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Unit Setting Method to Impose EBCs in Meshless Methods

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Abstract: Up to now, some methods have been proposed to impose essential boundary conditions (EBCs) in meshless methods to solve partial differential equations system. Based on the theory analysis about moving least square (MLS) approximation and numerical experimentation results, a very simple method to impose EBCs in element-free Galerkin methods, which is the same easy as in finite element methods, is posed here. Compared with Lagrange multiplier method, the new method is simple and gives better results at the distributed nodes. The new method dues to a view point, different from normal understanding, that taking generalized parameters in MLS approximations as the approximate values of the unknown field function at the nodes. This key view point leads to directly using unit setting method or large number setting method in the discrete equations system to impose EBCs in element-free Galerkin (EFG) method.

Keywords: moving least square approximation, generalized parameter, element-free Galerkin method, essential boundary conditions.

1 Introduction

The finite element method and the finite difference method which are based on meshes are the most widely used numerical techniques for modeling and engineering analysis. However, it is common knowledge that high quality mesh generation is a far more time-consuming task in these methods, so it has become necessary to explore other methods which are simple and efficient. Meshless methods, or meshfree methods, as another kind of numerical methods without the requirement of mesh generation, have attracted much attention in recent decades [Atluri (2004); Zhang and Liu (2004); Zhang, Song and Lu (2003)]. Having good adaptability on crack propagation simulation, analysis of large deformation and moving boundary problem, meshless methods use separated nodes, not mesh to represent the

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problem-domain. Element-free Galerkin (EFG) method [Belytschko, Lu and Gu (1994)] is one of the meshless methods which is based on moving least square (MLS) approximation [Lancaster and Salkauskas (1981)] and global Galerkin variation formulation. MLS approximation, initially used in data fitting and surface construction, can generate smoother approximate field functions than piecewise polynomial interpolation used in finite element methods. So, it is usually taken to construct the trial and test functions in meshless methods. In EFG, the approximate function $u^{\mathrm{ap}}(\mathbf{x})$ of the unknown field function can be defined by

$$u^{\mathrm{ap}}(\mathbf{x}) = \sum_{i=1}^{N} \phi_i(\mathbf{x}) u_i \tag{1}$$

where N is the number of nodes discreted in the solving domain; $\phi_i(\mathbf{x})(i = 1,2,\cdots, N)$ are nodal shape functions of MLS approximation; and data $u_i(i = 1,2,\cdots,N)$ are called generalized parameters in the existing literatures. But as a fitting method which does not force to realize interpolation, MLS approximation does not possess the Kronecker delta property generally, that is $u^{\mathrm{ap}}(\mathbf{x}_i) \neq u_i$. And therefore, it is not easy to impose essential boundary conditions (EBCs) directly. Many researchers have put their effort on it [Fernández-Méndez and Huerta (2004)], and some specific techniques have been developed for the implementation of essential boundary conditions in EFG method, such as, the Lagrange multiplier method [Belytschko, Lu and Gu (1994); Fernández-Méndez and Huerta (2004)], the penalty method [Zhu and Atluri (1998)], Nitsche's method [Griebel and Schweitzer (2002); Babuska, Banerjee and Osborn (2002)], method coupling to finite elements [Huerta and Fernández-Méndez (2000)] and so on.

As mentioned in the literature [Fernández-Méndez and Huerta (2004)], there are various difficulties on the above mentioned methods. For example, by introducing new multiplier variables, Lagrange multiplier method enlarges the dimension of linear equations, and at the same time, it is a difficult task to verify the LBB condition to ensure the uniqueness of the solution. The penalty method is in a dilemma of choosing proper penalty factor: if the value of the factor is too large, the matrix of the discrete linear equations is usually ill conditioned; however, if it is too small, the Dirichlet boundary condition is weakly imposed. Although Nitsche's method does not suffer of ill-conditioning, each particular problem has different modification, so this method is not easy to implement. Implementation of the method coupling to finite elements in meshless methods leads to a complex process. In a word, up to now, an ideal method to realize EBCs in meshless methods does not appear. Based on the theory analysis about MLS approximation and numerical experimentation results, and through changing our traditional view, this paper points out that EBCs in EFG method can be imposed the same way easily as in finite element methods.

