 CMES, vol.90, no.5, pp.359-378, 2013

Numerical solution of nonlinear fractional integral differential equations by using the second kind Chebyshev wavelets

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Abstract: By using the differential operator matrix and the product operation matrix of the second kind Chebyshev wavelets, a class of nonlinear fractional integral-differential equations is transformed into nonlinear algebraic equations, which makes the solution process and calculation more simple. At the same time, the maximum absolute error is obtained through error analysis. It also can be used under the condition that no exact solution exists. Numerical examples verify the validity of the proposed method.

Keywords: Nonlinear fractional integral-differential equation, the second kind Chebyshev wavelet, operational matrix, numerical solution, error analysis.

1 Introduction

Wavelet analysis is a new branch of science developed in the twentieth century. The main research is how to structure a wavelet base function to approximate the given function in a specific function space. Meanwhile, the wavelet operational matrix has been sucessfully applied in optimal control [Hsiao and Wang (1999);Karimi, Moshiri, Lohmann and Maralani (2005);Sadek, Abualrub and Abukhaled (2007)], system identification [Karimi, Lohmann, Maralani and Moshiri (2004);Pawlak and Hasiewicz (1998)], system analysis [Chen and Hsiao (1997);Bujurke, Salimath and Shiralashetti (2008)], and numerical solution of integral and differential equations [Bujurke, Shiralashetti and Salimath (2009);Babolian, Masouri and Hatamzadeh-Varmazyar (2009);Kajani and Vencheh (2008);Reihani and Abadi (2007);Khellat and Yousefi (2006);Razzaghi and Yousefi (2001)].

E.Babolian, F.Fattahzadeh used the first kind Chebyshev wavelet to solve the linear integer order differential equation [Baholian and Fattahzadeh (2007)]. M.Razzaghi [Razzaghi and Yousefi (2000)] adopted Legendre wavelets for variational problems. In Ref. [Maleknejad, Tavassoli Kajani and Mahmoudi (2003)], linear Fred-

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holm and Volterra integral equation of the second kind were solved by using Legendre wavelet. Han Danfu [Han and Shang (2007)] applied CAS wavelet for the integro-differential equation. Because the nonlinear fractional integral-differential equation can be used to be better simulate the physical process of nature and the dynamic system process, so it has been widely used in engineering mechanics, physics, and other fields of science. However, a lot of engineering problems being solved by using differential equation in the past can be better solved by using integral-differential equation. In the solution process, many ordinary differential equations and the partial differential equation can be transformed into integral-differential equations to solve.

In this paper, our study focuses on the following nonlinear integral-differential equation which will be discussed by using the second kind Chebyshev wavelet differential and integral operator matrix and product operator matrix. An error estimation expression will be given through error analysis. Numerical examples will be conducted to prove that the proposed method has higher precision and efficiency.

$$\begin{aligned}\sum_{i=1}^{N} a_i(x) D^{\alpha_i} y(x) &= y^j(x) + \lambda_1 \int_0^x k_1(x,t)\,[y(t)]^p\,dt \\ &\quad + \lambda_2 \int_0^1 k_2(x,t)\,[y(t)]^q\,dt + g(x),\end{aligned}$$

$$y^{(j)}(0) = b_j, \quad 0 \le x,t \le 1 \ , \quad n-1 < \alpha_i \ \le n. \tag{1}$$

2 The second kind Chebyshev wavelet

The second kind Chebyshev wavelet $\psi_{n,m}(t) = \psi(k,n,m,t)$involves four arguments, k is assumed any positive integer, m is the degree of the second kind Chebyshev polynomials and t is the normalized time. They are defined on the interval $[0,1)$ as

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k}{2}} \widetilde{T_m}(2^k t - 2n + 1)\ , \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}}, \\ 0\ , \quad otherwise, \end{cases} \tag{2}$$

where

$$\widetilde{T_m}(t) = \sqrt{\frac{2}{\pi}} T_m(t) \tag{3}$$

and $m = 0, 1, \cdots M-1$. In Eq. (3) the coefficients are used for orthogonality. Here $\widetilde{T_m}(t)$are the second kind Chebyshev polynomials of degree$m$which respect to the weight function $\omega(t) = \sqrt{1-t^2}$on the interval$[-1,1]$and satisfy the following recursive formula

$$T_0(t) = 1, \quad T_1(t) = 2t, \quad T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \qquad m = 1,2,\cdots .$$

We should note that in dealing with the second kind Chebyshev wavelet the weight function $\widetilde{\omega}(t) = \omega(2t-1)$ have to be dilated and translated as

$$\omega_n(t) = \omega\left(2^k t - 2n + 1\right).$$

It is easy to prove $\psi_{n,m}(t)$ are the standard orthonormal wavelet basis in$L^2_{\omega_n}[0,1]$,

$$\left(\psi_{n,m}(t), \psi_{n',m'}(t)\right) = \begin{cases} 1, & (m,n) = (m',n'), \\ 0, & (m,n) \neq (m',n'). \end{cases}$$

For function $f(t)$ defined over $[0,1)$ may be expanded as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m\in Z} c_{nm}\phi_{nm}(t), \tag{4}$$

where

$$c_{nm} = (f(t), \phi_{nm}(t))_{\omega_n} = \int_0^1 \omega_n(t)\,\phi_{nm}(t)\,f(t)dt, \tag{5}$$

in which $(\cdot,\cdot)$ denotes the inner product in $L^2_{\omega_n}[0,1]$. If the infinite series in Eq. (4) is truncated, then it can be written as

$$f(t) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\phi_{nm} = C^T\psi(t) \quad , \tag{6}$$

where C and $\psi(t)$are$2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \cdots c_{1(M-1)}, c_{20}, c_{21}, \cdots c_{2(M-1)}, \cdots c_{2^{k-1}0} \cdots c_{2^{k-1}(M-1)}]^T \tag{7}$$

and

$$\psi(t) = \left[\phi_{10}, \phi_{11}, \phi_{12} \cdots \phi_{1(M-1)}, \phi_{20}, \phi_{21}, \cdots \phi_{2(M-1)}, \cdots \phi_{2^{k-1}0}, \cdots \phi_{2^{k-1},(M-1)}\right]^T. \tag{8}$$

