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Numerical Study for a Class of Variable Order Fractional Integral-differential Equation in Terms of Bernstein Polynomials

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Abstract: The aim of this paper is to seek the numerical solution of a class of variable order fractional integral-differential equation in terms of Bernstein polynomials. The fractional derivative is described in the Caputo sense. Four kinds of operational matrixes of Bernstein polynomials are introduced and are utilized to reduce the initial equation to the solution of algebraic equations after dispersing the variable. By solving the algebraic equations, the numerical solutions are acquired. The method in general is easy to implement and yields good results. Numerical examples are provided to demonstrate the validity and applicability of the method.

Keywords: Bernstein polynomials, variable order fractional integral-differential equation, operational matrix, numerical solution, convergence analysis, the absolute error.

1 Introduction

In recent years, fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. One of the main advantages of the fractional calculus is that the fractional derivatives provide a superior approach for the description of memory and hereditary properties of various materials and processes [Galue, Kalla and Al-Saqabi (2007)]. Many numerical methods using different kinds of fractional derivative operators for solving different types of fractional differential equations have been proposed. The most commonly used ones are Adomian decomposition method (ADM) [EI-Sayed (1998)], Variational iteration method (VIM) [Odibat (2010)], Generalized differential transform method (GDTM) [Momani and Odibat (2007)], Generalized block pulse operational matrix method [Li and Sun. (2011)] and wavelet method [Yi and Chen (2012)] and so

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on.

Recently, more and more researchers are finding that numerous important dynamical problems exhibit fractional order behavior which may vary with space and time. This fact illustrates that variable order calculus provides an effective mathematical framework for the description of complex dynamical problems. The concept of a variable order operator is a much more recent development, which is a new orientation in science. Different authors have proposed different definitions of variable order differential operators, each of these with a specific meaning to suit desired goals. The variable order operator definitions recently proposed in the science include the Riemann-Liouvile definition, Caputo definition, Marchaud definition, Coimbra definition and Grünwald definition [Lorenzo and Hartley (2007); Coimbra (2003); Samko and Ross (1993); Samko. (1995); Soon, Coimbra and Kobayashi (2003)].

Since the kernel of the variable order operators is too complex for having a variable-exponent, the numerical solutions of variable order fractional differential equations are quite difficult to obtain, and have not attracted much attention. To the best of the authors' knowledge, there are few references appeared on the discussion the numerical of variable order fractional differential equation. Coimbra [Coimbra (2003)] employed a consistent approximation with first-order accurate for the solution of variable order differential equations. Soon et al. [Soon, Coimbra and Kobayashi (2003)] proposed a second-order Runge–Kutta method which is consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation. Lin et al. [Lin, Liu, and Anh (2009)] studied the stability and the convergence of an explicit finite-difference approximation for the variable-order fractional diffusion equation with a nonlinear source term. Zhuang et al. [Zhuang, Liu and Anh (2009)] obtained explicit and implicit Euler approximations for the fractional advection–diffusion nonlinear equation of variable-order. Aiming a variable-order anomalous subdiffusion equation, Chen et al. [Chen, Liu and Anh (2010)] employed two numerical schemes one fourth order spatial accuracy and with first order temporal accuracy, the other with fourth order spatial accuracy and second order temporal accuracy. However, as far as we know, no one had attempted to seek the numerical solution of the variable order fractional differential equations.

So in this paper, we introduce the Bernstein polynomials to seek the numerical solution of the variable order fractional equation. With the simple structure and perfect properties [Yousefi, Behroozifar and Dehghan (2011)], the Bernstein polynomials play an important role in various areas of mathematics and engineering. Those polynomials have been widely used in the solution of integral equations and differential equations [Chen, Yi and Chen (2011)].

In this paper, our study focuses on a class of variable order fractional integral-differential equation as follows:

$$D^{\alpha(t)}[u(x,t)\,g(x,t)]+\frac{\partial u(x,t)}{\partial t}+\int_0^t u(x,T)\,dT+\int_0^t u(x,T)k(x,T)\,dT=f(x,t) \tag{1}$$

Subject to the initial conditions

$$\begin{aligned} u(x,0)&=g(x) \quad x\in[0,1]\\ u(0,t)&=h(t) \quad\ t\in[0,1] \end{aligned} \tag{2}$$

where $D^{\alpha(x)}(u(x,t)\,g(x,t))$, $0<\alpha(x)\le 1$ is fractional derivative in Caputo sense, when $g(x,t)=u(x,t)$, the initial problem is changed to nonlinear equation. Among $f(x,t),g(x,t),u(x,t),k(x,t)$ are assumed to be casual functions of time and space on the section $(x,t)\in[0,1]\times[0,1]$, where $f(x,t),g(x,t),k(x,t)$ are known while $u(x,t)$ is the unknown.

The reminder of the paper is organized as follows: Sections 2 and 3 are preparative. In Section 4, four kinds of operational matrixes with Bernstein polynomials are derived and we applied the operational matrixes to solve the equation as given at beginning. In Section 5, we present some numerical examples to illustrative the method and to demonstrate efficiency of the method. We end the paper with a few concluding remarks in Section 6.

