eq. (4.67), the maximum value of Eq. (4.51) is obtained as follows:

$$x(m\pi_2)=\sqrt{1+0^2}+\sqrt{1+(-1)^2}=1+\sqrt{2}=2.414214\cdots \tag{4.74}$$

Domain (4): We consider the case where condition (i) is satisfied in Domain (4): $\frac{3\pi_2}{2}+2m\pi_2 \le t < 2\pi_2+2m\pi_2$. As shown in Fig. 24, the curve of function *sleaf*$_2$*(t)* does not intersect with the curve of function c*leaf*$_2$*(t)*. Therefore, *t* does not satisfy condition (i).

Next, we consider the case where condition (ii) is satisfied in Domain (4): $\frac{3\pi_2}{2}+2m\pi_2 \le t < 2\pi_2+2m\pi_2$. Using Eq. (4.65), the minimum value of Eq. (4.51) is obtained as follows:

$$x\left((8k+7)\frac{\pi_2}{4}\right)=\sqrt{1+\left(-\sqrt{\sqrt{2}-1}\right)^2}+\sqrt{1+\left(-\sqrt{\sqrt{2}-1}\right)^2}=2^{\frac{5}{4}} \tag{4.75}$$

Next, we consider the case where condition (iii) is satisfied in Domain (4): $\frac{3\pi_2}{2}+2m\pi_2 \le t < 2\pi_2+2m\pi_2$. Using Eq. (4.68), the maximum value of Eq. (4.51) is obtained as follows:

$$x\left((2m+1)\frac{\pi_2}{2}\right)=\sqrt{1+(-1)^2}+\sqrt{1+0^2}=1+\sqrt{2}=2.414214\cdots \tag{4.76}$$

As stated above, the range of *x(t)* is as follows:

$$2^{\frac{5}{4}} \le x(t) \le 1+\sqrt{2} \tag{4.77}$$

The centers of displacement and amplitude are $\frac{1+\sqrt{2}+2^{\frac{5}{4}}}{2}$ and $\frac{1+\sqrt{2}-2^{\frac{5}{4}}}{2}$, respectively.

We now analyze Figs. 25 and 26. The variables *Φ=0* and *ω=1* are set, whereas amplitude *A* is varied. Under these conditions, the waves obtained by the type (V) exact solution are shown in Fig. 25. The range of displacement *x(t)* can be obtained by the following inequality:

$$2^{\frac{5}{4}}A \le x(t) \le \left(1+\sqrt{2}\right)A \quad A>0 \tag{4.78}$$

$$\left(1+\sqrt{2}\right)A \le x(t) \le 2^{\frac{5}{4}}A \quad A<0 \tag{4.79}$$

The center of displacement *x(t)* is obtained as follows:

$$(Center\ of\ displacement)=\frac{2^{\frac{5}{4}}+1+\sqrt{2}}{2}A \tag{4.80}$$

The amplitude is obtained as follows:

$$(Amplitude)=\left(1+\sqrt{2}\right)|A|-\frac{2^{\frac{5}{4}}+1+\sqrt{2}}{2}|A|=\frac{1+\sqrt{2}-2^{\frac{5}{4}}}{2}|A| \tag{4.81}$$

Next, the variables *Φ=0* and *A=1* are set, whereas angular frequency *ω* is varied. The waves obtained by the type (V) exact solution are shown in Fig. 26. The period of the waves varies according to the absolute value *ω*; as *ω* increases, the period decreases. For *ω=±1*, the period becomes constant $\pi_2/2$, for *ω=±2*, it is $\pi_2/4$, and for *ω=±3*, it is $\pi_2/6$. By using *ω*, period *T* is obtained as follows:

$$T=\frac{\pi_2}{2|\omega|} \tag{4.82}$$

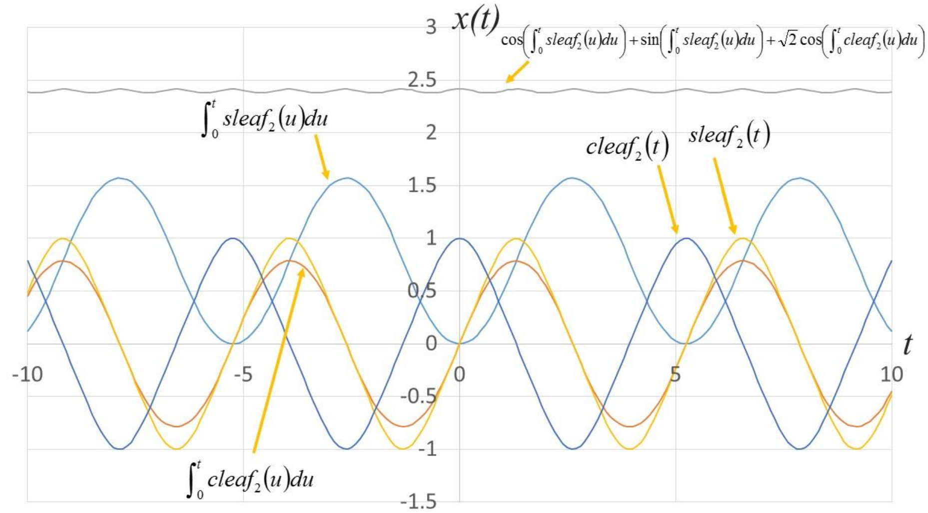

Figure 22: Waves obtained by the type (V) exact solution; leaf function $sleaf_2(t)$ and function $\int_0^t sleaf_2(u)du$

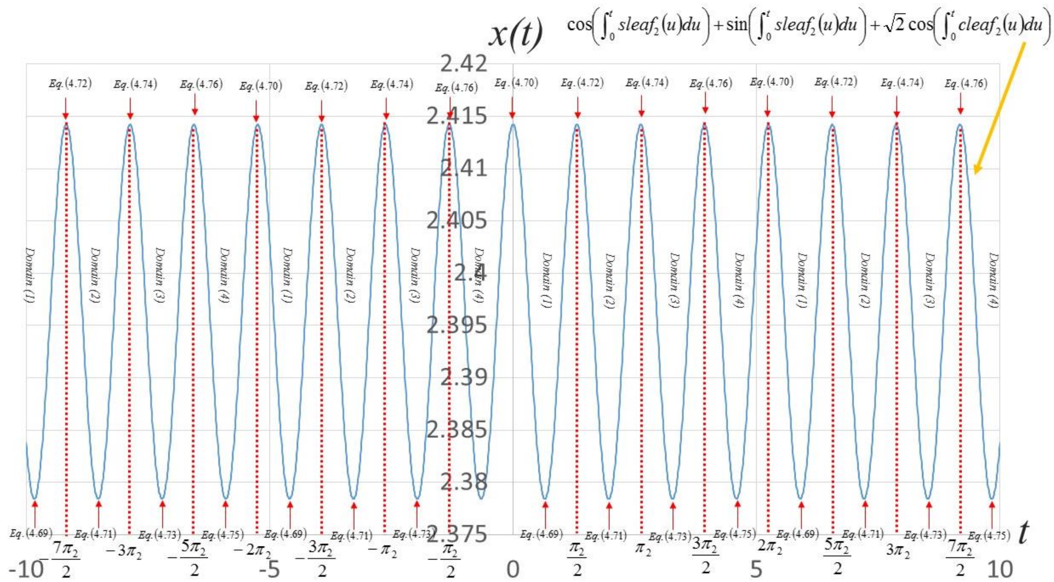

Figure 23: Enlargement of Fig. 22

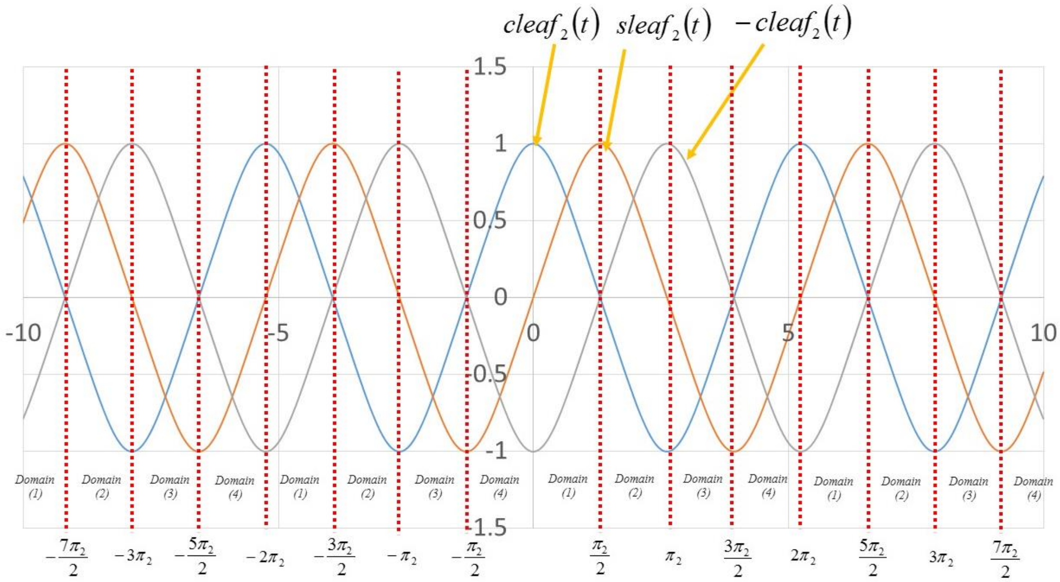

Figure 24: Waves obtained by functions *cleaf₂(t)*, *sleaf₂(t)*, and -*cleaf₂(t)*