A function$k(x,t) \in L^2_\omega([0,1]\times[0,1])$may be approximated as

$$k(x,t) = \psi^T(x)K\psi(t), \tag{9}$$

where $K = (K_{ij})$ is a $2^{k-1}M \times 2^{k-1}M$ matrix with

$$K_{ij} = (\phi_i(x), (k(x,t), \phi_j(t))). \tag{10}$$

3 The second kind Chebyshev wavelet integral operational matrix

The integration of the vector $\psi(t)$ defined in Eq. (8) can be obtained as

$$\int_0^t \psi\left(t'\right)dt' = P\psi(t), \tag{11}$$

where P is the $2^{k-1}M \times 2^{k-1}M$ operational matrix for integration as [Babollane and Fatfahzadehf (2007)]

$$P = \frac{1}{2^k}\begin{bmatrix} L & F & F & \cdots & F \\ 0 & L & F & \cdots & F \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & F \\ 0 & 0 & \cdots & \cdots & L \end{bmatrix},$$

where F and L are $M \times M$ matrices given by

$$F = \begin{bmatrix} 2 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 \\ -\frac{3}{4} & 0 & \frac{1}{4} & 0 & 0 & \cdots & 0 \\ \frac{1}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & 0 & \cdots & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{8} & 0 & -\frac{1}{8} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{M-2}\frac{1}{M-1} & 0 & 0 & 0 & -\frac{1}{2(M-1)} & 0 & \frac{1}{2(M-1)} \\ (-1)^{M-1}\frac{1}{M} & 0 & 0 & 0 & 0 & -\frac{1}{2M} & 0 \end{bmatrix}$$

The integration of two second kind Chebyshev wavelet vector product is $I = \int_0^1 \psi(t)\psi^T(t)\,dt$, where I is a unit matrix.

4 The second kind Chebyshev wavelet product operational matrix

Let

$$\psi(t)\,\psi^T(t)\,C \simeq \widetilde{C}\psi(t), \tag{12}$$

where $\widetilde{C}$ is a $2^{k-1}M \times 2^{k-1}M$ product operational matrix. To expound the derivation process of product operational matrix, we take $k=2, M=3$, then

$C = \left[c_{10}, c_{11}, c_{12}, c_{20}, c_{21}, c_{22}\right]^T$,

$\psi(t) = \left[\phi_{10}, \phi_{11}, \phi_{12}, \phi_{20}, \phi_{21}, \phi_{22}\right]^T$.

We have

$$\psi(t)\,\psi^T(t) = 2\sqrt{\frac{2}{\pi}}\begin{bmatrix} \phi_{10} & \phi_{11} & \phi_{12} & 0 & 0 & 0 \\ \phi_{11} & \phi_{10}+\phi_{12} & \phi_{11} & 0 & 0 & 0 \\ \phi_{12} & \phi_{11} & \phi_{10}+\phi_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_{20} & \phi_{21} & \phi_{22} \\ 0 & 0 & 0 & \phi_{21} & \phi_{20}+\phi_{22} & \phi_{21} \\ 0 & 0 & 0 & \phi_{22} & \phi_{21} & \phi_{20}+\phi_{22} \end{bmatrix}, \tag{13}$$

so the vector $\widetilde{C}$ is

$$\widetilde{C} = 2\sqrt{\frac{2}{\pi}}\begin{bmatrix} \widetilde{C_1} & 0 \\ 0 & \widetilde{C_2} \end{bmatrix}, \tag{14}$$

where $\widetilde{C_i}, i=1,2$ are 3×3 matrices as

$$\widetilde{C_i} = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} \\ c_{i1} & c_{i0}+c_{i2} & c_{i1} \\ c_{i2} & c_{i1} & c_{i0}+c_{i2} \end{bmatrix}. \tag{15}$$

5 The second kind Chebyshev wavelet fractional integral and differential operational matrix

In Eq. (8), taking the collocation points as following

$$t_i = \frac{2i-1}{2^k M}, \qquad i = 1,2,\cdots,2^{k-1}M\ ,$$

we define the second kind Chebyshev wavelet matrix $\psi_{m\times m}$as [Li (2010)]

$$\psi_{m\times m} \triangleq \left[\psi\left(\frac{1}{2m}\right), \psi\left(\frac{3}{2m}\right), \cdots \psi\left(\frac{2m-1}{2m}\right)\right], \tag{16}$$

where, $m = 2^{k-1}M$. For example, when $M=3$ and $k=2$, the second kind Chebyshev wavelet is express as

$$\psi_{6\times 6} = \begin{bmatrix} 1.5958 & 1.5958 & 1.5958 & 0 & 0 & 0 \\ -2.1277 & 0 & 2.1277 & 0 & 0 & 0 \\ 1.2412 & -1.5958 & 1.2412 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.5958 & 1.5958 & 1.5958 \\ 0 & 0 & 0 & -2.1277 & 0 & 2.1277 \\ 0 & 0 & 0 & 1.2412 & -1.5958 & 1.2412 \end{bmatrix}.$$

Also, we define an m-set of Block Pulse Function (BPF) [Maleknejad and Mahmoudi (2004)], the set of these functions over the interval$[0,T)$is defined as

$$\varphi_i(t)=\begin{cases}1, & (i-1)\frac{T}{m}\le t<i\frac{T}{m}\\ 0, & otherwise,\end{cases} \tag{17}$$

where $i=1,2,\cdots,m,\ \ \frac{T}{m}=h.$

The BPF have two useful properties which will be used further,

1. Disjointness

$$\varphi_i(t)\,\varphi_j(t)=\begin{cases}0, & i\neq j,\\ \varphi_i(t), & i=j.\end{cases}\qquad t\in[0,T),\quad i,j=1,2,\cdots,m\quad . \tag{18}$$

2. Orthogonality

$$\int_0^1 \varphi_i(t)\varphi_j(t)\,dt=\begin{cases}0, & i\neq j,\\ h, & i=j.\end{cases}\qquad t\in[0,T),\quad i,j=1,2,\cdots,m\quad . \tag{19}$$