2 Basic definitions and properties of the variable order fractional derivatives

In this section, we firstly provide some basic definitions and properties of the variable order fractional order derivatives [Lorenzo and Hartley (2007); Coimbra (2003)].

Definition 2.1: Captuo's fractional derivate with order $\alpha(t)$, $(0<\alpha(t)\le 1)$

$$D^{\alpha(t)}u(t)=\frac{1}{\Gamma(1-\alpha(t))}\int_{0+}^t (t-\tau)^{-\alpha(t)}u'(\tau)\,d\tau \quad (0<\alpha(t)<1) \tag{3}$$

With the definition above, we can get the following formula $(0<\alpha(t)\le 1)$:

$$D_*^{\alpha(t)}c=0 \tag{4}$$

$$D_*^{\alpha(t)}x^\beta=\begin{cases} 0 & \beta=0\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(t))}x^{\beta-a(t)} & \beta=1,2,3\cdots \end{cases} \tag{5}$$

3 Bernstein polynomials and their properties

3.1 The definition of Bernstein polynomials basis

The Bernstein polynomials of degree n are defined by

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} \tag{6}$$

By using the binomial expansion of $(1-x)^{n-i}$, Eq. (6) can be expressed as:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i} = \sum_{k=0}^{n-i} (-1)^k \binom{n}{i} \binom{n-i}{k} x^{i+k} \tag{7}$$

Now, we define:

$$\boldsymbol{\Phi}(x) = [B_{0,n}(x), B_{1,n}(x), \cdots, B_{n,n}(x)]^T \tag{8}$$

where we can have

$$\boldsymbol{\Phi}(x) = \boldsymbol{A}\boldsymbol{T}_n(x) \tag{9}$$

where

$$\boldsymbol{A} = \begin{bmatrix} (-1)^0\binom{n}{0} & (-1)^1\binom{n}{0}\binom{n-0}{1} & \cdots & (-1)^{n-0}\binom{n}{0}\binom{n-0}{n-0} \\ 0 & (-1)^0\binom{n}{1}\binom{n-1}{0} & \cdots & (-1)^{n-1}\binom{n}{1}\binom{n-1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (-1)^0\binom{n}{n} \end{bmatrix} \tag{10}$$

$$\boldsymbol{T}_n(x) = \left[1, x, x^2, \cdots, x^n\right]^T \tag{11}$$

Clearly

$$\boldsymbol{T}_n(x) = \boldsymbol{A}^{-1}\boldsymbol{\Phi}(x) \tag{12}$$

3.2 Function approximation

A function $f(x) \in L^2(0,1)$ can be expressed in terms of the Bernstein Polynomials basis. In practice, only the first $(n+1)$ terms of Bernstein Polynomials are considered. Hence

$$f(x) \simeq \sum_{i=0}^{n} c_i B_{i,n}(x) = c^T \boldsymbol{\Phi}(x) \tag{13}$$

where $\boldsymbol{c} = [c_0, c_1, \cdots, c_n]^T$.
Then we have

$$\boldsymbol{c} = \boldsymbol{Q}^{-1}(f, \boldsymbol{\Phi}(x)) \tag{14}$$

where $\boldsymbol{Q}$ is an $(n+1)\times(n+1)$ matrix, which is called the dual matrix of $\boldsymbol{\Phi}(x)$.

$$\begin{aligned}\boldsymbol{Q} &= \int_0^1 \boldsymbol{\Phi}(x)\boldsymbol{\Phi}^T(x)\,dx \\ &= \boldsymbol{A}\left(\int_0^1 \boldsymbol{T}_n(x)\boldsymbol{T}_n{}^T(x)\,dx\right)\boldsymbol{A}^T \\ &= \boldsymbol{AHA}^T\end{aligned} \tag{15}$$

where $\boldsymbol{H}$ is the Hilbert matrix:

$$\boldsymbol{H} = \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n+1} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1} & \frac{1}{n+2} & \cdots & \frac{1}{2n+1} \end{bmatrix} \tag{16}$$

We can also approximate the function $u(x,t) \in L^2([0,1]\times[0,1])$as following:

$$u(x,t) \simeq \sum_{i=0}^{n}\sum_{j=0}^{n} u_{i,j} B_{i,n}(x) B_{j,n}(t) = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(t) \tag{17}$$

where

$$\boldsymbol{U} = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ u_{10} & u_{11} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n0} & u_{n1} & \cdots & u_{nn} \end{bmatrix} \tag{18}$$

And $\boldsymbol{U}$ can be obtained as following:

$$\boldsymbol{U} = \boldsymbol{Q}^{-1}(\boldsymbol{\Phi}(x), (\boldsymbol{\Phi}(t), u(x,t)))\boldsymbol{Q}^{-1} \tag{19}$$

3.3 Convergence analysis

Suppose that the function $f : [0,1] \to \mathbb{R}$ is $m+1$ times continuously differentiable,$f \in C^{m+1}[0,1]$, and $\mathbb{Y} = Span\{B_{0,n}, B_{1,n}, B_{2,n} \cdots, B_{n,n}\}$. If $\boldsymbol{c}^T\boldsymbol{\Phi}(x)$

is the best approximation of f out of $\mathbb{Y}$, then the mean error bound is presented as follows:

$$\left\|f-\boldsymbol{c}^T\boldsymbol{\Phi}\right\|_2 \leq \frac{\sqrt{2}MS^{\frac{2m+3}{2}}}{(m+1)!\sqrt{2m+3}} \tag{20}$$

where $M=\max_{x\in[0,1]}\left|f^{(m+1)}(x)\right|$, $S=\max\{1-x_0,x_0\}$.