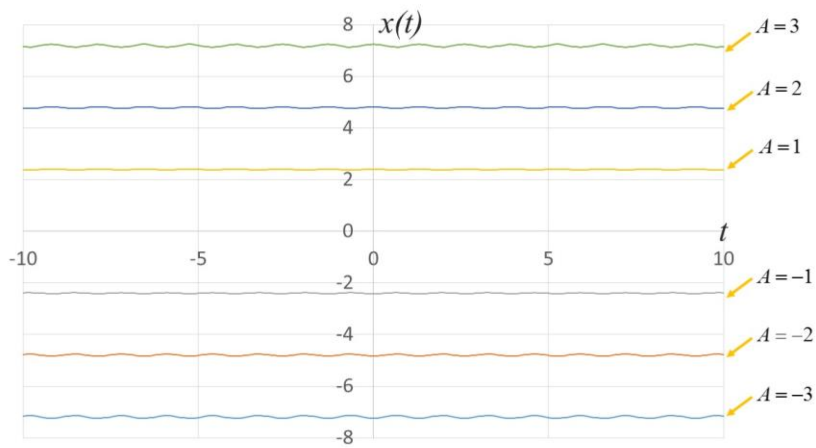

Figure 25: Waves obtained by the type (V) exact solution at varying amplitude *A*

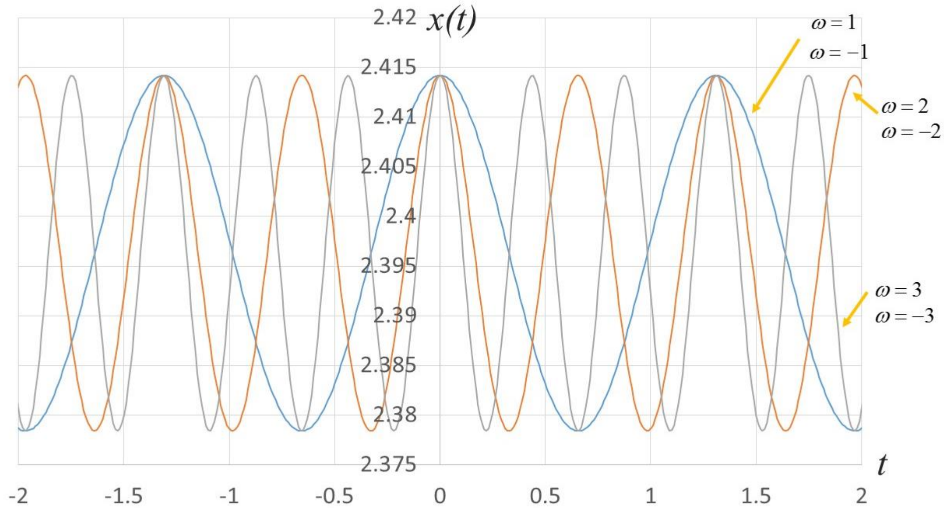

Figure 26: Wave obtained by the type (V) exact solution at varying angular frequency ω

4.6 Numerical results of exact solution of type(VI)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.22), the waves of the exact solution of type (VI), leaf functions $sleaf_2(t)$ and $cleaf_2(t)$, and functions $\int_0^t sleaf_2(u)du$ and $\int_0^t cleaf_2(u)du$ are shown in Fig. 27. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. The numerical data of the type (VI) solution can be obtained from the data given in Tabs. 5 and 7. To discuss the range of $x(t)$ in the type (VI) exact solution, the following equation is transformed using leaf functions $sleaf_2(t)$ and $cleaf_2(t)$:

$$x(t)=\sqrt{1+(sleaf_2(t))^2}-\sqrt{1+(cleaf_2(t))^2} \tag{4.83}$$

The above equation can be obtained by the same operation using Eqs. (4.46) -(4.51). To discuss the range of $x(t)$ in the type (VI) exact solution, the first-order differential is obtained. The sign of the first-order differential with respect to leaf functions $sleaf_2(t)$ and $cleaf_2(t)$ depends on domain t of variable $x(t)$. The first-order differential of Eq. (4.83) is discussed by dividing the domains (1)-(4) as shown in Fig. 24.

Domain (1): $2m\pi_2 \le t < \frac{\pi_2}{2}+2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt}=sleaf_2(t)\sqrt{1-(sleaf_2(t))^2}+cleaf_2(t)\sqrt{1-(cleaf_2(t))^2} \tag{4.84}$$

Domain (2): $\frac{\pi_2}{2}+2m\pi_2 \le t < \pi_2+2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt}=-sleaf_2(t)\sqrt{1-(sleaf_2(t))^2}+cleaf_2(t)\sqrt{1-(cleaf_2(t))^2} \tag{4.85}$$

Domain (3): $\pi_2 + 2m\pi_2 \le t < \frac{3\pi_2}{2} + 2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt} = -sleaf_2(t)\sqrt{1-(sleaf_2(t))^2} - cleaf_2(t)\sqrt{1-(cleaf_2(t))^2} \tag{4.86}$$

Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$ *(m: integer)*

$$\frac{dx(t)}{dt} = sleaf_2(t)\sqrt{1-(sleaf_2(t))^2} - cleaf_2(t)\sqrt{1-(cleaf_2(t))^2} \tag{4.87}$$

The extreme value of the type (VI) solution is obtained by *dx(t)/dt=0* with Eqs. (4.84)-(4.87), which is the same equation as (4.56). Therefore, to satisfy *dx(t)/dt=0*, it is necessary to satisfy any of the conditions (i)-(iii), as shown in Eqs. (4.57)-(4.59).

Domain (1): We consider the case where condition (i) is satisfied in Domain (1): $2m\pi_2 \le t < \frac{\pi_2}{2} + 2m\pi_2$. In condition (i), variable *t* obtained by Eq. (4.84) does not satisfy dx(t)/dt=0. Therefore, the extreme value of Eq. (4.83) cannot be obtained under condition (i).

Next, we consider the case where condition (ii) is satisfied in Domain (1): $2m\pi_2 \le t < \frac{\pi_2}{2} + 2m\pi_2$. As shown in Fig. 24, the curve of function $sleaf_2(t)$ does not intersect with the curve of function $-cleaf_2(t)$. Therefore, variable t does not satisfy condition (ii), and the extremal value cannot be obtained by Eq. (4.83) under condition (ii).

Next, we consider the case where condition (iii) is satisfied in Domain (1): $2m\pi_2 \le t < \frac{\pi_2}{2} + 2m\pi_2$. Using Eq. (4.67), the minimum value of Eq. (4.83) is obtained as follows:

$$x(m\pi_2) = -\sqrt{3-2\sqrt{2}} = -(\sqrt{2}-1) = -0.414214\cdots \textit{(m: integer)} \tag{4.88}$$

Domain (2): We consider the case where condition (i) is satisfied in Domain (2): $\frac{\pi_2}{2} + 2m\pi_2 \le t < \pi_2 + 2m\pi_2$. As shown in Fig. 24, the curve of function $sleaf_2(t)$ does not intersect with the curve of function $cleaf_2(t)$. Therefore, *t* does not satisfy condition (i), and the extreme value of Eq. (4.83) cannot be obtained under condition (i). Next, we consider the case where condition (ii) is satisfied in Domain (2): $\frac{\pi_2}{2} + 2m\pi_2 \le t < \pi_2 + 2m\pi_2$. In condition (ii), *t* satisfying *dx(t)/dt=0* cannot be obtained by Eq. (4.85), and the extreme value of Eq. (4.83) cannot be obtained under condition (ii).

Next, we consider the case where condition (iii) is satisfied in Domain (2): $\frac{\pi_2}{2} + 2m\pi_2 \le t < \pi_2 + 2m\pi_2$. Using Eq. (4.68), the maximum value of Eq. (4.83) is obtained as follows:

$$x\left((2m+1)\frac{\pi_2}{2}\right) = \sqrt{2} - 1 = 0.414214\cdots \textit{(m: integer)} \tag{4.89}$$

Domain (3): We consider the case where condition (i) is satisfied in Domain (3):

$\pi_2 + 2m\pi_2 \le t < \frac{3\pi_2}{2} + 2m\pi_2$. In condition (i), t satisfying $dx(t)/dt=0$ cannot be obtained by Eq. (4.86), and the extreme value of Eq. (4.83) cannot be obtained under condition (i). Next, we consider the case where condition (ii) is satisfied in Domain (3): $\pi_2 + 2m\pi_2 \le t < \frac{3\pi_2}{2} + 2m\pi_2$. As shown in Fig. 24, the curve of function $sleaf_2(t)$ does not intersect with the curve of function $-cleaf_2(t)$. Therefore, t does not satisfy condition (ii).