Define $\varphi(t)=[\varphi_1(t),\varphi_2(t),\cdots,\varphi_m(t)]^T$, for the BPF properties, $\varphi(t)$ satisfies the relationship as follows

$$\varphi(t)\,\varphi^T(t)=\begin{bmatrix}\varphi_1(t) & 0 & \cdots & 0\\ 0 & \varphi_2(t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \varphi_m(t)\end{bmatrix}. \tag{20}$$

For $\sum\limits_{i=1}^{m}(\varphi_i(t))^2=\sum\limits_{i=1}^{m}\varphi_i(t)=1$, then

$$\varphi^T(t)\,\varphi(t)=1. \tag{21}$$

Now, we derive the second kind Chebyshev wavelet operational matrix of the fractional integration. Let

$$I^\alpha\psi_m(t)\approx P^\alpha_{m\times m}\psi_m(t),\qquad n-1<\alpha\le n\ , \tag{22}$$

where matrix $P^\alpha_{m\times m}$ is called the second kind Chebyshev wavelet operational matrix of the fractional integration.

The second kind Chebyshev wavelet may be expanded into an m-term BPF as

$$\psi_m(t) = \psi_{m\times m}\varphi(t), \tag{23}$$

for the second kind Chebyshev wavelet, we have

$$I^{\alpha}\psi_m(t) \approx I^{\alpha}\psi_{m\times m}\varphi(t). \tag{24}$$

Kilicman and Al Zhour in Ref. [Kilicman and Al Zhour (2007)] have given the Block Pulse operational matrix of the fractional integration F_α as following

$$I^{\alpha}\varphi(t) \approx F_\alpha\varphi(t), \tag{25}$$

where

$$F_\alpha = \left(\frac{1}{m}\right)^{\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{26}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. As for the second kind Chebyshev wavelet operational matrix of the fractional differential D_α as

$$D_\alpha = m^{\alpha}\Gamma(\alpha+2)\begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{-1}. \tag{27}$$

Using Eq. (25) and Eq. (24), we have

$$I^{\alpha}\psi_m(t) \approx I^{\alpha}\psi_{m\times m}\varphi(t) = \psi_{m\times m}I^{\alpha}\varphi(t) = \psi_{m\times m}F_\alpha\varphi(t). \tag{28}$$

From Eq. (22) and Eq. (24), we get

$$I^{\alpha}\psi_m(t) = P^{\alpha}_{m\times m}\psi_m(t) = P^{\alpha}_{m\times m}\psi_{m\times m}\varphi_m(t) = \psi_{m\times m}F_\alpha\varphi_m(t). \tag{29}$$

Then, the second kind Chebysev wavelet operational matrix of the fractional integration $P^{\alpha}_{m\times m}$ is given by

$$P^{\alpha}_{m\times m} = \psi_{m\times m}F_\alpha\psi^{-1}_{m\times m}. \tag{30}$$

Let $D^{\alpha}_{m\times m}$ is the second kind Chebyshev wavelet operational matrix of the fractional differential, according to the properties of the fractional integration and differential

$D^{\alpha}P^{\alpha}=I,$

where I is the unit matrix, we can receive D^{α} by the inverse of P^{α},

$$D^{\alpha}_{m\times m}=\left(P^{\alpha}_{m\times m}\right)^{-1}=\psi_{m\times m}D_{\alpha}\psi^{-1}_{m\times m}. \tag{31}$$

In particular, for $k=2, M=3, \alpha=0.5$ the second kind Chebyshev wavelet operational matrix of fractional order integration $P^{\alpha}_{m\times m}$ is given by

$$P^{0.5}_{6\times 6}=\begin{bmatrix} 0.1513 & -0.2077 & -0.1558 & -3.7364 & -1.5403 & -0.0746 \\ 0.2077 & 0.5841 & 0.2077 & 1.8244 & 0.1826 & 0.0033 \\ -0.1212 & -0.1615 & 0.1860 & -0.7450 & -0.2871 & -0.0096 \\ 0 & 0 & 0 & 0.1513 & -0.2077 & -0.1558 \\ 0 & 0 & 0 & 0.2077 & 0.5841 & 0.2077 \\ 0 & 0 & 0 & -0.1212 & -0.1615 & 0.1860 \end{bmatrix},$$

and the operational matrix of fractional order differential $D^{\alpha}_{m\times m}$ is

$$D^{0.5}_{6\times 6}=\begin{bmatrix} 4.9090 & 2.2029 & 1.6529 & 60.0271 & 37.3483 & 10.6012 \\ -2.2023 & 0.3196 & -2.2023 & -42.5606 & -24.6517 & -9.1326 \\ 1.2853 & 1.7130 & 4.5414 & 18.4721 & 12.3276 & 2.4296 \\ 0 & 0 & 0 & 4.9090 & 2.2029 & 1.6529 \\ 0 & 0 & 0 & -2.2023 & 0.3196 & -2.2023 \\ 0 & 0 & 0 & 1.2853 & 1.7130 & 4.5414 \end{bmatrix}.$$

To verity the exactness of P^{α} and D^{α} further, take the function $u(t)=t$, then the fractional integration of function $u(t)=t$ is $I^{\alpha}_{*}u(t)=\frac{\Gamma(2)}{\Gamma(\alpha+2)}t^{\alpha+1}$, the fractional differential is $D^{\alpha}_{*}u(t)=\frac{\Gamma(2)}{\Gamma(2-\alpha)}t^{1-\alpha}$.
When $k=5, M=2, \alpha=0.5$, image of the approximate solutions for the fractional integration and the fractional differential are shown in Fig.1 and Fig.2, respectively.