Proof: We consider the Taylor polynomials

$$f_1(x)=f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2}+\cdots+f^{(m)}(x_0)\frac{(x-x_0)^m}{m!}$$

which we know

$$|f(x)-f_1(x)|=\left|f^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!} \quad \exists\varepsilon\in(0,1)$$

Since $\boldsymbol{c}^T\boldsymbol{\Phi}(x)$ is the best approximation of f, so we have

$$\begin{aligned}\left\|f-c^T\Phi\right\|_2^2 &\leq \|f-f_1\|_2^2=\int_0^1 (f(x)-f_1(x))^2dx\\ &=\int_0^1\left(\left|f^{(m+1)}(\varepsilon)\right|\frac{(x-x_0)^{m+1}}{(m+1)!}\right)^2dx\\ &\leq\frac{2M^2S^{2m+3}}{[(m+1)!]^2(2m+3)}\end{aligned}$$

4 The operational matrix in terms of Bernstein polynomials

4.1 The operational matrix of the section as $\frac{\partial u(x,t)}{\partial t}$ in terms of Bernstein polynomials

The differentiation of vector $\boldsymbol{\Phi}(t)$in Eq. (9) can be expressed as:

$$\boldsymbol{\Phi}'(t)=\boldsymbol{D\Phi}(t) \tag{21}$$

where $\boldsymbol{D}$ is the $(n+1)\times(n+1)$ operational matrix of derivatives for Bernstein polynomials. Form Eq. (9) we have

$$\boldsymbol{\Phi}'(t)=\boldsymbol{A}\begin{bmatrix}0\\1\\\vdots\\nt^{n-1}\end{bmatrix} \tag{22}$$

Define the $(n+1)\times(n)$ matrix $\boldsymbol{V}_{(n+1)\times n}$ and vector $\boldsymbol{T}_n^*(t)$ as:

$$\boldsymbol{V}_{(n+1)\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{bmatrix}, \quad \boldsymbol{T}_n^*(x) = \begin{bmatrix} 1 \\ t \\ \vdots \\ t^{n-1} \end{bmatrix}_{(n\times 1)} \tag{23}$$

Eq. (22) may then be restated as

$$\boldsymbol{\Phi}'(t) = \boldsymbol{A}\boldsymbol{V}_{(n+1)\times n}\boldsymbol{T}_n^*(t) \tag{24}$$

We now expand vector $\boldsymbol{T}_n^*(t)$ in terms of $\boldsymbol{\Phi}(t)$. Form Eq. (12), then we get

$$\boldsymbol{T}_n^*(t) = \boldsymbol{B}^*\boldsymbol{\Phi}(t) \tag{25}$$

where

$$\boldsymbol{B}^* = \begin{bmatrix} \boldsymbol{A}_{[1]}^{-1} \\ \boldsymbol{A}_{[2]}^{-1} \\ \vdots \\ \boldsymbol{A}_{[n]}^{-1} \end{bmatrix} \tag{26}$$

$\boldsymbol{A}_{[k]}^{-1}$ is kth row of $\boldsymbol{A}^{-1}$, $k = 1,2,\cdots,n$.
Then we have

$$\boldsymbol{\Phi}'(t) = \boldsymbol{A}\boldsymbol{V}_{(n+1)\times n}\boldsymbol{B}^*\boldsymbol{\Phi}(t) \tag{27}$$

Therefore we get the operational matrix of the section as $\frac{\partial u(x,t)}{\partial t}$ as follows:

$$\frac{\partial u(x,t)}{\partial t} = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}'(t) = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\boldsymbol{V}_{(n+1)\times n}\boldsymbol{B}^*\boldsymbol{\Phi}(t) \tag{28}$$

4.2 *The operational matrix of the section as $D^{\alpha(t)}(u(x,t)\,g(x,t))$ in terms of Bernstein polynomials*