Next, we consider the case where condition (ii) is satisfied in Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$. In condition (ii), variable t satisfying $dx(t)/dt=0$ cannot be obtained by Eq. (4.87), and the extreme value of Eq. (4.83) cannot be obtained under condition (ii). Next, we consider the case where condition (iii) is satisfied in Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$. Using Eq. (4.67), the minimum value of Eq. (4.83) is obtained as follows:

$$x(m\pi_2) = -\left(\sqrt{2}-1\right) = -0.414214\cdots \quad \textit{(m: integer)} \tag{4.90}$$

Domain (4): We consider the case where condition (i) is satisfied in Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$. As shown in Fig. 24, the curve of function $sleaf_2(t)$ does not intersect with the curve of function $cleaf_2(t)$. Therefore, t does not satisfy condition (ii).

Next, we consider the case where condition (ii) is satisfied in Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$. In condition (ii), variable t satisfying $dx(t)/dt=0$ cannot be obtained by Eq. (4.87), and the extreme value of Eq. (4.83) cannot be obtained under condition (ii). Next, we consider the case where condition (iii) is satisfied in Domain (4): $\frac{3\pi_2}{2} + 2m\pi_2 \le t < 2\pi_2 + 2m\pi_2$. Using Eq. (4.68), the maximum value of Eq. (4.83) is obtained as follows:

$$x\left((2m+1)\frac{\pi_2}{2}\right) = \sqrt{2}-1 = 0.414214\cdots \quad \textit{(m: integer)} \tag{4.91}$$

As stated above, the range of variable $x(t)$ is as follows:

$$-\left(\sqrt{2}-1\right) \le x(t) \le \sqrt{2}-1 \tag{4.92}$$

The amplitude becomes $\sqrt{2}-1$.

We now analyze Figs. 29-31. The variables $\Phi=0$ and $\omega=1$ are set, whereas amplitude A is varied. Under these conditions, the waves obtained by the type (VI) exact solution are shown in Figs. 29 and 30. The range of displacement $x(t)$ can be obtained by the following inequality:

$$-\left(\sqrt{2}-1\right)A \le x(t) \le \left(\sqrt{2}-1\right)A \tag{4.93}$$

The centers of displacement $x(t)$ and amplitude are 0.0 and $\left(\sqrt{2}-1\right)A$, respectively.

Next, the variables $\Phi=0$ and $A=1$ are set, whereas angular frequency ω is varied. The waves obtained by the type (VI) exact solution are shown in Fig. 31. The period of the waves vary according to the absolute value ω. As ω increases, the period decreases. For $\omega=\pm 1$, the period becomes constant π_2, for $\omega=\pm 2$, it is $\pi_2/2$, and for $\omega=\pm 3$, it is $\pi_2/3$. By using ω, period T is obtained as follows:

$$T = \frac{\pi_2}{|\omega|} \tag{4.94}$$

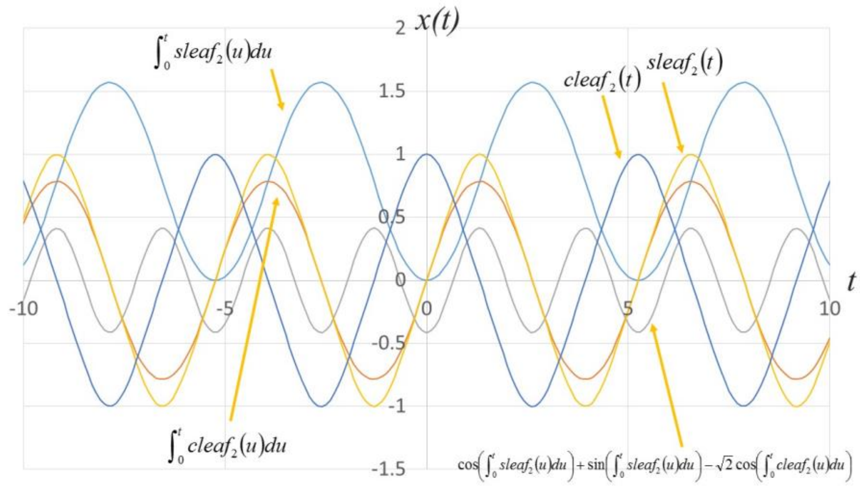

Figure 27: Waves obtained by the type (VI) exact solution; leaf function $sleaf_2(t)$ and function $\int_0^t sleaf_2(u)du$

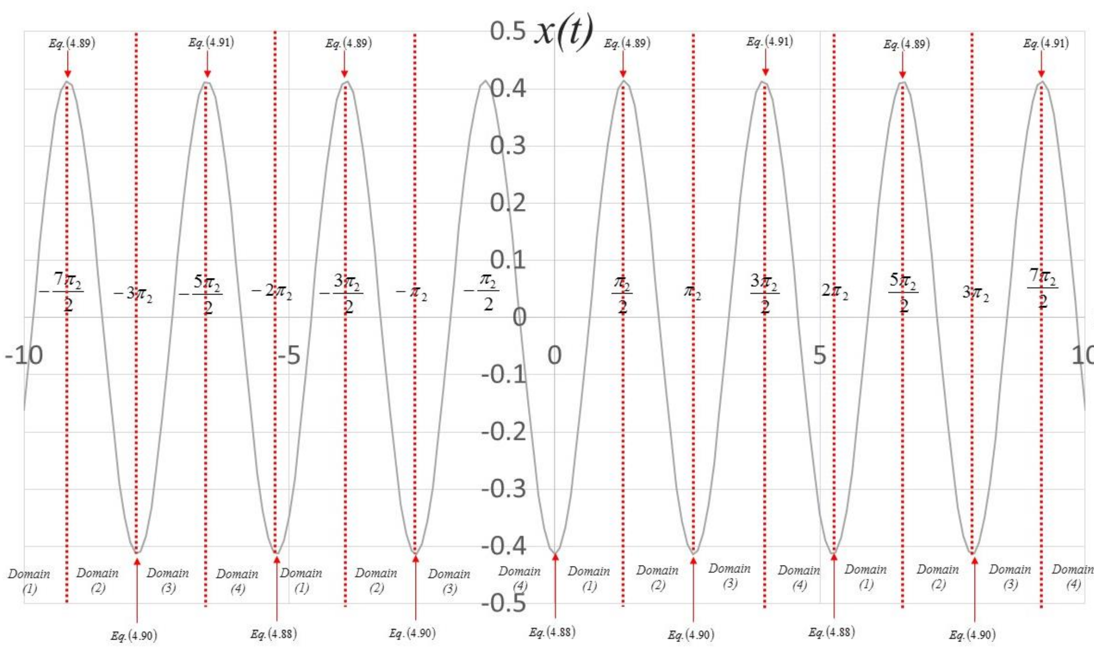

Figure 28: Waves obtained by the type (VI) exact solution

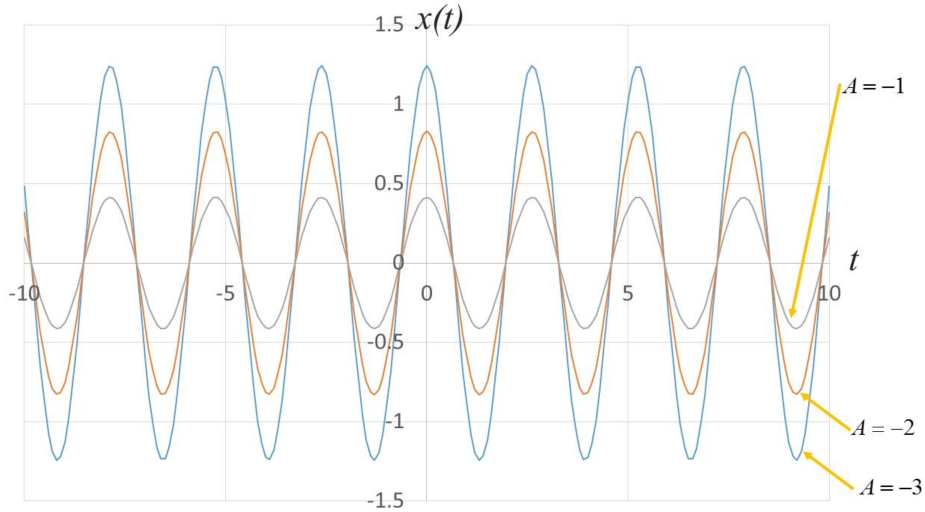

Figure 29: Wave obtained by the type (VI) exact solution at varying amplitude *A* (*A=-1,-2,-3*)

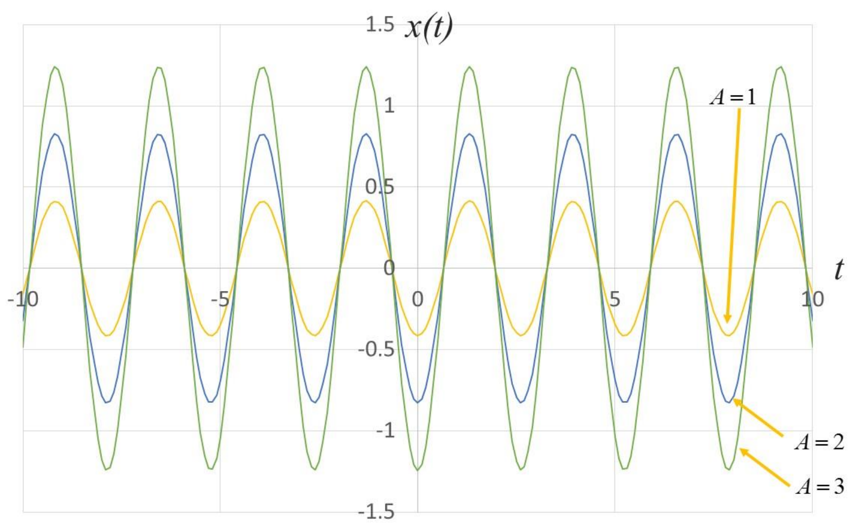

Figure 30: Wave obtained by the type (VI) exact solution at varying amplitude *A* (*A=1,2,3*)