6 Numerical Algorithms

The second kind Chebyshev wavelet operational matrix has been given, we consider a class of nonlinear fractional integral-differential equations as follows

$$\begin{gathered}\sum_{i=1}^{N}a_i(x)D^{\alpha_i}y(x)=y^{j}(x)+\lambda_1\int_0^x k_1(x,t)\left[y(t)\right]^p dt+\lambda_2\int_0^1 k_2(x,t)\left[y(t)\right]^q dt+g(x) \\ y^{(j)}(0)=b_j \quad 0\le x,t\le 1 \qquad n-1<\alpha_i\ \le n\end{gathered},$$

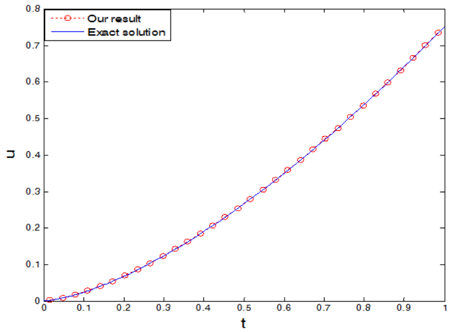


Figure 1: The integration of $u(t) = t$ of $\alpha = 0.5$

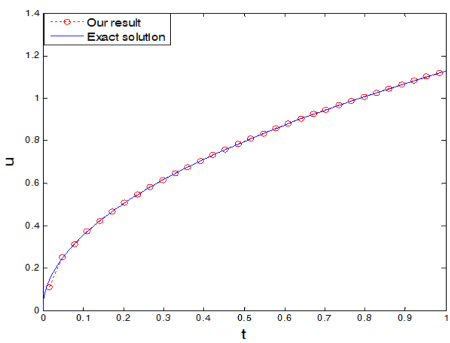


Figure 2: The differential of $u(t) = t$ of $\alpha = 0.5$

where $N, j \in Z_+, \lambda_1, \lambda_2$ are arbitrary parameters, p, q are nonnegative integers, $g(x)$ is a known function, $k_1(x,t), k_2(x,t) \in L^2([0,1] \times [0,1])$, $a_i(x) > 0$ and $a_i(x) \in C[0,1]$. Let

$$\begin{array}{ll} y(x) \approx C^T \psi(x) = \psi^T(x) C, & g(x) \approx G^T \psi(x) = \psi^T(x) G \ , \\ k_1(x,t) = \psi^T(x) K_1 \psi(t), & k_2(x,t) = \psi^T(x) K_2 \psi(t), \\ y^j(x) \approx C_j^{*T} \psi(x) = \psi^T(x) C_j^* & \\ [y(t)]^p \approx C_p^{*T} \psi(t) = \psi^T(t) C_p^*, & [y(t)]^q \approx C_q^{*T} \psi(t) = \psi^T(t) C_q^*, \end{array}$$

the variable coefficients $a_i(x)$ can be dispersed, let

$$A_i = \begin{bmatrix} a_i(x_0) & 0 & 0 & 0 & 0 \\ 0 & a_i(x_1) & 0 & 0 & 0 \\ 0 & 0 & a_i(x_2) & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & a_i(x_{2^{k-1}M-1}) \end{bmatrix}.$$

Substituting the above expanded forms into Eq. (1), we have

$$\sum_{i=1}^{N} a_i(x) D^{\alpha_i} y(x) \approx \sum_{i=1}^{N} A_i D^{\alpha_i} C^T \psi(x) \approx C^T \sum_{i=1}^{N} A_i D^{\alpha_i} \psi(x),$$

$$\begin{aligned} & y^j(x) + \lambda_1 \int_0^x k_1(x,t)[y(t)]^p dt + \lambda_2 \int_0^1 k_2(x,t)[y(t)]^q dt + g(x) \\ & \approx \psi^T(x) C_j^* + \lambda_1 \int_0^x \psi^T(x) K_1 \psi(t) \psi^T(t) C_p^* dt \\ & \quad + \lambda_2 \int_0^1 \psi^T(x) K_2 \psi(t) \psi^T(t) C_q^* dt + G^T \psi(x) \\ & \approx \psi^T(x) C_j^* + \lambda_1 \psi^T(x) K_1 \int_0^x \psi(t) \psi^T(t) C_p^* dt \\ & \quad + \lambda_2 \psi^T(x) K_2 \int_0^1 \psi(t) \psi^T(t) C_q^* dt + G^T \psi(x) \\ & \approx \psi^T(x) C_j^* + \lambda_1 \psi^T(x) K_1 \int_0^x \widetilde{C_p^*} \psi(t) dt + \lambda_2 \psi^T(x) K_2 I C_q^* + G^T \psi(x) \\ & \approx \psi^T(x) C_j^* + \lambda_1 \psi^T(x) K_1 \widetilde{C_p^*} P \psi(x) + \lambda_2 \psi^T(x) K_2 C_q^* + G^T \psi(x), \end{aligned}$$

then

$$C^T \sum_{i=1}^{N} A_i D^{\alpha_i} \psi(x) = \psi^T(x) C_j^* + \lambda_1 \psi^T(x) K_1 \widetilde{C_p^*} P \psi(x) + \lambda_2 \psi^T(x) K_2 C_q^* + G^T \psi(x), \tag{32}$$

where $\widetilde{C_p^*}$ is product operation matrix for C_p^*, P is wavelet integral operational matrix, C_j^*, C_p^*, C_q^* are column vectors, linear combination of the element of C,they can be received by the product operational matrix of wavelet. The D^{α_i} can be obtained from the differential operational matrix.

We can see, when j,p,q are taken as different value $k, k \geq 2$, then $C_k^* = \left[\left(\widetilde{C}\right)^{k-1}\right]^T C$.

Let us put the collocation points $\{x_n\}_{n=1}^{2^{(k-1)}M}$ in the interval $[0,1]$into Eq. (32), the equation will be transformed as following

$$\begin{aligned} C^T \sum_{i=1}^{N} A_i D^{\alpha_i} \psi(x_n) \quad = \quad & \psi^T(x_n) C_j^* + \lambda_1 \psi^T(x_n) K_1 \widetilde{C_p^*} P \psi(x_n) \\ & + \lambda_2 \psi^T(x_n) K_2 C_q^* + G^T \psi(x_n) \end{aligned}$$

Solving the nonlinear algebraic equations by using the Newton iteration method, we get the matrix of$C$, and then we obtain the approximate solution$y(x) = C^T \psi(x)$.