If we approximate the function $u(x,t), g(x,t)$ with Bernstein polynomials, it can be written as $u(x,t) = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(t)$, $g(x,t) = \boldsymbol{\Phi}^T(x)\boldsymbol{G}\boldsymbol{\Phi}(t)$, where $\boldsymbol{U}$ is unknown and $\boldsymbol{G}$ is known. So we have:

$$\begin{aligned} & D^{\alpha(t)}[u(x,t)\,g(x,t)] \\ & = D^{\alpha(t)}\left[\boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^T(t)\boldsymbol{G}\boldsymbol{\Phi}(x)\right] \\ & = \boldsymbol{\Phi}^T(x)\boldsymbol{U}D^{\alpha(t)}\left[\boldsymbol{\Phi}(t)\boldsymbol{\Phi}^T(t)\right]\boldsymbol{G}\boldsymbol{\Phi}(x) \\ & = \boldsymbol{\Phi}^T(x)\boldsymbol{U}D^{\alpha(t)}\left[\boldsymbol{A}\boldsymbol{T}_n^*(t)\left(\boldsymbol{A}\boldsymbol{T}_n^*(t)\right)^T\right]\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x) \\ & = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}D^{\alpha(t)}\left[\boldsymbol{T}_n^*(t)\left(\boldsymbol{T}_n^*(t)\right)^T\right]\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x) \end{aligned} \tag{29}$$

$$= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}D^{\alpha(t)}\begin{bmatrix} 1 & t & \dots & t^n \\ t & t^2 & \dots & t^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ t^n & t^{2n} & \dots & t^{2n} \end{bmatrix}\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x)$$

$$= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))}t^{2n-\alpha(t)} \end{bmatrix}\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x)$$

$$= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\boldsymbol{M}\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x)$$

Now we define

$$\boldsymbol{M} = \begin{bmatrix} 0 & \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \dots & \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} \\ \frac{\Gamma(2)}{\Gamma(2-\alpha(t))}t^{1-\alpha(t)} & \frac{\Gamma(3)}{\Gamma(3-\alpha(t))}t^{2-\alpha(t)} & \dots & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha(t))}t^{n-\alpha(t)} & \frac{\Gamma(n+2)}{\Gamma(n+2-\alpha(t))}t^{n+1-\alpha(t)} & \dots & \frac{\Gamma(2n+1)}{\Gamma(2n+1-\alpha(t))}t^{2n-\alpha(t)} \end{bmatrix} \tag{30}$$

$\boldsymbol{M}$ is called the operational matrix of the section as $D^{\alpha(t)}(u(x,t)\,g(x,t))$ with Bernstein polynomials.

So we have

$$D^{\alpha(t)}[u(x,t)\,g(x,t)] = \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\boldsymbol{M}\boldsymbol{A}^T\boldsymbol{G}\boldsymbol{\Phi}(x) \tag{31}$$

4.3 The operational matrix of the section as $\int_0^t u(x,T)\,dT$ in terms of Bernstein polynomials

The integration of the vector $\boldsymbol{\Phi}(t)$ in Eq. (9) can be expressed as:

$$\int_0^t \boldsymbol{\Phi}(t)dT = \boldsymbol{P}\boldsymbol{\Phi}(t) \tag{32}$$

Where $\boldsymbol{P}$ is the $(n+1)\times(n+1)$ operational matrix of integration for Bernstein polynomials. So we have

$$\int_0^t \boldsymbol{\Phi}(T)dT = \int_0^t \boldsymbol{A}\boldsymbol{T}_n(T)\,dT = \boldsymbol{A}\int_0^t \boldsymbol{T}_n(T)\,dT$$

$$= \boldsymbol{A}\int_0^t \begin{bmatrix} 1 \\ T \\ \vdots \\ T^n \end{bmatrix} dT = \boldsymbol{A}\begin{bmatrix} t \\ \frac{1}{2}t^2 \\ \vdots \\ \frac{1}{n+1}t^{n+1} \end{bmatrix} = \boldsymbol{A}_p\boldsymbol{T}_p \tag{33}$$

where $\boldsymbol{A}_p$ is an $(n+1)\times(n+1)$ matrix:

$$\boldsymbol{A}_p = \boldsymbol{A}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n+1} \end{bmatrix}, \quad \boldsymbol{T}_p = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^{n+1} \end{bmatrix} \tag{34}$$

where $\boldsymbol{A}_p$ is an $(n+1)\times(n+1)$ matrix:

$$\boldsymbol{A}_p = \boldsymbol{A}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{n+1} \end{bmatrix}, \quad \boldsymbol{T}_p = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^{n+1} \end{bmatrix} \tag{35}$$

Now we approximate the elements of vector $\boldsymbol{T}_p$ in terms of $\boldsymbol{\Phi}(t)$. By Eq. (12), then we have:

$$t^k = \boldsymbol{A}_{[k+1]}^{-1}\boldsymbol{\Phi}(t), \quad k = 1, \cdots, n \tag{36}$$

where $\boldsymbol{A}_{[k+1]}^{-1}$ is the $k+1\ th$ row of $\boldsymbol{A}^{-1}$ for $k = 1, \cdots, n$. We just need to approximate $t^{n+1} = \boldsymbol{c}_{n+1}^T\boldsymbol{\Phi}(t)$. By using Eq. (14), we have

$$\begin{aligned} \boldsymbol{c}_{n+1} =& \boldsymbol{Q}^{-1}\int_0^1 t^{n+1}\boldsymbol{\Phi}(t)\,dt = \boldsymbol{Q}^{-1}\begin{bmatrix} \int_0^1 t^{n+1}B_{0,n}(t)\,dt \\ \int_0^1 t^{n+1}B_{1,n}(t)\,dt \\ \vdots \\ \int_0^1 t^{n+1}B_{n,n}(t)\,dt \end{bmatrix} \\ =& \frac{\boldsymbol{Q}^{-1}}{2n+2}\begin{bmatrix} \frac{\binom{n}{0}}{\binom{2n+1}{n+1}} & \frac{\binom{n}{1}}{\binom{2n+1}{n+2}} & \cdots & \frac{\binom{n}{n}}{\binom{2n+1}{2n+1}} \end{bmatrix}^T \end{aligned} \tag{37}$$