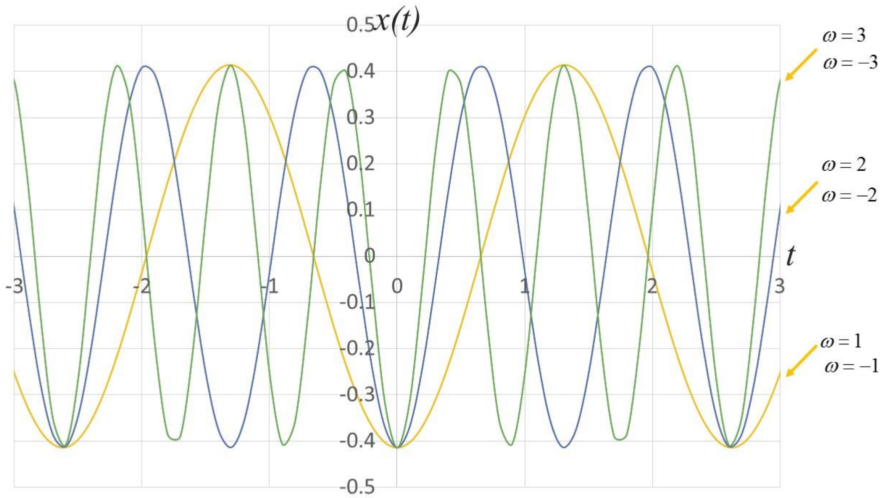

Figure 31: Wave obtained by the type (VI) exact solution at varying angular frequency ω

4.7 Numerical results of exact solution of type (VII)

For $A=1$, $\omega=1$, and $\Phi=0$ in Eq. (3.26), the waves of the exact solution of type (VII) and leaf functions $sleaf_2(t)$ and $cleaf_2(t)$ are shown in Fig. 32. The horizontal and vertical axes represent time and displacement $x(t)$, respectively. The numerical data of the type (VII) solution can be obtained by using data given in Tab. 4. Next, the variables $\Phi=0$ and $\omega=1$ are set, whereas amplitude A is varied. Under these conditions, the waves obtained by the type (VII) exact solution are shown in Figs. 33 and 34. To obtain the extreme values, the first-order differential is derived as follows:

$$\frac{dx(t)}{dt} = A\cdot cleaf_2(t)\cdot\sqrt{1-(sleaf_2(t))^4} - A\cdot sleaf_2(t)\sqrt{1-(cleaf_2(t))^4} = 0 \tag{4.95}$$

Then, the following equation is derived:

$$cleaf_2(t) = \pm sleaf_2(t) \tag{4.96}$$

By using Eqs. (4.96) and (B1) (see Appendix B), functions $sleaf_2(t)$ and $cleaf_2(t)$ are obtained as follows:

$$sleaf_2(t) = \pm\sqrt{-1+\sqrt{2}} \quad cleaf_2(t) = \pm\sqrt{-1+\sqrt{2}} \tag{4.97}$$

Substituting the above values for the Eq. (3.26), the value is obtained as follows:

$$x\left(\frac{\pi_2}{4}(4m+1)\right) = A\left(\sqrt{-1+\sqrt{2}}\right)^2 = A\cdot\left(-1+\sqrt{2}\right) \quad (m=0,\pm1,\pm2\cdots) \tag{4.98}$$

Substituting the above values for the Eq. (3.26), the value is obtained as follows:

$$x\left(\frac{\pi_2}{4}(4m+3)\right) = -A\left(\sqrt{-1+\sqrt{2}}\right)^2 = -A\cdot\left(-1+\sqrt{2}\right) \quad (m=0,\pm1,\pm2\cdots) \tag{4.99}$$

The range of displacement $x(t)$ can be obtained by following inequality:

$$-\left(-1+\sqrt{2}\right)A \le x(t) \le \left(-1+\sqrt{2}\right)A \tag{4.100}$$

The angular frequency ω is varied. The waves obtained by the type (VII) exact solution are shown in Figs. 35 and 36. As shown in the figures, period T is obtained from angular frequency ω as follows:

$$T = \frac{\pi_2}{|\omega|} \tag{4.101}$$

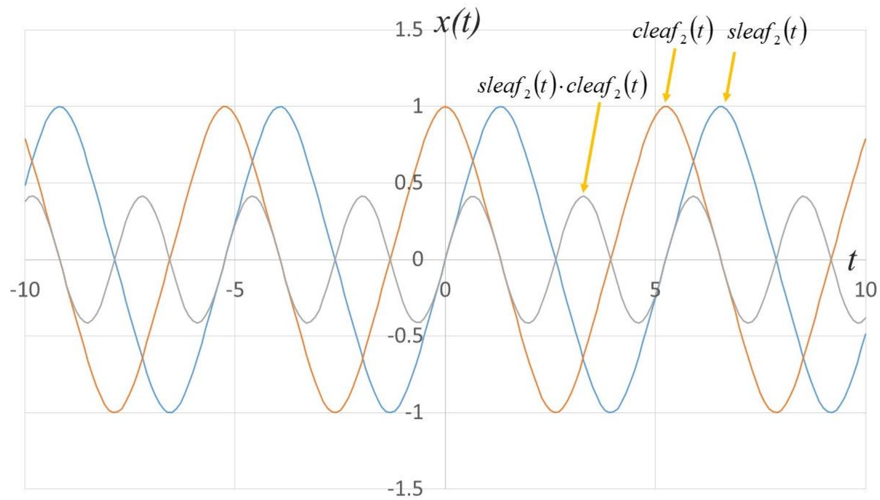

Figure 32: Waves obtained by the type (VII) exact solution; Leaf functions $sleaf_2(t)$ and $cleaf_2(t)$

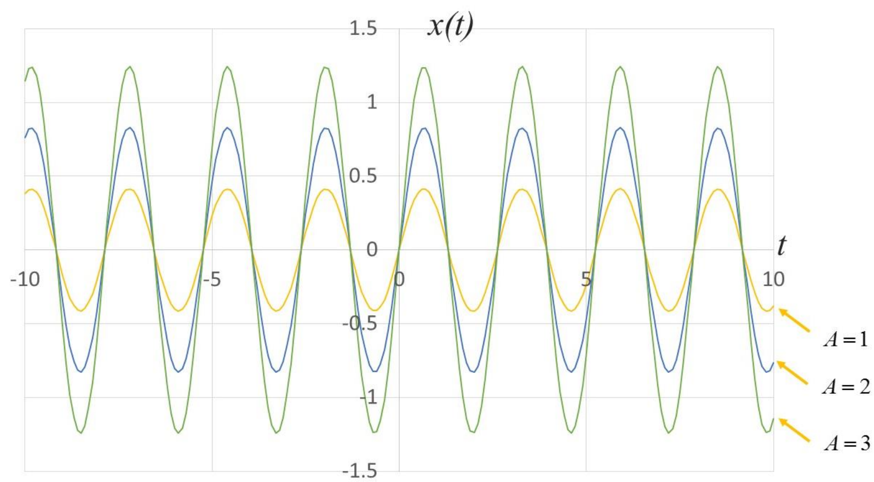

Figure 33: Wave obtained by the type(VII) exact solution at varying amplitude *A(A=1,2,3)*

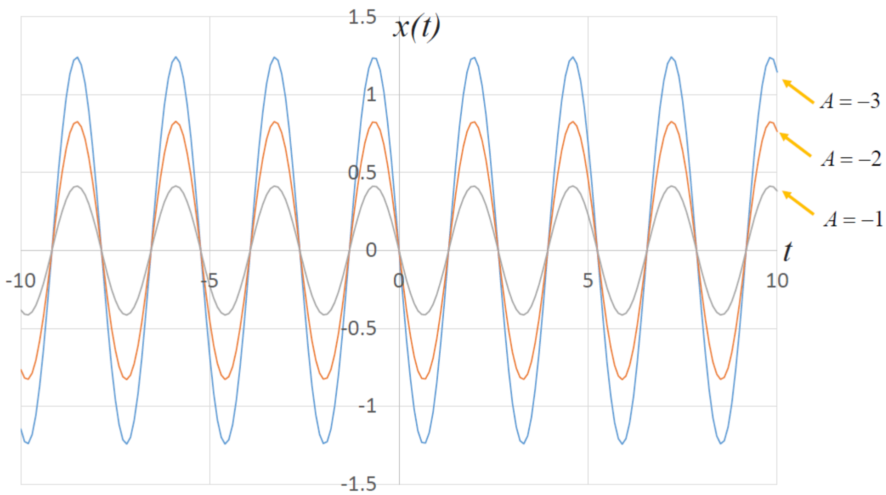

Figure 34: Wave obtained by the type(VII) exact solution at varying amplitude *A (A=-1, -2, -3)*