7 Error analysis

Theorem 1. Suppose that the function $u:[0,1]\to R$ is m times continuously differentiable and $u\in C^m[0,1]$. Then $\widetilde{u}(x)$ approximates $u(x)$ with mean error bounded as follows [Wang and Fan (2012)]

$$\|u(x)-\widetilde{u}(x)\|\leq\frac{2}{2^{m(k-1)}4^m m!}\sup_{x\in[0,1]}\left|u^{(m)}(x)\right|.$$

Proof. We divided the interval $[0,1]$ into subintervals $I_{k,n}=\left[\frac{n-1}{2^{k-1}},\frac{n}{2^{k-1}}\right],n=1,\cdots,$ 2^{k-1} with the restriction that $\widetilde{u}(x)$ is a polynomial of degree less than m that approximates u with minimum mean error. The approximate solution approaches the exact solution as k approaches ∞. Use the maximum error estimate for the polynomial which interpolates u at Chebyshev notes of order m on $I_{k,n}$. Then we have

$$\begin{aligned}&\|u(x)-\widetilde{u}(x)\|^2=\int_0^1[u(x)-\widetilde{u}(x)]^2dx=\\&\sum_n\int_{I_{k,n}}[u(x)-\widetilde{u}(x)]^2dx\leq\sum_n\int_{I_{k,n}}[u(x)-\widehat{u}(x)]^2dx\\&\leq\sum_n\int_{I_{k,n}}\left[\frac{2}{2^{m(k-1)}4^m m!}\sup_{x\in I_{k,n}}|u^m(x)|\right]^2dx\leq\left[\frac{2}{2^{m(k-1)}4^m m!}\sup_{x\in[0,1]}|u^m(x)|\right]^2.\end{aligned}\tag{33}$$

It gives the upper bound of the square roots. In Eq. (31), $\widehat{u}(x)$ denotes the interpolating polynomial of degree m which agrees with $u(x)$ at the Chebyshev nodes of order$m$ on $I_{k,n}$. Here, we use the well-known maximum error bound for Chebyshev interpolation. Therefore, the error between the approximation $\widetilde{u}(x)$ and $u(x)$ decays at speed of $2^{-m(k-1)}$. Meanwhile, we notice that the number of wavelet is $m'=2^{k-1}M$, where M presents the degree of the second kind Chebyshev polynomials, usually being taken as a small value in computation. When M is fixed, the larger the value of k is, the more accurate the approximation solution is.

Also, we present error estimation for the second kind Chebyshev wavelet series solution with the residual error function. We can get a theorem as follows.

Theorem 2. For the problem in Eq. (1), the maximum absolute error can be estimated approximately

$$E_{k,M,\alpha}=\max\left\{\left|e'_{k,M,\alpha}(x)\right|,0\leq x\leq 1\right\},$$

where $e'_{k,M,\alpha}=\sum_{n=1}^{2^k-1}\sum_{m=0}^{M-1}c^*_{n,m}\phi_{n,m}(x)$. Here, the coefficients $c^*_{n,m}$ are determined by solving the error problem.

Proof. Firstly, we named the error function as

$$e_{k,M,\alpha}(x) = y(x) - y_{k,M,\alpha}(x), \tag{34}$$

where$y(x)$is the exact solution of the Eq. (1).

Therefore, $y_{k,M,\alpha}(x)$satisfies the following problem

$$\begin{aligned}\sum_{i=1}^{N} a_i(x) D^{\alpha_i} y_{k,M,\alpha}(x) = y^j_{k,M,\alpha}(x) + \lambda_1 \int_0^x k_1(x,t)\left[y_{k,M,\alpha}(t)\right]^p dt \\ +\lambda_2 \int_0^1 k_2(x,t)\left[y_{k,M,\alpha}(t)\right]^q dt + g(x) + R_{k,M}(x) \ , \\ 0 \le x,t \le 1, \quad n-1 < \alpha_i \ \le n.\end{aligned} \tag{35}$$

Here, $R_{k,M}(x)$ is the residual function and the Eq. (33) is obtained by substituting the approximate solution $y_{k,M,\alpha}(x)$ into Eq. (1).

Also, we note that $k\alpha - th$order fractional derivative of the approximate solution $y_{k,M,\alpha}(x)$ is computed by using the Caputo fractional derivative.

Now, let us subtract Eq. (33) from Eq. (1). Hence, we obtain the error problem

$$\begin{aligned}\sum_{i=1}^{N} a_i(x) D^{\alpha_i} e_{k,M,\alpha}(x) = \left(y^j(x) - y^j_{k,M,\alpha}(x)\right) + \lambda_1 \int_0^x k_1(x,t)\left[y^p(t) - y^p_{k,M,\alpha}(t)\right] dt \\ +\lambda_2 \int_0^1 k_2(x,t)\left[y^q(t) - y^q_{k,M,\alpha}(t)\right] dt - R_{k,M}(x) \quad .\end{aligned} \tag{36}$$

For example,$j = p = q = 2$, Eq. (34) can be transformed as

$$\begin{aligned}\sum_{i=1}^{N} a_i(x) D^{\alpha_i} e_{k,M,\alpha}(x) = \left(2e_{k,M,\alpha}(x)\, y_{k,M,\alpha}(x) + e^2_{k,M,\alpha}(x)\right) \\ +\lambda_1 \int_0^x k_1(x,t)\left[2e_{k,M,\alpha}(t)\, y_{k,M,\alpha}(t) + e^2_{k,M,\alpha}(t)\right] dt \\ +\lambda_2 \int_0^1 k_2(x,t)\left[2e_{k,M,\alpha}(t)\, y_{k,M,\alpha}(t) + e^2_{k,M,\alpha}(t)\right] dt - R_{k,M}(x).\end{aligned} \tag{37}$$

By solving the above error problem in the same way as that in Section 5, we get the approximation

$$e'_{k,M,\alpha}(x) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c^*_{n,m} \phi_{n,m}(x),$$

where, the coefficients $c^*_{n,m}$ are determined by solving the error Eq. (35). Hence, the maximum absolute error can be estimated as

$$E_{k,M,\alpha} = \max\left\{\left|e'_{k,M,\alpha}(x)\right|, 0 \le x \le 1\right\}.$$

We can see that if the exact solution unknown, this error estimation can be used to test the reliability of the results.