We define

$$\boldsymbol{P}_1 = \begin{bmatrix} \boldsymbol{A}_{[2]}^{-1} \\ \boldsymbol{A}_{[3]}^{-1} \\ \vdots \\ \boldsymbol{A}_{[n+1]}^{-1} \\ \boldsymbol{c}_{n+1} \end{bmatrix} \tag{38}$$

Then we can get $\boldsymbol{T}_p = \boldsymbol{P}_1\boldsymbol{\Phi}(t)$. Therefore we have the operational matrix of integration as follows:

$$\boldsymbol{P} = \boldsymbol{A}_p\boldsymbol{P}_1 \tag{39}$$

So we have

$$\begin{aligned}
&\int_0^t u(x,T)dT \\
&= \int_0^t \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(T)dT \\
&= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}_p\boldsymbol{P}_1\boldsymbol{\Phi}(t)
\end{aligned} \tag{40}$$

4.4 The operational matrix of the section as $\int_0^t u(x,T)k(x,T)dT$ in terms of Bernstein polynomials

Firstly, we approximate the function $k(x,t)$ with Bernstein polynomials, it can be written as $k(x,t) = \boldsymbol{\Phi}^T(x)\boldsymbol{K}\boldsymbol{\Phi}(t)$, and $\boldsymbol{K}$ is known. So we have

$$\begin{aligned}
&\int_0^t u(x,T)k(x,t)\,dT \\
&= \int_0^t \left(\boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{\Phi}(T)\boldsymbol{\Phi}^T(T)\boldsymbol{K}\boldsymbol{\Phi}(x)\right)dT \\
&= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\int_0^t \left(\boldsymbol{\Phi}(T)\boldsymbol{\Phi}^T(T)\right)dT\boldsymbol{K}\boldsymbol{\Phi}(x) \\
&= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\int_0^t \begin{bmatrix} 1 & T & \cdots & T^n \\ T & T^2 & \cdots & T^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ T^n & T^{n|+1} & \cdots & T^{2n} \end{bmatrix} dT\boldsymbol{A}^T\boldsymbol{K}\boldsymbol{\Phi}(x) \\
&= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\begin{bmatrix} t & \frac{1}{2}t^2 & \cdots & \frac{1}{n+1}t^{n+1} \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 & \cdots & \frac{1}{n+2}t^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1}t^{n+1} & \frac{1}{n+2}t^{n+2} & \cdots & \frac{1}{2n+1}t^{2n+1} \end{bmatrix}\boldsymbol{A}^T\boldsymbol{K}\boldsymbol{\Phi}(x) \\
&= \boldsymbol{\Phi}^T(x)\boldsymbol{U}\boldsymbol{A}\boldsymbol{R}\boldsymbol{A}^T\boldsymbol{K}\boldsymbol{\Phi}(x)
\end{aligned} \tag{41}$$

We define

$$\boldsymbol{R} = \begin{bmatrix} t & \frac{1}{2}t^2 & \cdots & \frac{1}{n+1}t^{n+1} \\ \frac{1}{2}t^2 & \frac{1}{3}t^3 & \cdots & \frac{1}{n+2}t^{n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n+1}t^{n+1} & \frac{1}{n+2}t^{n+2} & \cdots & \frac{1}{2n+1}t^{2n+1} \end{bmatrix} \tag{42}$$

$\boldsymbol{R}$ is called the operational matrix of the section as $\int_0^t u(x,T)\,k(x,T)\,dT$ in terms of Bernstein polynomials.

So the initial equation is transformed into the products of several dependent matrixes as follows:

$$\begin{aligned}&\boldsymbol{\Phi}^T(x)\boldsymbol{UAMA}^T\boldsymbol{G\Phi}(x)+\boldsymbol{\Phi}^T(x)\boldsymbol{UP\Phi}(t)+\boldsymbol{\Phi}^T(x)\boldsymbol{UAV}_{(n+1)\times n}\boldsymbol{B}^*\boldsymbol{\Phi}(t)\\&+\boldsymbol{\Phi}^T(x)\boldsymbol{UARA}^T\boldsymbol{K\Phi}(x)=f(x,t)\end{aligned}\tag{43}$$

Dispersing Eq. (43) with (x_i,t_j), $(i=1,2,\cdots,n;j=1,2,\cdots,n)$ by using a symbolic software such as "Mathematica", we can obtain $\mathbf{U}$. So the numerical solution of the original problem is obtained ultimately.