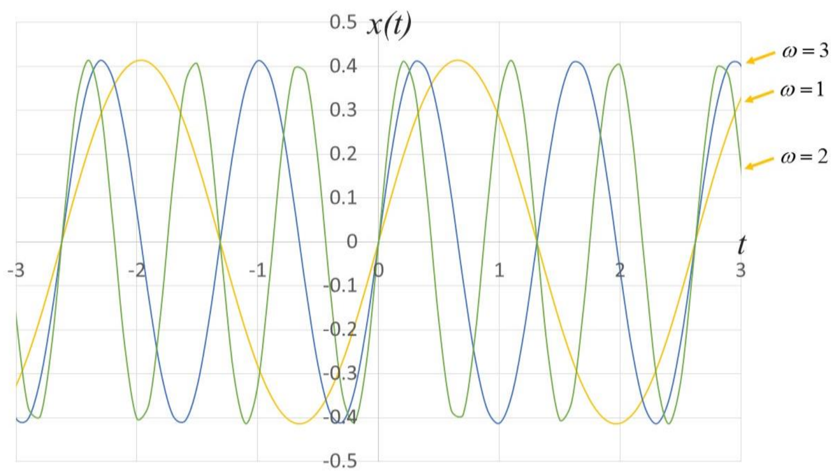

Figure 35: Wave obtained by the type (VII) exact solution at varying angular frequency *ω (ω=1, 2, 3)*

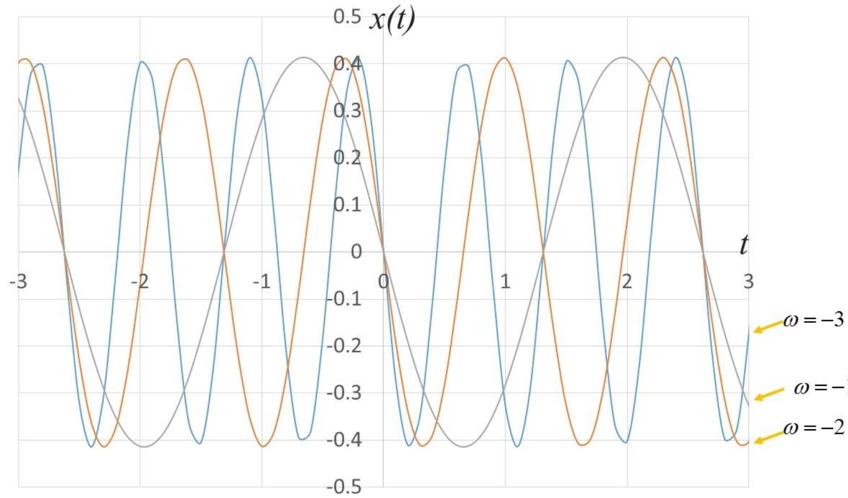

Figure 36: Wave obtained by the type (VII) exact solution at varying angular frequency *ω (ω=-1, -2, -3)*

5 Conclusions

By using leaf functions, the exact solution of the cubic Duffing equation can be derived under certain conditions. The waves obtained by the exact solutions are graphically visualized. The conclusions are summarized as follows:

・Through leaf functions, seven types of exact solutions can be derived from the cubic Duffing equation.

・The seven types of exact solutions have two parameters, namely, angular frequency ω and amplitude A, which indicate the characteristics of the wave. The coefficients of the terms x and x^3 in the cubic Duffing equation can be described by both wave amplitude A and wave frequency parameter ω in the leaf functions. Amplitude A of the wave becomes constant, even though these coefficients vary according to variation in ω. In contrast, wave frequency ω of the wave becomes constant, even though these coefficients vary according to variation in A. Since parameters A and ω do not affect the characteristics of the wave, they are independent variables in the ordinary differential equation.

・As amplitude A increases (decreases), the height of the wave also increases (decreases). As the frequency parameter ω increases (decreases), the period of the waves decreases (increases). The waveform obtained by the nonlinear spring can be controlled by adjusting these variables. Several new waveforms satisfying the cubic Duffing equation can be constructed by combining both trigonometric functions and leaf functions.

In the future research, the relation between trigonometric functions and hyperbolic functions can be obtained by using imaginary numbers. The analogy also exists for leaf functions. These extended leaf functions are defined as hyperbolic leaf functions [Shinohara (2016)]. By using these hyperbolic leaf functions, leaf functions and exponential functions, we are able to derive more exact solutions of Duffing equation. It is in that future research that exact solutions can be presented.

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Appendix I

The type (I) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{I.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\frac{dx(t)}{dt} = -A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\omega t+\phi) \tag{I.2}$$

With the above equation, the second-order differential is obtained as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(cleaf_2(\omega t+\phi)\right)^2 \\ &- A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(-\sqrt{1-\left(cleaf_2(\omega t+\phi)\right)^4}\right)\end{aligned} \tag{I.3}$$

Using Eq. (A4) (see Appendix A), the following equation is obtained:

$$\left(cleaf_2(\omega t+\phi)\right)^2 = \cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{I.4}$$

The above equation is transformed as follows:

$$\sqrt{1-\left(cleaf_2(\omega t+\phi)\right)^4} = \sqrt{1-\left(\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right)^2}$$
$$= \sin\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{I.5}$$

Substituting the above two equations into Eq. (I.3), the following equation is obtained:

$$\frac{d^2x(t)}{dt^2} = -A\omega^2 \cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)$$
$$+A\omega^2 \sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)$$
$$= -A\omega^2 \cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du + 2\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{I.6}$$
$$= -A\omega^2 \cos\left(3\int_0^{\omega t+\phi} cleaf_2(u)du\right)$$
$$= 3\omega^2 A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) - 4\left(\frac{\omega}{A}\right)^2 \left\{A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^3$$

Substituting Eq. (3.2) into the above equation, the following equation is obtained:

$$\frac{d^2x(t)}{dt^2} = 3\omega^2 x(t) - 4\left(\frac{\omega}{A}\right)^2 \{x(t)\}^3 \tag{I.7}$$

Eq. (3.3) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2} - 3\omega^2 x(t) + 4\left(\frac{\omega}{A}\right)^2 x(t)^3 = 0 \tag{I.8}$$

The coefficients α and β in Eq. (3.1) are expressed as follows:

$$\alpha = -3\omega^2 \tag{I.9}$$

$$\beta = 4\left(\frac{\omega}{A}\right)^2 \tag{I.10}$$

Appendix II

The type (II) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{II.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\frac{dx(t)}{dt} = A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\omega t+\phi) \tag{II.2}$$

With the above equation, the second-order differential is obtained as follows:

$$\frac{d^2x(t)}{dt^2} = -A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(cleaf_2(\omega t+\phi)\right)^2 + A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(-\sqrt{1-\left(cleaf_2(\omega t+\phi)\right)^4}\right) \tag{II.3}$$

Substituting Eqs. (I.4) and (I.5) into the above equations, the following equation is obtained:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &- A\omega^2\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &= -A\omega^2\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du + 2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &= -A\omega^2\sin\left(3\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &= -3\omega^2A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) + 4\left(\frac{\omega}{A}\right)^2\left\{A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^3\end{aligned} \tag{II.4}$$

Substituting Eq. (II.1) into the above equations, the following equation is obtained:

$$\frac{d^2x(t)}{dt^2} = -3\omega^2x(t) + 4\left(\frac{\omega}{A}\right)^2\{x(t)\}^3 \tag{II.5}$$

Eq. (3.7) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2} + 3\omega^2x(t) - 4\left(\frac{\omega}{A}\right)^2x(t)^3 = 0 \tag{II.6}$$

The coefficients α and β in Eq. (3.1) are expressed as follows:

$$\alpha = 3\omega^2 \tag{II.7}$$

$$\beta = -4\left(\frac{\omega}{A}\right)^2 \tag{II.8}$$

Appendix III

The type (III) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) + A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{III.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\frac{dx(t)}{dt} = -A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi) + A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi) \tag{III.2}$$

With the above equation, the second-order differential is obtained as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2 \\ &- A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4} \\ &- A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2 \\ &+ A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}\end{aligned} \tag{III.3}$$

By using Eq. (63) given in Shinohara [Shinohara (2015)], the following equation is obtained:

$$\left(sleaf_2(\omega t+\phi)\right)^2 = \sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{III.4}$$

The above equation is transformed as follows:

$$\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4} = \sqrt{1-\left(\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2} = \cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{III.5}$$

Substituting the above two equations into Eq. (III.3), the following equation is obtained:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &+ A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\end{aligned} \tag{III.6}$$

The above equation is summarized by the addition theorem:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= A\omega^2\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du + \int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du + \int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &= A\omega^2\cos\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\omega^2\sin\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)\end{aligned} \tag{III.7}$$

The above equation is transformed as follows:

$$\frac{d^2x(t)}{dt^2}=4A\omega^2\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+4A\omega^2\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$-3A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)-3A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{III.8}$$

Here, the following equation is derived.