2 Analysis on MLS approximation scheme and the new method

2.1 The MLS approximation scheme

Moving least squares may be considered as one of the most effective methods to approach unknown field function. The nodal shape functions $\{\phi_i(\mathbf{x})\}_{i=1}^N$ are obtained by minimizing a quadratic functional, defined as

$$J(\mathbf{a}) = \sum_{i=1}^{N} w_i(\mathbf{x})[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - u_i]^2 \tag{2}$$

for each point $\mathbf{x}$ in the domain. Here, the approximate function is as follow

$$u^{\mathrm{ap}}(\mathbf{x}) = \sum_{i=1}^{m} a_i(\mathbf{x})p_i(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{3}$$

where $\mathbf{p}^{\mathrm{T}}(\mathbf{x}) = [p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots, p_m(\mathbf{x})]$ is a complete monomial basis of order m; $\mathbf{a}(\mathbf{x}) = [a_1(\mathbf{x}), a_2(\mathbf{x}), \cdots, a_m(\mathbf{x})]^{\mathrm{T}}$ is the coefficient vector; $w_i(\mathbf{x})$ is the weight function associated with node i. If $u(\mathbf{x}_i) = u_i (i = 1, 2, \cdots, n)$ are known, Eq. 2 stands for minimizing weighted discrete error of quadratic functional J with respect to $\mathbf{a}$. The stationarity of J with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear equations

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{u} = 0 \tag{4}$$

that is

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u} \tag{5}$$

where $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are the matrices defined by

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{N} w_i(\mathbf{x})\mathbf{p}(\mathbf{x}_i)\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)$$

$$\mathbf{B}(\mathbf{x}) = [w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2), \cdots, w_N(\mathbf{x})\mathbf{p}(\mathbf{x}_N)]$$

Solving for $\mathbf{a}(\mathbf{x})$ from Eq. 5, we obtained

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \cdot \mathbf{u} \tag{6}$$

From Eq. 6, we can see that through minimizing the quadratic functional J, $\mathbf{a}(\mathbf{x})$ is a linear combination of generalized parameters $\{u_i\}_{i=1}^n$. Substituting $\mathbf{a}(\mathbf{x})$ into Eq. 3 gives the approximate expression Eq. 1 for $u(\mathbf{x})$ at the neighborhood of any point $\mathbf{x}$. Because the minimum value of J may not be zero, that is, approximation function $u^{\mathrm{ap}}(\mathbf{x})$ may not equal to generalized parameters $u_i = u(\mathbf{x}_i)$ at the node $\mathbf{x}_i$, so MLS approximation is a method of data fitting but not interpolation.

2.2 Analysis on MLS approximation

The approximate function developed by MLS method can be used in the following way if the sampling data $\{u_i\}_{i=1}^N$ are known: (1) evaluate approximate value of the non-sampling point by Eq. 1 naturally because of no other choice; (2) in the oversight of the observation error, the data satisfy $u_i = u(\mathbf{x}_i)(i = 1,2,\cdots,N)$, so it is wise to use these sampling data $\{u_i\}_{i=1}^N$ directly instead of evaluation by Eq. 1 like non-sampling points, and the idea help to avoid further error induced by MLS approximation. That is to say sampling data are more reliable than approximate value calculated by MLS approximation. The viewpoint is in coincidence with the presenter's who propose the MLS approximation.

While MLS method is taken to approximate the unknown field function which yields to partial differential equations system or variational principle, the sampling data $u_i = u(\mathbf{x}_i)(i = 1,2,\cdots,N)$ are unknown. Thus we can only take them as undetermined parameters to develop approximate expression, i.e. Eq. 1, by MLS method. Through solving the linear equations system developed by variational principle with finite freedom, we can only get the approximate values $\{\bar{u}_i\}_{i=1}^N$ of $\{u_i\}_{i=1}^N$ which are described in Eq. 1. The error of $\{\bar{u}_i\}_{i=1}^N$ is induced by approximate of unknown field function, numerical integration, and discretization of variational principle etc. The fact means that when the undetermined parameters are evaluated from the linear equations system, the MLS approximate function is developed by minimizing the following quadratic functional

$$J(\mathbf{a}) = \sum_{i=1}^{N} w_i(\mathbf{x})[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - \bar{u}_i]^2 \tag{7}$$

The approximate function can also be expressed similarly as Eq. 3, that is

$$u^{\mathrm{ap}}(\mathbf{x}) = \sum_{i=1}^{m} a_i(\mathbf{x})p_i(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{8}$$

where with substituting Eq. 7 into Eq. 4, the same calculation process(Eq. 4-Eq. 6) is used for evaluating $\mathbf{a}(\mathbf{x})$, and we get

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}) \cdot \bar{\mathbf{u}} \tag{9}$$

The implementation of the EFG method on partial differential equations can be carried out according to the following chart.