5 Numerical examples

Example1

$$D^{\frac{t}{3}}[u(x,t)(x+t+1)]+\frac{\partial u(x,t)}{\partial t}+\int_0^t u(x,T)\,dT+\int_0^t u(x,T)(x+t)\,dT=f(x,t)$$

$$u(x,0)=x^2 \quad u(0,t)=t^2 \qquad [x,t]\in[0,1]\times[0,1]$$

where

$$f(x,t)=2t+\frac{t^3}{3}+\frac{t^4}{4}+\frac{t^3x}{3}+tx^2+\frac{t^2x^2}{2}+tx^3-\frac{3t^{1-\frac{t}{3}}\left[6t(9+8t)-6(-9+t)tx\right]}{(-9+t)(-6+t)(-3+t)\Gamma\left(1-\frac{t}{3}\right)}$$

The exact solution of the above equation is $u(x,t)=x^2+t^2$.

Taking $n=2$, dispersing $x_i=\frac{k_i}{3}-\frac{1}{6}$, $t_j=\frac{k_j}{3}-\frac{1}{6}$ $(k_i=1,2,3;k_j=1,2,3)$, we can obtain the matrix $\boldsymbol{U}$ as follows:

$$\boldsymbol{U}=\begin{bmatrix}0 & 0 & 1\\ 0 & 0.00217 & 0.99905\\ 1 & 1.00036 & 1.99974\end{bmatrix}$$

The numerical solution is $u(x,t)=\boldsymbol{\Phi}(x)\boldsymbol{U\Phi}(t)$, as the matrix $\boldsymbol{U}$ is given above, and $\boldsymbol{\Phi}(x)=\left[\begin{array}{ccc}(1-x)^2 & 2(1-x)x & x^2\end{array}\right]^T$, $\boldsymbol{\Phi}(t)=\left[\begin{array}{ccc}(1-t)^2 & 2(1-t)t & t^2\end{array}\right]^T$.

The absolute error between the exact solution and the numerical solution is displayed as **Figure 1**.

Taking $n=3$, dispersing $x_i=\frac{k_i}{4}-\frac{1}{8}$, $t_j=\frac{k_j}{4}-\frac{1}{8}$ $(k_i=1,2,3,4;k_j=1,2,3,4)$, we can obtain the matrix $\boldsymbol{U}$ as follows:

$$\boldsymbol{U}=\frac{1}{3}\begin{bmatrix}0 & 0 & 1 & 3\\ 0 & 0 & 1 & 3\\ 1 & 1 & 2 & 4\\ 3 & 3 & 4 & 6\end{bmatrix}$$

The numerical solution is $u(x,t) = \mathbf{\Phi}(x)\boldsymbol{U}\mathbf{\Phi}(t)$, as the matrix $\boldsymbol{U}$ is given above, and $\mathbf{\Phi}(x) = \left[\begin{array}{cccc} (1-x)^3 & 3(1-x)^2x & 3(1-x)x^2 & x^3 \end{array}\right]^T$, $\mathbf{\Phi}(t) = \left[\begin{array}{cccc} (1-t)^3 & 3(1-t)^2t & 3(1-t)t^2 & t^3 \end{array}\right]^T$The absolute error between the exact solution and the numerical solution is displayed as **Figure 2**.

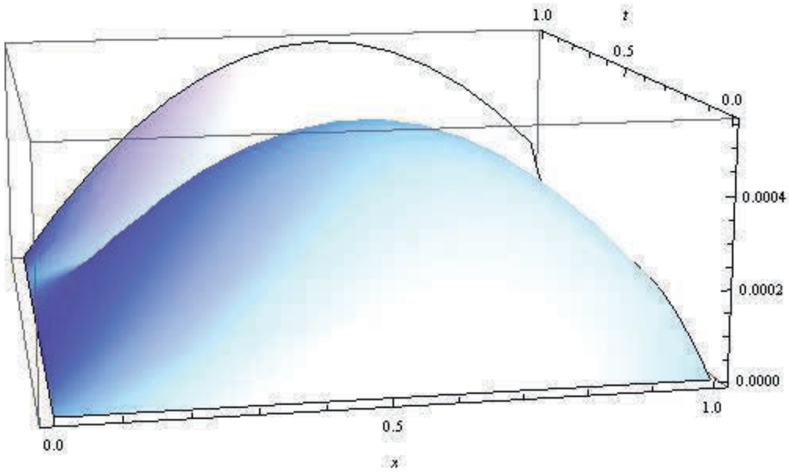

Figure 1: The absolute error when$n = 2$.

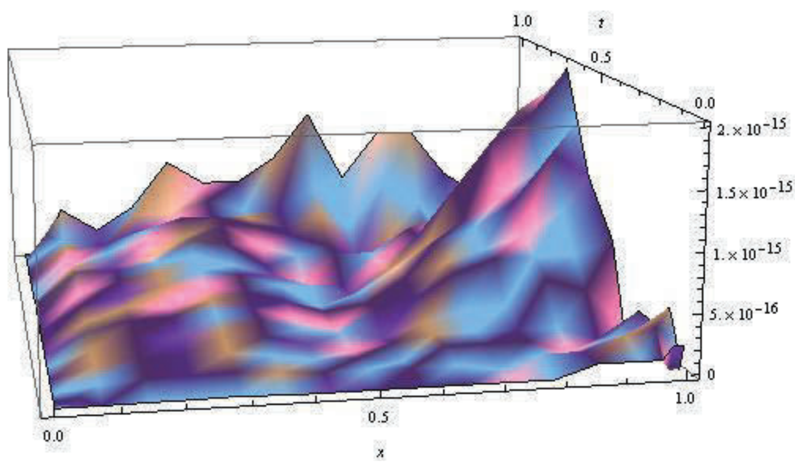

Figure 2: The absolute error when$n = 3$.