$$\{x(t)\}^2=A^2\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2+A^2\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2$$
$$+2A^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{III.9}$$
$$=A^2+2A^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$

The following equation is derived from the above equation:

$$\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)=\frac{\{x(t)\}^2-A^2}{2A^2} \tag{III.10}$$

Next, the following equation is derived.

$$\{x(t)\}^3=\left\{A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$=A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3+A^3\cdot 3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$+A^3\cdot 3\cdot\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2+A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3 \tag{III.11}$$
$$=A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3+A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3$$
$$+A^2\cdot 3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$\cdot\left(A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)$$

The above equation is transformed to get

$$A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3+A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3$$
$$=\{x(t)\}^3-A^2\cdot 3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$\cdot\left(A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right) \tag{III.12}$$
$$=\{x(t)\}^3-A^2\cdot 3\frac{\{x(t)\}^2-A^2}{2A^2}x(t)=-\frac{1}{2}\{x(t)\}^3+\frac{3}{2}A^2x(t)$$

Substituting Eq. (III.12) into Eq. (III.8), the following equation is derived:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= 4\frac{\omega^2}{A^2}\left[A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3 + A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3\right] \\ &-3\omega^2\left[A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) + A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right] \\ &= 4\frac{\omega^2}{A^2}\left[-\frac{1}{2}\{x(t)\}^3 + \frac{3}{2}A^2x(t)\right] - 3\omega^2x(t) \\ &= -2\frac{\omega^2}{A^2}\{x(t)\}^3 + 3\omega^2x(t)\end{aligned} \tag{III.13}$$

Eq. (3.11) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2} - 3\omega^2x(t) + 2\left(\frac{\omega}{A}\right)^2 x(t)^3 = 0 \tag{III.14}$$

The coefficients α and β in Eq. (3.1) are expressed as follows:

$$\alpha = -3\omega^2 \tag{III.15}$$

$$\beta = 2\left(\frac{\omega}{A}\right)^2 \tag{III.16}$$

The initial conditions of Eqs. (3.12) and (3.13) are given as follows:

$$\begin{aligned}x(0) &= A\cos\left(\int_0^{\phi} sleaf_2(u)du\right) + A\sin\left(\int_0^{\phi} sleaf_2(u)du\right) \\ &= \sqrt{2}A\cos\left(\int_0^{\phi} sleaf_2(u)du - \frac{\pi}{4}\right)\end{aligned} \tag{III.17}$$

$$\begin{aligned}\frac{dx(0)}{dt} &= -A\sin\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi) \\ &+ A\cos\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi) \\ &= A\cdot\omega\cdot sleaf_2(\phi)\cdot\left\{\cos\left(\int_0^{\phi} sleaf_2(u)du\right) - \sin\left(\int_0^{\phi} sleaf_2(u)du\right)\right\} \\ &= \sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^{\phi} sleaf_2(u)du + \frac{\pi}{4}\right)\end{aligned} \tag{III.18}$$

Appendix IV

The type (IV) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \tag{IV.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\begin{aligned}\frac{dx(t)}{dt} &= -A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi) \\ &- A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi)\end{aligned} \tag{IV.2}$$

With the above equation, the second-order differential is obtained as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2 \\ &- A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4} \\ &+ A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2 \\ &- A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}\end{aligned} \tag{IV.3}$$

Substituting Eqs. (III.4) and (III.5) into the above equations, the following equation is obtained:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &+ A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\end{aligned} \tag{IV.4}$$

The above equation is summarized by the addition theorem:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du + \int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &- A\omega^2\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du + \int_0^{\omega t+\phi} sleaf_2(u)du\right) \\ &= -A\omega^2\cos\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\omega^2\sin\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)\end{aligned} \tag{IV.5}$$

The above equation is transformed as follows:

$$\frac{d^2x(t)}{dt^2} = -4A\omega^2\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3 + 4A\omega^2\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$+3A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - 3A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \quad \text{(IV.6)}$$

Here, the following equation is derived.

$$\{x(t)\}^2 = A^2\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2 + A^2\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2$$
$$-2A^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \quad \text{(IV.7)}$$
$$= A^2 - 2A^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$

The following equation is derived from the above equation:

$$\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) = \frac{A^2 - \{x(t)\}^2}{2A^2} \quad \text{(IV.8)}$$

Next, the following equation is derived.

$$\{x(t)\}^3 = \left\{A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$= A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3 - A^3\cdot 3\cdot\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2$$
$$+A^3\cdot 3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^2\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3$$
$$= A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3 - A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3$$
$$-A^2\cdot 3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$\cdot\left(A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)$$
$$\text{(IV.9)}$$

The above equation is transformed to get

$$\begin{aligned}
&A^3\left(\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3 - A^3\left(\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right)^3 \\
&= \{x(t)\}^3 + A^2\cdot 3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) \\
&\cdot\left(A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right) \\
&= \{x(t)\}^3 + A^2\cdot 3\frac{A^2-\{x(t)\}^2}{2A^2}x(t) = -\frac{1}{2}\{x(t)\}^3 + \frac{3}{2}A^2x(t)
\end{aligned} \tag{IV.10}$$

The following equation is derived from Eq. (IV.6).

$$\begin{aligned}
\frac{d^2x(t)}{dt^2} &= -4\frac{\omega^2}{A^2}\left[A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3 - A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3\right] \\
&+ 3\omega^2\left[A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right] \\
&= -4\frac{\omega^2}{A^2}\left[-\frac{1}{2}\{x(t)\}^3 + \frac{3}{2}A^2x(t)\right] + 3\omega^2x(t) \\
&= 2\frac{\omega^2}{A^2}\{x(t)\}^3 - 3\omega^2x(t)
\end{aligned} \tag{IV.11}$$

Eq. (3.15) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2} + 3\omega^2x(t) - 2\left(\frac{\omega}{A}\right)^2x(t)^3 = 0 \tag{IV.12}$$

The coefficients α and β in Eq. (3.1) are expressed as follows:

$$\alpha = 3\omega^2 \tag{IV.13}$$

$$\beta = -2\left(\frac{\omega}{A}\right)^2 \tag{IV.14}$$

The initial conditions of Eqs. (3.16) and (3.17) are given as follows:

$$\begin{aligned}
x(0) &= A\cos\left(\int_0^{\phi} sleaf_2(u)du\right) - A\sin\left(\int_0^{\phi} sleaf_2(u)du\right) \\
&= \sqrt{2}A\cos\left(\int_0^{\phi} sleaf_2(u)du + \frac{\pi}{4}\right)
\end{aligned} \tag{IV.15}$$

$$\frac{dx(0)}{dt} = -A\sin\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi)$$
$$-A\cos\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi)$$
$$= A\cdot\omega\cdot sleaf_2(\phi)\cdot\left\{\cos\left(\int_0^{\phi} sleaf_2(u)du\right)+\sin\left(\int_0^{\phi} sleaf_2(u)du\right)\right\} \tag{IV.16}$$
$$= \sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^{\phi} sleaf_2(u)du-\frac{\pi}{4}\right)$$

Appendix V

The type (V) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{V.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\frac{dx(t)}{dt} = -A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi)$$
$$+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi) \tag{V.2}$$
$$-\sqrt{2}A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\omega t+\phi)$$

With the above equation, the second-order differential is obtained as follows:

$$\frac{d^2x(t)}{dt^2} = -A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2$$
$$-A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}$$
$$-A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2 \tag{V.3}$$
$$+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}$$
$$-\sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(cleaf_2(\omega t+\phi)\right)^2$$
$$+\sqrt{2}A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(cleaf_2(\omega t+\phi)\right)^4}$$

The above equation can be transformed as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &+ A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- \sqrt{2}A\omega^2\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &+ \sqrt{2}A\omega^2\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\end{aligned} \tag{V.4}$$

The above equation is summarized by the addition theorem:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= A\omega^2\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du+\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- A\omega^2\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du+\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- \sqrt{2}A\omega^2\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du+2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\ &= A\omega^2\cos\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)-A\omega^2\sin\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &- \sqrt{2}A\omega^2\cos\left(3\int_0^{\omega t+\phi} cleaf_2(u)du\right)\end{aligned} \tag{V.5}$$

The above equation can be transformed as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= 4\left(\frac{\omega}{A}\right)^2\left\{A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+4\left(\frac{\omega}{A}\right)^2\left\{A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3\\ &-4\sqrt{2}\left(\frac{\omega}{A}\right)^2\left\{A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^3-3A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\ &-3A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+3\sqrt{2}\omega^2A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\end{aligned} \tag{V.6}$$

Here, the following equation is derived from Eq. (66) given in Shinohara [Shinohara (2015)].

$$(sleaf_2(\omega t+\phi))^2+(cleaf_2(\omega t+\phi))^2+(sleaf_2(\omega t+\phi))^2(cleaf_2(\omega t+\phi))^2=1 \tag{V.7}$$

The following equation is obtained from the above equation:

$$\begin{aligned}&\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\&+\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)=1\end{aligned} \tag{V.8}$$

$$\begin{aligned}&\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)+2\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2-1\\&+\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\left[2\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2-1\right]=1\end{aligned} \tag{V.9}$$

$$\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2\left\{1+\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}=1 \tag{V.10}$$

$$\begin{aligned}&\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2\cdot\\&\left[\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2+\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2+2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right]=1\end{aligned} \tag{V.11}$$

$$\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2\cdot\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2=1 \tag{V.12}$$