8 Numerical examples

Example1. Consider the following nonlinear fractional integral-differential equation

$$x^2D^{1.5}y(x)+xD^{0.5}y(x)=y^4(x)+\int_0^x(x-t)y^2(t)\,dt+\int_0^1(1+t)y^3(t)\,dt$$
$$+\left(-\frac{2539}{280}-\frac{\sqrt{x}}{\sqrt{\pi}}+30x^2+\frac{20x^{5/2}}{3\sqrt{\pi}}-\frac{71x^4}{3}+\frac{239x^6}{30}-x^8\right)$$

the exact solution is $y(x)=x^2-2,\quad 0<x<1.$

From the above derivation, we can see the equation can be transformed into the matrix from as follows

$$C^TA_1D^{1.5}\psi(x_n)+C^TA_2D^{0.5}\psi(x_n)=\psi^T(x_n)C_4^*+\psi^T(x_n)K_1\widetilde{C_2^*}P\psi(x_n)$$
$$+\psi^T(x_n)K_2C_3^*+G^T\psi(x_n).$$

For example $k=2, M=3$,

where $C^T=[c_{10},c_{11},c_{12},c_{20},c_{21},c_{22}]$, take point $\{x_n\}_{n=1}^6=\left\{\frac{1}{12},\frac{3}{12},\frac{5}{12},\frac{7}{12},\frac{9}{12},\frac{11}{12}\right\}$, we have

$$A_1=diag\left[\begin{array}{cccccc}\frac{1}{144} & \frac{9}{144} & \frac{25}{144} & \frac{49}{144} & \frac{81}{144} & \frac{121}{144}\end{array}\right],$$
$$A_2=diag\left[\begin{array}{cccccc}\frac{1}{12} & \frac{3}{12} & \frac{5}{12} & \frac{7}{12} & \frac{9}{12} & \frac{11}{12}\end{array}\right]$$

$$D^{1.5}_{6\times6}=\begin{bmatrix}73.6229 & 33.0401 & 24.7792 & 900.2823 & 560.1802 & 158.9866\\ -33.0379 & 4.7916 & -33.0379 & -638.5036 & -369.8640 & -137.0056\\ 19.2729 & 25.6981 & 68.1166 & 277.0901 & 184.9339 & 36.4373\\ 0 & 0 & 0 & 73.6229 & 33.0401 & 24.7792\\ 0 & 0 & 0 & -33.0379 & 4.7916 & -33.0379\\ 0 & 0 & 0 & 19.2729 & 25.6981 & 68.1166\end{bmatrix},$$

$$D^{0.5}_{6\times6}=\begin{bmatrix}4.9090 & 2.2029 & 1.6529 & 60.0271 & 37.3483 & 10.6012\\ -2.2023 & 0.3196 & -2.2023 & -42.5606 & -24.6517 & -9.1326\\ 1.2853 & 1.7130 & 4.5414 & 18.4721 & 12.3276 & 2.4296\\ 0 & 0 & 0 & 4.9090 & 2.2029 & 1.6529\\ 0 & 0 & 0 & -2.2023 & 0.3196 & -2.2023\\ 0 & 0 & 0 & 1.2853 & 1.7130 & 4.5414\end{bmatrix},$$

$$C_4^*=\left[\left(\widetilde{C}\right)^3\right]^TC,\quad C_3^*=\left[\left(\widetilde{C}\right)^2\right]^TC,\quad C_2^*=\left(\widetilde{C}\right)^TC,$$

$$P = 1/4\begin{bmatrix} 1 & 1/2 & 0 & 2 & 0 & 0 \\ -3/4 & 0 & 1/4 & 0 & 0 & 0 \\ 1/3 & -1/6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & -3/4 & 0 & 1/4 \\ 0 & 0 & 0 & 1/3 & -1/6 & 0 \end{bmatrix} \quad G = \begin{bmatrix} -0.5246 \\ -1.2080 \\ -3.9058 \\ 0.8386 \\ -4.0813 \\ 4.0827 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0 & 0.0123 & 0 & -0.0069 & 0.0169 & -0.0345 \\ -0.0123 & 0 & -0.0613 & -0.0077 & -0.0031 & -0.0383 \\ 0 & 0.0613 & 0 & -0.0345 & 0.0843 & -0.1725 \\ 0.0069 & 0.0077 & 0.0345 & 0 & 0.0123 & 0 \\ -0.0169 & 0.0031 & -0.0843 & -0.0123 & 0 & -0.0613 \\ 0.0345 & 0.0383 & 0.1725 & 0 & 0.0613 & 0 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} 0.0172 & -0.0238 & 0.0862 & 0.0241 & -0.0284 & 0.1207 \\ -0.0115 & 0.0158 & -0.0575 & -0.0161 & 0.0189 & -0.0805 \\ 0.0862 & -0.1188 & 0.4312 & 0.1207 & -0.1418 & 0.6037 \\ 0.0172 & -0.0238 & 0.0862 & 0.0241 & -0.0284 & 0.1207 \\ -0.0115 & 0.0158 & -0.0575 & -0.0161 & 0.0189 & -0.0805 \\ 0.0862 & -0.1188 & 0.4312 & 0.1207 & -0.1418 & 0.6037 \end{bmatrix}.$$

Using the above known conditions, absolute error for $M = 3$ and different values of k is given in Tab.1. Using error estimation according the error analysis in Section 6, the maximum absolution error is given in Tab.2.