Example 2

$$D^{\frac{\sin t}{3}}\left[u(x,t)(xt)\right]+\frac{\partial u(x,t)}{\partial t}+\int_0^t u(x,T)\,dT+\int_0^t u(x,T)(x+t)\,dT=f(x,t)$$
$$u(x,0)=(1+x)^2 \quad u(0,t)=(1+t)^2 \quad [x,t]\in[0,1]\times[0,1]$$

Where

$$f(x,t)=2(1+x+t)+t+\frac{3t^2}{2}+t^3+\frac{t^4}{4}+3tx+3t^2x+t^3x+3tx^2+\frac{3t^2x^2}{2}+tx^3$$
$$-\frac{3t^{1-\frac{\sin t}{3}}x\left[54(1+t+x)^2+(1+x)\sin t\left(-3(5+4t+5x)+(1+x)\sin t\right)\right]}{\Gamma\left(1-\frac{\sin t}{3}\right)(-9+\sin t)(-6+\sin t)(-3+\sin t)}$$

The exact solution of the above problem is $u(x,t)=(1+x+t)^2$.
Taking $n=3$, dispersing $x_i=\frac{k_i}{4}-\frac{1}{8}$, $t_j=\frac{k_j}{4}-\frac{1}{8}$ $(i=j=1,2,\cdots,4)$, the matrix $\boldsymbol{U}$ is displayed as follows:

$$\boldsymbol{U}=\begin{bmatrix} 1 & \frac{5}{3} & \frac{8}{3} & 4 \\ \frac{5}{3} & \frac{23}{9} & \frac{34}{9} & \frac{16}{3} \\ \frac{8}{3} & \frac{34}{9} & \frac{47}{9} & 7 \\ 4 & \frac{16}{3} & 7 & 9 \end{bmatrix}$$

The absolute error between the exact solution and the numerical solution is displayed in **Table 1**.

Table 1: The absolute error when $n=3$.

	t=0.1	t=0.3	t=0.5	t=0.7	t=0.9
x=0.0	0	0	0	0	0
x=0.1	1.78E-15	2.44E-15	4.44E-16	3.11E-15	4.77E-15
x=0.2	2.22E-15	2.66E-15	1.33E-15	5.10E-15	7.33E-15
x=0.3	2.89E-15	3.99E-15	4.44E-16	4.44E-15	5.99E-15
x=0.4	1.78E-15	2.66E-15	4.44E-16	5.33E-15	5.77E-15
x=0.5	1.78E-15	2.66E-15	4.44E-16	1.33E-15	3.55E-15
x=0.6	1.34E-15	3.11E-15	8.88E-16	1.33E-15	8.88E-15
x=0.7	2.23E-15	1.78E-15	1.78E-16	3.11E-15	2.66E-15
x=0.8	8.89E-16	2.67E-15	0	2.22E-15	4.44E-16
x=0.9	4.45E-16	1.78E-15	8.88E-16	4.44E-15	5.10E-15
x=1.0	1.78E-15	2.66E-15	8.88E-16	2.66E-15	5.55E-15

Taking $n=4$, dispersing $x_i=\frac{k_i}{5}-\frac{1}{10}$, $t_j=\frac{k_j}{5}-\frac{1}{10}$ $(i=1,2,\cdots,5; j=1,2,\cdots,5)$, the matrix $\boldsymbol{U}$ is displayed as follows:

$$\boldsymbol{U}=\begin{bmatrix} 1 & \frac{3}{2} & \frac{13}{6} & 3 & 4 \\ \frac{3}{2} & 2.125 & 2.91667 & \frac{31}{8} & 5 \\ \frac{13}{6} & 2.91667 & 3.83333 & 4.91667 & 6.1667 \\ 3 & 3.875 & 4.91667 & 6.125 & 7.5 \\ 4 & 5 & \frac{37}{6} & 7.5 & 9 \end{bmatrix}$$

The absolute error between the exact solution and the numerical solution is displayed as **Table 2**.

Table 2: The absolute error when $n=4$.

	t=0.1	t=0.3	t=0.5	t=0.7	t=0.9
x=0.0	0	0	0	0	0
x=0.1	4.22E-15	2.22E-16	3.55E-15	1.25E-14	1.70E-14
x=0.2	2.44E-15	4.44E-16	2.22E-15	1.14E-14	5.93E-15
x=0.3	2.22E-16	1.78E-15	3.33E-15	1.77E-15	1.23E-14
x=0.4	3.11E-15	1.78E-15	1.33E-15	1.77E-15	1.75E-14
x=0.5	2.22E-15	3.11E-15	1.77E-15	3.10E-15	1.02E-14
x=0.6	1.78E-15	2.66E-15	1.33E-15	4.88E-15	1.37E-14
x=0.7	1.78E-15	4.44E-16	3.102E-15	4.44E-15	4.23E-14
x=0.8	5.33E-15	2.66E-15	2.22E-15	1.55E-14	5.90E-14
x=0.9	7.99E-15	7.99E-15	3.10E-15	3.30E-14	4.001E-14
x=1.0	7.11E-15	1.51E-14	0	5.99E-14	5.37E-14

When $g(x,t)=u(x,t)$, the initial equation becomes nonlinear equation. **Example 3** describes the situation.