As shown in Fig. 6, the following inequalities are satisfied.

$$-\frac{\pi}{4}\le\int_0^t cleaf_2(u)du\le\frac{\pi}{4} \tag{V.13}$$

$$0\le\int_0^t sleaf_2(u)du\le\frac{\pi}{2} \tag{V.14}$$

Therefore, the following equation is obtained from Eq. (V.12).

$$\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}=1 \tag{V.15}$$

Here, the following is expanded as

$$\left\{A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$=A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+A^3\cdot 3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2\cdot\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$+A^3\cdot 3\cdot\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2+A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3 \quad \text{(V.16)}$$
$$=A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$+3A^3\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)-3A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$
$$+3A^3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)-3A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3$$

Next, $x(t)^3$ is expanded as follows:

$$\{x(t)\}^3=A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+2\sqrt{2}A^3\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^3$$
$$+3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^2$$
$$+3\left(\sqrt{2}\right)^2A^3\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^2\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}$$
(V.17)

Substituting Eq. (V.15) into the above equations, the following equation is obtained:

$$\{x(t)\}^3=A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+3A^3\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$-3A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+3A^3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)-3A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3 \quad \text{(V.18)}$$
$$+2\sqrt{2}A^3\left(\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right)^3+3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+3\sqrt{2}A^3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$+6A^3\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)$$

The following equation is obtained from the above equation:

$$\{x(t)\}^3=-2A^3\left\{\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3-2A^3\left\{\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\right\}^3+2\sqrt{2}A^3\left\{\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\right\}^3$$
$$+3A^3\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+3A^3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)$$
$$+3\sqrt{2}A^3\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)+6A^3\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)$$
(V.19)

Eq. (3.19) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2} - 3\omega^2\left(1+2\sqrt{2}\right)x(t) + 2\frac{\omega^2}{A^2}x(t)^3 = 0 \tag{V.20}$$

The initial conditions of Eqs. (3.20) and (3.21) are given as follows:

$$x(0) = A\cos\left(\int_0^{\phi} sleaf_2(u)du\right) + A\sin\left(\int_0^{\phi} sleaf_2(u)du\right) + \sqrt{2}A\cos\left(\int_0^{\phi} cleaf_2(u)du\right)$$
$$= \sqrt{2}A\left\{\sin\left(\int_0^{\phi} sleaf_2(u)du + \frac{\pi}{4}\right) + \cos\left(\int_0^{\phi} cleaf_2(u)du\right)\right\} \tag{V.21}$$

$$\frac{dx(0)}{dt} = -A\sin\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi)$$
$$+ A\cos\left(\int_0^{\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi)$$
$$-\sqrt{2}A\sin\left(\int_0^{\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\phi)$$
$$= \sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^{\phi} sleaf_2(u)du + \frac{\pi}{4}\right) - \sqrt{2}A\sin\left(\int_0^{\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\phi)$$
$$= \sqrt{2}A\cdot\omega\cdot\left\{\cos\left(\int_0^{\phi} sleaf_2(u)du + \frac{\pi}{4}\right)\cdot sleaf_2(\phi) - \sin\left(\int_0^{\phi} cleaf_2(u)du\right)\cdot cleaf_2(\phi)\right\} \tag{V.22}$$

Appendix VI

The type (VI) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) + A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right) - \sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right) \tag{VI.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\frac{dx(t)}{dt} = -A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi)$$
$$+ A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\omega t+\phi) \tag{VI.2}$$
$$+\sqrt{2}A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\omega t+\phi)$$

With the above equation, the second-order differential is obtained as follows:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2\\
&- A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}\\
&- A\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\left(sleaf_2(\omega t+\phi)\right)^2\\
&+ A\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(sleaf_2(\omega t+\phi)\right)^4}\\
&+ \sqrt{2}A\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\left(cleaf_2(\omega t+\phi)\right)^2\\
&- \sqrt{2}A\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\omega^2\cdot\sqrt{1-\left(cleaf_2(\omega t+\phi)\right)^4}\end{aligned} \tag{VI.3}$$

The following equation is derived from the above equation:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= -A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&- A\omega^2\sin\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&+ A\omega^2\cos\left(\int_0^{\omega t+\phi} sleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&+ \sqrt{2}A\omega^2\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\cos\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\
&- \sqrt{2}A\omega^2\sin\left(\int_0^{\omega t+\phi} cleaf_2(u)du\right)\cdot\sin\left(2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\end{aligned} \tag{VI.4}$$

The above equation is summarized by the addition theorem:

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= A\omega^2\cos\left(2\int_0^{\omega t+\phi} sleaf_2(u)du+\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&- A\omega^2\sin\left(2\int_0^{\omega t+\phi} sleaf_2(u)du+\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&+ \sqrt{2}A\omega^2\cos\left(\int_0^{\omega t+\phi} cleaf_2(u)du+2\int_0^{\omega t+\phi} cleaf_2(u)du\right)\\
&= A\omega^2\cos\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)-A\omega^2\sin\left(3\int_0^{\omega t+\phi} sleaf_2(u)du\right)\\
&+ \sqrt{2}A\omega^2\cos\left(3\int_0^{\omega t+\phi} cleaf_2(u)du\right)\end{aligned} \tag{VI.5}$$

The above equation can be transformed as follows:

$$\frac{d^2x(t)}{dt^2}=4\left(\frac{\omega}{A}\right)^2\left\{A\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3+4\left(\frac{\omega}{A}\right)^2\left\{A\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3$$
$$+4\sqrt{2}\left(\frac{\omega}{A}\right)^2\left\{A\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\right\}^3-3A\omega^2\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right) \tag{VI.6}$$
$$-3A\omega^2\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)-3\sqrt{2}\omega^2A\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)$$

Next, $x(t)^3$ is expanded as follows:

$$\{x(t)\}^3=A^3\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3-2\sqrt{2}A^3\left\{\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\right\}^3$$
$$-3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^2$$
$$+3\left(\sqrt{2}\right)^2A^3\left\{\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\right\}^2\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}$$
$$\tag{VI.7}$$

Substituting Eq. (V.15) into the above equations, the following equation is obtained:

$$\{x(t)\}^3=A^3\left\{\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3+A^3\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3+3A^3\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)$$
$$-3A^3\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3+3A^3\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)-3A^3\left\{\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3 \tag{VI.8}$$
$$-2\sqrt{2}A^3\left\{\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\right\}^3-3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)-3\sqrt{2}A^3\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)$$
$$+6A^3\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)$$

The following equation is obtained from the above equation:

$$\{x(t)\}^3=-2A^3\left\{\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3-2A^3\left\{\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)\right\}^3-2\sqrt{2}A^3\left\{\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)\right\}^3$$
$$+3A^3\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+3A^3\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)-3\sqrt{2}A^3\cos\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)$$
$$-3\sqrt{2}A^3\sin\left(\int_0^{\omega t+\phi}sleaf_2(u)du\right)+6A^3\cos\left(\int_0^{\omega t+\phi}cleaf_2(u)du\right)$$
$$\tag{VI.9}$$

Eq. (3.23) is obtained from the above equation.

$$\frac{d^2x(t)}{dt^2}+3\omega^2\left(2\sqrt{2}-1\right)x(t)+2\frac{\omega^2}{A^2}x(t)^3=0 \tag{VI.10}$$

The initial conditions of Eqs. (3.24) and (3.25) is expressed as follows:

$$\begin{aligned} x(0) &= A\cos\left(\int_0^\phi sleaf_2(u)du\right) + A\sin\left(\int_0^\phi sleaf_2(u)du\right) - \sqrt{2}A\cos\left(\int_0^\phi cleaf_2(u)du\right) \\ &= \sqrt{2}A\left\{\sin\left(\int_0^\phi sleaf_2(u)du + \frac{\pi}{4}\right) - \cos\left(\int_0^\phi cleaf_2(u)du\right)\right\} \end{aligned} \tag{VI.11}$$

$$\begin{aligned} \frac{dx(0)}{dt} &= -A\sin\left(\int_0^\phi sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi) \\ &+ A\cos\left(\int_0^\phi sleaf_2(u)du\right)\cdot\omega\cdot sleaf_2(\phi) \\ &+ \sqrt{2}A\sin\left(\int_0^\phi cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\phi) \\ &= \sqrt{2}A\cdot\omega\cdot sleaf_2(\phi)\cdot\cos\left(\int_0^\phi sleaf_2(u)du + \frac{\pi}{4}\right) + \sqrt{2}A\sin\left(\int_0^\phi cleaf_2(u)du\right)\cdot\omega\cdot cleaf_2(\phi) \\ &= \sqrt{2}A\cdot\omega\cdot\left\{\cos\left(\int_0^\phi sleaf_2(u)du + \frac{\pi}{4}\right)\cdot sleaf_2(\phi) + \sin\left(\int_0^\phi cleaf_2(u)du\right)\cdot cleaf_2(\phi)\right\} \end{aligned} \tag{VI.12}$$