In Fig. 1, compound Gauss quadrature formula is used in the integration scheme to evaluate the integrals. Usually, Gauss points are not sampling points, so we have to evaluate the integrals on Gauss points by virtue of the approximate function, i.e. Eq. 1.

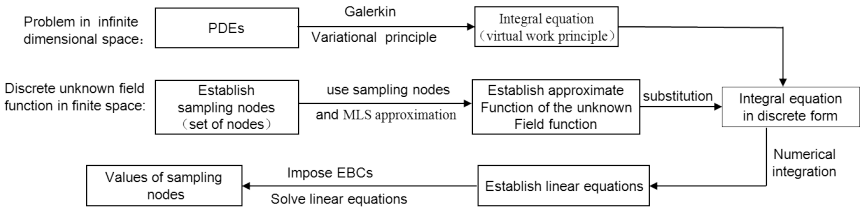

Figure 1: The flow chart of EFG method

In the usual literature, they take nodes value which calculated from approximate function, i.e. Eq. 8 as approximate value of nodes. For this reason, with the fact that $u^{\mathrm{ap}}(\mathbf{x}_i) \neq \bar{u}_i$, EFG method complicates the imposition of essential boundary condition. Like the statement in the first paragraph of section 2.2, generalized parameters $\{\bar{u}_i\}_{i=1}^N$ are the better approximate value of nodes compared with $\{u^{\mathrm{ap}}(\mathbf{x}_i)\}_{i=1}^N$, for its avoidance error from MLS approximation. Furthermore, the above analysis conclusion is tested and illustrated by the latter numerical examples.

3 The method of imposing EBCs and numerical examples

In this section, we consider the following simple two-dimensional boundary value problem of Poisson equation on the domain D:

$$\begin{cases} -\Delta u = f(x,y) & (x,y) \in D \\ u|_{\partial D} = g(x,y) & (x,y) \in \partial D \end{cases} \tag{10}$$

and EFG method is used to find the numerical solution of the problem. In order to compare the accuracy of generalized parameters $\{\bar{u}_i\}_{i=1}^N$ and MLS approximation values $\{u^{\mathrm{ap}}(\mathbf{x}_i)\}_{i=1}^N$, in the first example Lagrange multiplier method is used to impose EBCs as in usual literatures. Subsequently, a new method of imposing EBCs is proposed and some numerical examples are shown. From the computing results, the accuracy and efficiency of the new method and Lagrange multiplier method are displayed.

Example 1 Consider a Poisson problem in a two-dimensional domain as show in Fig. 2 with right hand side $f(x,y) = -2(x^2+y^2+x+y+2)$ in Eq. 10 and the boundary condition

$$\begin{cases} u(0,y) = y^2+y+1, u(8,y) = 73(y^2+y+1) & y \in [-1,1] \\ u(x,-1) = x^2+x+1, u(x,1) = 3(x^2+x+1) & x \in [0,8] \end{cases}$$

The exact solution of the problem is

$$u(x,y) = (x^2+x+1)(y^2+y+1) \tag{11}$$

The 5×11 uniform nodes are used to discrete the domain (Fig. 2). The EFG method with linear bases and cubic spline weight function are employed to solve this Poisson problem. The relative errors of generalized parameters at five nodes which are chosen arbitrarily and the corresponding MLS approximation solutions are shown in Tab. 1.

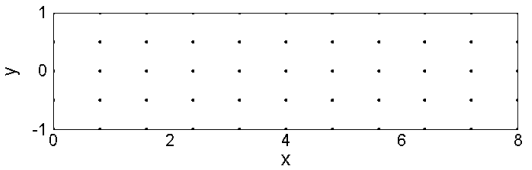

Figure 2: Distribution of 5×11 uniform nodes

From the data in Tab. 1, we can see clearly that the accuracy of generalized parameters as the approximation values at the nodes are better than that of evaluated from MLS approximate function. This phenomenon is inevitable: approximation function $u^{\mathrm{ap}}(\mathbf{x})$ is the optimal approximate which aims at approximating the general parameters, so the value of generalized parameters at the nodes is of course better than MLS approximate function. The fact is the same as the analysis result mentioned above.