Table 1: Absolute error for $M = 3$ and different values of k

x	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
0.1	3.5026e-003	3.0723e-003	2.1658e-004	1.0023e-004	5.7362e-005
0.2	1.0539e-003	5.4603e-004	2.3718e-004	8.7216e-005	2.9271e-005
0.3	5.2937e-003	4.7523e-004	1.7362e-004	7.6937e-005	2.0325e-005
0.4	2.2549e-003	2.9357e-004	8.7342e-005	4.5981e-005	6.9725e-006
0.5	4.7608e-003	1.9306e-004	7.6247e-005	3.2716e-005	4.8163e-006
0.6	5.6092e-004	2.1708e-004	8.7062e-005	2.5937e-005	3.0718e-006
0.7	4.9183e-003	3.7243e-004	1.0358e-004	7.9347e-005	1.8572e-005
0.8	3.5127e-003	4.0762e-004	1.2371e-004	6.8192e-005	9.3528e-006
0.9	2.7017e-004	1.9873e-004	7.6981e-005	3.6971e-005	6.0541e-006

Table 2: Maximum absolution error for $M = 3$ and different values of k

x	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	3.8715e-003	3.4872e-003	2.8402e-004	1.6827e-004	6.2508e-005
0.2	1.6217e-003	6.3781e-004	3.0125e-004	9.5718e-005	4.0021e-005
0.3	6.1753e-003	5.1671e-004	2.5761e-004	8.2601e-005	3.2109e-005
0.4	3.0506e-003	3.8762e-004	9.5603e-005	5.2702e-005	7.6305e-006
0.5	5.2179e-003	2.7705e-004	8.5706e-005	4.6339e-005	5.6218e-006
0.6	6.2871e-004	3.1587e-004	9.1655e-005	4.0208e-005	5.3369e-006
0.7	5.5621e-003	4.3365e-004	1.9802e-004	9.1125e-005	3.2614e-005
0.8	4.6284e-003	5.1306e-004	2.6415e-004	7.5134e-005	1.9246e-005
0.9	3.5514e-004	2.4676e-004	8.3362e-005	4.7759e-005	8.1306e-006

Example2. Consider the following equation

$$\begin{aligned}&\left(x^2-1\right)D^{1.5}y(x)+2xD^{0.75}y(x)+(x+2)D^{0.25}y(x)\\&=y^3(x)+\tfrac{1}{4}\int_0^x(x-t)y^2(t)\,dt+\int_0^1(1+t)y^4(t)\,dt+g(x)\ ,\end{aligned}$$

where

$$\begin{aligned}g(x)\quad=\quad&-\frac{88}{15}-\frac{2}{\sqrt{\pi}\sqrt{x}}+\frac{2x^{3/2}}{\sqrt{\pi}}-8x^3-\frac{x^4}{12}+\frac{16x^{5/4}}{\Gamma(1/4)}\\&+\frac{16x^{3/4}}{3\Gamma(3/4)}+\frac{8x^{7/4}}{3\Gamma(3/4)},0<x<1,\end{aligned}$$

the exact solution is $y(x) = 2x$. When $M = 2, k = 3,4,5,6$, the comparison of numerical solutions and the exact solution is shown in Fig.3, Fig.4, Fig.5 and Fig.6, respectively.

Example3. Consider the following equation

$$xD^{1.25}y(x)+(x+1)D^{0.5}y(x)=y^2(x)+\int_0^x(x+t)y^3(t)dt-\frac{1}{2}\int_0^1(x-t)y^2(t)dt+g(x),$$

where

$$\begin{aligned}g(x)=&-0.2679x^8+0.9286x^7-1.1000x^6+0.4500x^5-x^4+2x^3+1.5045x^{2.5}\\&-x^2+2.1761x^{1.75}+0.3761x^{1.5}+0.0167x-1.1284x^{0.5}-0.0083\\&0<x<1,\end{aligned}$$

the exact solution is $y(x) = x^2 - x$. When $M = 2, k = 4,5,6$, the comparison of numerical solutions and the exact solution is shown in Fig.7.