Appendix VII

The type (VII) exact solution satisfies the cubic Duffing equation in Eq. (3.1). The exact solution is given as follows:

$$x(t) = A\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi) \tag{VII.1}$$

With the above equation, the first-order differential is obtained as follows:

$$\begin{aligned} \frac{dx(t)}{dt} &= A\omega\cdot cleaf_2(\omega\cdot t+\phi)\cdot\sqrt{1-(sleaf_2(\omega\cdot t+\phi))^4} \\ &- A\omega\cdot sleaf_2(\omega\cdot t+\phi)\sqrt{1-(cleaf_2(\omega\cdot t+\phi))^4} \end{aligned} \tag{VII.2}$$

With the above equation, the second-order differential is obtained as follows:

$$\begin{aligned} \frac{d^2x(t)}{dt^2} &= -A\omega^2\cdot\sqrt{1-(cleaf_2(\omega\cdot t+\phi))^4}\cdot\sqrt{1-(sleaf_2(\omega\cdot t+\phi))^4} \\ &+ A\omega^2\cdot cleaf_2(\omega\cdot t+\phi)\cdot\left\{-2(sleaf_2(\omega\cdot t+\phi))^3\right\} \\ &- A\omega^2\cdot\sqrt{1-(sleaf_2(\omega\cdot t+\phi))^4}\sqrt{1-(cleaf_2(\omega\cdot t+\phi))^4} \\ &- A\omega^2\cdot sleaf_2(\omega\cdot t+\phi)\left\{2(cleaf_2(\omega\cdot t+\phi))^3\right\} \\ &= -2A\omega^2\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)\left\{(cleaf_2(\omega\cdot t+\phi))^2 + (sleaf_2(\omega\cdot t+\phi))^2\right\} \\ &- 2A\omega^2\cdot\sqrt{1-(cleaf_2(\omega\cdot t+\phi))^4}\cdot\sqrt{1-(sleaf_2(\omega\cdot t+\phi))^4} \end{aligned} \tag{VII.3}$$

The following equation is obtained from Eq. (V.7).

$$\sqrt{1-\left(sleaf_2(\omega\cdot t+\phi)\right)^4}=\sqrt{1-\left\{\frac{1-\left(cleaf_2(\omega\cdot t+\phi)\right)^2}{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2}\right\}^2}$$
$$=\sqrt{\frac{\left\{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2\right\}^2-\left\{1-\left(cleaf_2(\omega\cdot t+\phi)\right)^2\right\}^2}{\left\{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2\right\}^2}} \tag{VII.4}$$
$$=\sqrt{\frac{4\left(cleaf_2(\omega\cdot t+\phi)\right)^2}{\left\{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2\right\}^2}}=\frac{2cleaf_2(\omega\cdot t+\phi)}{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2}$$

Similarly, the following equation is derived.

$$\sqrt{1-\left(cleaf_2(\omega\cdot t+\phi)\right)^4}=\frac{2sleaf_2(\omega\cdot t+\phi)}{1+\left(sleaf_2(\omega\cdot t+\phi)\right)^2} \tag{VII.5}$$

The following equation is derived by using these above equations.

$$\sqrt{1-\left(cleaf_2(\omega\cdot t+\phi)\right)^4}\cdot\sqrt{1-\left(sleaf_2(\omega\cdot t+\phi)\right)^4}=\frac{2sleaf_2(\omega\cdot t+\phi)}{1+\left(sleaf_2(\omega\cdot t+\phi)\right)^2}\frac{2cleaf_2(\omega\cdot t+\phi)}{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2}$$
$$=\frac{4sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)}{1+\left(cleaf_2(\omega\cdot t+\phi)\right)^2+\left(sleaf_2(\omega\cdot t+\phi)\right)^2+\left(cleaf_2(\omega\cdot t+\phi)\right)^2\left(sleaf_2(\omega\cdot t+\phi)\right)^2} \tag{VII.6}$$
$$=2sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)$$

The following equation is derived by using these above equations.

$$\frac{d^2x(t)}{dt^2}=-2A\omega^2\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)\left\{1-\left(cleaf_2(\omega\cdot t+\phi)\right)^2\left(sleaf_2(\omega\cdot t+\phi)\right)^2\right\}$$
$$-4A\omega^2\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi) \tag{VII.7}$$
$$=-6A\omega^2\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)+2A\omega^2\cdot\left(cleaf_2(\omega\cdot t+\phi)\right)^3\left(sleaf_2(\omega\cdot t+\phi)\right)^3$$
$$=-6\omega^2\cdot\left\{A\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)\right\}+2\frac{\omega^2}{A^2}\cdot\left\{A\cdot sleaf_2(\omega\cdot t+\phi)\cdot cleaf_2(\omega\cdot t+\phi)\right\}^3$$

The above equation is transformed as follows:

$$\frac{d^2x(t)}{dt^2}+6\omega^2x(t)-2\left(\frac{\omega}{A}\right)^2x(t)^3=0 \tag{VII.8}$$

The coefficients α and β in Eq. (3.1) are expressed as follows:

$$\alpha=6\omega^2 \tag{VII.9}$$

$$\beta=-2\left(\frac{\omega}{A}\right)^2 \tag{VII.10}$$

The initial conditions of Eqs. (3.28) and (3.29) are given as follows:

$$x(0)=A\cdot sleaf_2(\phi)\cdot cleaf_2(\phi) \tag{VII.11}$$

$$\begin{aligned}\frac{dx(0)}{dt} &= A\omega \cdot cleaf_2(\phi)\cdot\sqrt{1-(sleaf_2(\phi))^4} - A\omega\cdot sleaf_2(\phi)\sqrt{1-(cleaf_2(\phi))^4}\\ &= A\omega\cdot cleaf_2(\phi)\cdot\frac{2cleaf_2(\phi)}{1+(cleaf_2(\phi))^2} - A\omega\cdot sleaf_2(\phi)\frac{2sleaf_2(\phi)}{1+(sleaf_2(\phi))^2}\\ &= A\omega\cdot(cleaf_2(\phi))^2\cdot\left(1+(sleaf_2(\phi))^2\right) - A\omega\cdot(sleaf_2(\phi))^2\left(1+(cleaf_2(\phi))^2\right)\\ &= A\omega\left\{(cleaf_2(\phi))^2-(sleaf_2(\phi))^2\right\}\end{aligned} \tag{VII.12}$$

Appendix A

We discuss the following derivative with respect to variable *u*.

$$\frac{d}{du}\arccos(cleaf_n(u))^n = -\frac{1}{\sqrt{1-(cleaf_n(u))^{2n}}}n(cleaf_n(u))^{n-1}\left\{-\sqrt{1-(cleaf_n(u))^{2n}}\right\} = n(cleaf_n(u))^{n-1} \tag{A.1}$$

The above equation is integrated from *0* to variable *t*.

$$\left[\arccos(cleaf_n(u))^n\right]_0^t = \int_0^t n(cleaf_n(u))^{n-1}du \tag{A.2}$$

$$\begin{aligned}\left[\arccos(cleaf_n(u))^n\right]_0^t &= \arccos\ (cleaf_n(t))^n - \arccos\ (cleaf_n(0))^n\\ &= \arccos\ (cleaf_n(t))^n - \arccos(1) = \arccos\ (cleaf_n(t))^n\end{aligned} \tag{A.3}$$

Therefore, the following relation is obtained:

$$(cleaf_n(t))^n = \cos\left(n\int_0^t (cleaf_n(u))^{n-1}du\right) \tag{A.4}$$

In case of *n=2*, the above equation becomes

$$(cleaf_2(t))^2 = \cos\left(2\int_0^t cleaf_2(u)du\right) \tag{A.5}$$

Substituting $t=\frac{\pi_2}{2}(2m-1)$ into the above equation, we get

$$\cos\left(2\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du\right) = \left(cleaf_2\left(\frac{\pi_2}{2}(2m-1)\right)\right)^2 = 0 \tag{A.6}$$

Eq. (2.9) is applied to the above equation. As shown in Fig. 6, the range of $\int_0^t cleaf_2(u)du$ is as follows:

$$-\frac{\pi}{4} \le \int_0^t cleaf_2(u)du \le \frac{\pi}{4} \tag{A.7}$$

Therefore, the following equation based on the relation (A.6) can be obtained:

$$2\int_0^{\frac{\pi_2}{2}(2m-1)} cleaf_2(u)du = \pm\frac{\pi}{2} \tag{A.8}$$

Eq. (4.1) can be obtained. Next, substituting $t=m\pi_2$ into Eq. (A.5), we get

$$\cos\left(2\int_0^{m\pi_2} cleaf_2(t)dt\right) = (cleaf_2(m\pi_2))^2 = (\pm 1)^2 = 1 \tag{A.9}$$

Eq. (2.10) or (2.11) is applied to the above equation. For the inequality (A.7) to be satisfied, the integral $\int_0^{m\pi_2} cleaf_2(t)dt$ should be zero.

Appendix B

The relation between functions $sleaf_2(t)$ and $cleaf_2(t)$ is given by Eq. (66) in Shinohara [Shinohara (2015)]):

$$(sleaf_2(t))^2 + (cleaf_2(t))^2 + (sleaf_2(t))^2 (cleaf_2(t))^2 = 1 \tag{B.1}$$