Table 1: Relative errors of generalized parameters and approximated solutions

node coordinates	Generalized parameters(%)	MLS Approximated solutions(%)
(1.6,-0.5)	3.3989	7.8464
(1.6,0.5)	1.3007	3.3553
(4.0,0.0)	0.7158	4.0266
(6.4,0.5)	0.1287	1.7799
(7.2,0.0)	1.3595	2.0389

Now that both the theory analysis and numerical results indicate that so-called generalized parameters in literatures are the best approximation values at the discrete nodes, we should try to use this fact. Generalized parameters are obtained by solving the linear equations system, and they are the good approximation at nodes, but some nodes value at the essential boundary are given in the problem. How to realize the EBCs at the boundary nodes, as a natural choice, let the so-called generalized parameters at the boundary nodes equal to the given value. Thus, we can reference

finite element methods to realize the EBCs at boundary nodes. For example, unit setting method or large number setting method can be used in the discrete equations system to impose EBCs in EFG method. However, we should note that the EBCs just be satisfied at the given nodes not the whole boundary because the form of MLS approximation function is a kind of rational expression.

In the following examples, unit setting method is used to realize EBCs at boundary nodes. Giving by existing literature [Zhang and Liu (2004)], the overall discrete relative error (ODRE) is defined as follow

$$|u-u'| = \sqrt{\frac{\sum_{i=1}^{n} |u(x_i)-u'(x_i)|^2}{\sum_{i=1}^{n} u^2(x_i)}} \tag{12}$$

Example 2 Taking the same nodes distribution, bases function, and weight function as in Example 1, the right-hand items of Poisson equation are defined as follows:

$a)$ $f(x,y) = 2(x^2+y^2+x+y+2)$
$b)$ $f(x,y) = -2\cos x \sin y$
$c)$ $f(x,y) = 2e^{x+y}$

The corresponding exact solutions of the above problems are:

$$u = (x^2+x+1)(y^2+y+1) \tag{13}$$

$$u = \cos x \sin y \tag{14}$$

$$u = e^{x+y} \tag{15}$$

where the boundary conditions are prescribed on all boundaries according to Eq. 13, Eq. 14 and Eq. 15.

The EBCs are imposed by Lagrange multiplier method and unit setting method in the example. The corresponding overall discrete relative errors are shown in Tab. 2. The data indicate that unit setting method gives better results compared with Lagrange multiplier method for all the three different right-hand items. In addition, unit setting method do not introduce extra unknown variables or adjusting parameters, and it can be realized easily and save much more time cost, so it is a simple and reliable method like finite element methods.

Example 3 This example is to test the property of unit setting method while nonuniform nodes are used. Here the domain of Poisson equation is discretized by 106 irregular nodes as shown in Fig. 3.

The exact solution of Poisson equation is given by

$$u(x,y) = \sin^2 x \cos y + e^{xy^2} \tag{16}$$

Table 2: ODRE of Lagrange multiplier method and unit setting method

case	Lagrange multiplier method(%)	Unit setting method(%)
a)	2.6707	0.4521
b)	1.0035	0.9184
c)	3.8108	2.6333

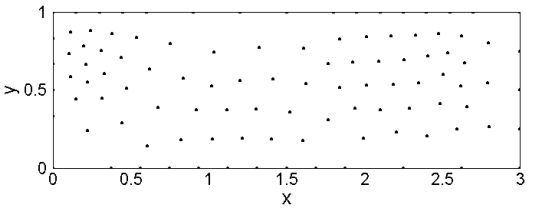

Figure 3: Distribution of 106 non-uniform nodes

Once again, both linear bases function and cubic spline weight function are used. From the data showed in Tab. 3, we can reach the same conclusion as in Example 2.

Table 3: Relative errors of Lagrange multiplier method and unit setting method

Node coordinates	Lagrange multiplier method(%)	Unit setting method(%)
(0.676,0.387)	0.3388	0.1455
(2.104,0.374)	1.5132	0.3433
(0.473,0.512)	0.2193	0.1442
(1.414,0.569)	0.8519	0.7702
ODRE(%)	3.7280	2.4361

4 Conclusions

Both theoretical analysis and numerical results indicate that generalized parameters give better approximate values at the discrete nodes when MLS method is used to approach the unknown field function. So, different from conventional thought, we naturally take generalized parameters in MLS approximations as the approximate values of the unknown field function at the nodes. Based on the above facts, generalized parameters at the boundary nodes can be assigned directly to be the given

boundary values, so imposing EBCs in EFG method can be the same as in finite element methods, such as large number setting method or unit setting method.

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