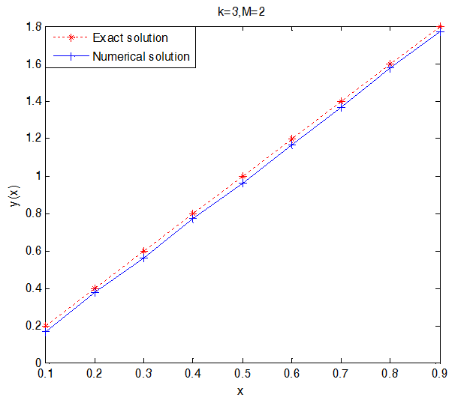


Figure 3: Exact solution and Numerical solution of $k = 3, M = 2$

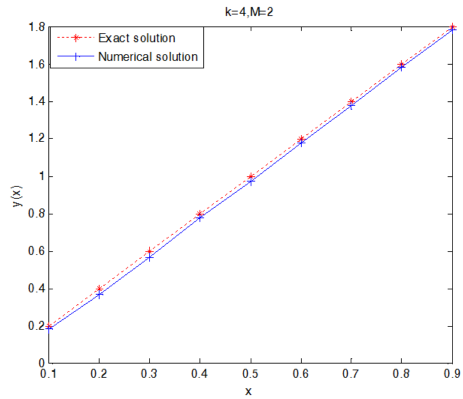


Figure 4: Exact solution and Numerical solution of $k = 4, M = 2$

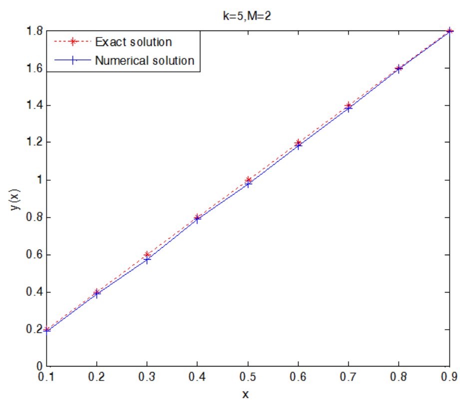


Figure 5: Exact solution and Numerical solution of$k = 5, M = 2$

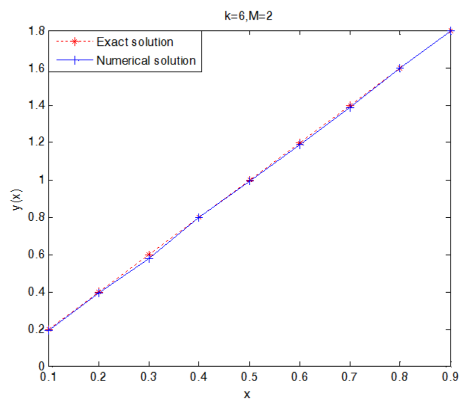


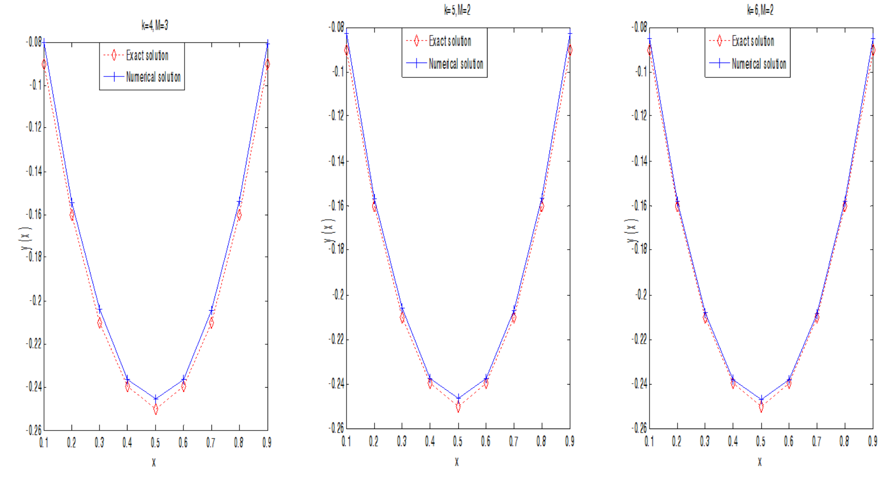
Figure 6: Exact solution and Numerical solution of$k = 6, M = 2$

Example4. Consider the following equation

$$x^2 D^{0.8} y(x) + (x+1) D^{0.2} y(x) = y^4(x) - \int_0^x (1+t) y^2(t)\,dt + \frac{3}{4}\int_0^1 (x+t) y^3(t)\,dt + g(x),$$

where

$$g(x) = -x^8 + 0.1665x^6 + 0.1998x^5 + 1.8152x^{3.2} + 1.193x^{2.8} + 1.1930x^{1.8} - 0.1071x - 0.0938 \qquad 0 < x < 1,$$



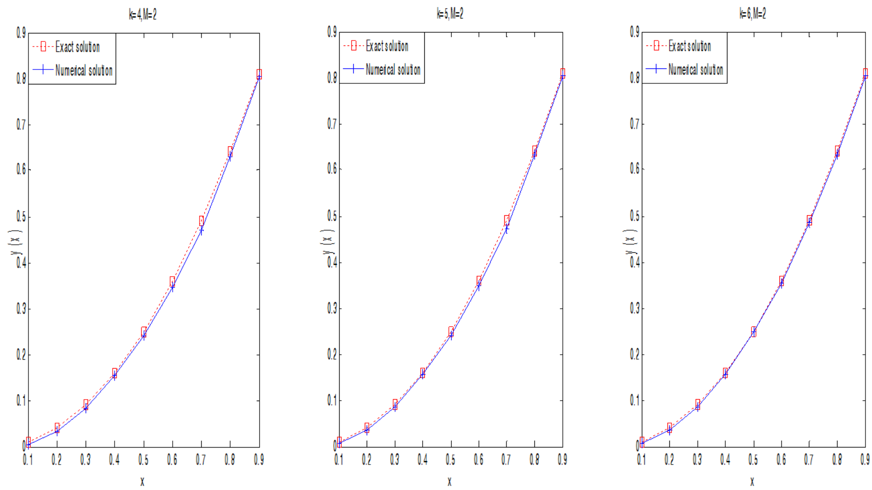
(a) (b) (c)

Figure 7: The comparison between exact solution and approximate for $M = 2, k = 4, 5, 6$

the exact solution is $y(x) = x^2$. When $M = 3$ and k is taken as different values, the absolute error is given in Tab.3. When $M = 2, k = 4, 5, 6$, the comparison of numerical solutions and the exact solution is shown in Fig.8.

Table 3: Absolute error for $M = 3$ and different values of k

x	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
0.1	2.7354e-004	1.5426e-005	3.2891e-006	5.4641e-007	6.0781e-008
0.2	5.0371e-005	3.8072e-006	5.7631e-007	2.0301e-007	3.5409e-008
0.3	4.8716e-005	8.5348e-006	2.0368e-006	7.6345e-007	1.2937e-007
0.4	3.7931e-004	6.2108e-005	1.2031e-005	6.0349e-006	4.9632e-007
0.5	6.7681e-005	3.0718e-006	5.3419e-007	1.1038e-007	5.7603e-008
0.6	2.3703e-004	4.1926e-005	8.2907e-006	2.4091e-006	6.4309e-007
0.7	7.0325e-005	2.4315e-005	4.0723e-006	8.4236e-007	2.3109e-007
0.8	3.0829e-004	6.0321e-005	1.1243e-005	5.2308e-006	8.5471e-007
0.9	4.5371e-005	1.3452e-005	5.0789e-006	6.5021e-007	7.4917e-008



(a) (b) (c)

Figure 8: The comparison between exact solution and approximate for $M = 2, k = 4, 5, 6$

Calculations results show that the second kind Chebyshev wavelet can be used to well solve the nonlinear fractional integral-differential equations with high precision. The larger the value of k is, the more accurate the approximation solution is. At the same time, with the increase of k, the error decreases gradually. And the absolute error between numerical solutions and the exact solution is slightly less than the maximum absolute error obtained by according the error estimation. So the method has the higher feasibility and effectiveness.

9 Conclusion

Using the second kind Chebyshev wavelet, a class of nonlinear fractional integral-differential equations is studied. The fractional differential operational matrix of the second kind Chebyshev wavelet and numerical algorithm are given. Through the numerical examples, it shows that the second kind Chebyshev wavelet has a very good approximation effect and high accuracy. It is an effective and simple algorithm. An expression of maximum absolute error is derived, which is slightly higher than the absolute error between numerical solution and exact solution. The presented method can also be used to solve the two-dimensional nonlinear fractional partial differential equations.

Acknowledgement: This research was supported by the Natural Science Foundation of Hebei province grant A2012203047.

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