Example 3:

$$D^{\frac{t}{3}}u^2(x,t)+D^{\frac{t}{4}}u(x,t)+\frac{\partial^2 u(x,t)}{\partial t^2}=f(x,t)$$
$$u(x,0)=x^2,\quad u(0,t)=t^2\quad (x,t)\in[0,1]\times[0,1]$$

where

$$\begin{aligned} f(x,t)=&2+\frac{36t^{2-\frac{t}{3}}\left(54t^2+(-12+t)(-9+t)x^2\right)}{(-12+t)(-9+t)(-6+t)(-3+t)\Gamma\left(1-\frac{t}{3}\right)} \\ &+\frac{32t^{2-\frac{t}{4}}}{(32-12t+t^2)\Gamma\left(1-\frac{t}{4}\right)} \end{aligned}$$

The exact solution of the above equation is $u(x,t) = x^2 + t^2$

This is a nonlinear variable order fractional differential equation, the numerical solution can also be gained with the method proposed in Section 4 when $n \leq 2$.

Taking $n = 2$, dispersing $x_i = \frac{k_i}{2} - \frac{1}{4}$, $t_j = \frac{k_j}{2} - \frac{1}{4}$ $(k_i = 1,2; k_j = 1,2)$, we can obtain the matrix $\boldsymbol{U}$ as follows:

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

The

numerical solution is $u(x,t) = \boldsymbol{\Phi}(x)\boldsymbol{U}\boldsymbol{\Phi}(t)$, as the matrix $\boldsymbol{U}$ is given above, where $\boldsymbol{\Phi}(x) = \begin{bmatrix} (1-x)^2 & 2(1-x)x & x^2 \end{bmatrix}^T$, $\boldsymbol{\Phi}(t) = \begin{bmatrix} (1-t)^2 & 2(1-t)t & t^2 \end{bmatrix}^T$.

The absolute error between the exact solution and the numerical solution is displayed as **Figure 3**.

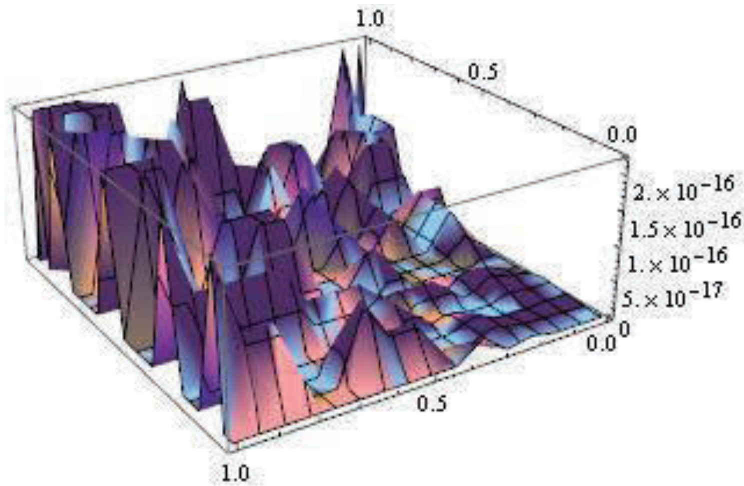

Figure 3: The absolute error for Example 3 of $n = 2$.

When $n \geq 3$, the computation is very large, getting the numerical solution is a very difficult thing.

From **Figures 1-3, Tables 1-2**, we can see that the absolute error is very tiny and only a small number of Bernstein polynomials are needed when $n \geq 3$. The calculating results also show that combining with Bernstein polynomials, the method in this paper can be effectively used in the numerical solution of the fractional equation. At the same time the feasibility of the method can be also proved. From the above results, the numerical solutions are in good agreement with the exact solution.

6 Conclusions

This article uses Bernstein polynomials method to solve a class of the variable order fractional integral-differential equation by combining Bernstein polynomials with the properties of fractional differentiation. Actually we derive four kinds of operational matrixes using Bernstein polynomials. The matrixes are used to solve the numerical solutions of a class of fractional integral-differential equations effectively. We translate the initial equation into the product of some relevant matrixes which can also be regarded as the system of linear equations after dispersing the variable. And it is easy to solve by the least square method. Numerical examples illustrate the powerful of the proposed method. The solutions obtained using the suggested method show that numerical solutions are in very good coincidence with the exact solution. The method can be applied by developing for the other fractional problem.

However, there are many issues to be resolved, such as the section $\Omega_1 = [0,1] \times [0,1]$ is transformed to $\Omega_2 = [0,X] \times [0,T]$, or the equations are nonlinear and so on. This requires the efforts of all of us.

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