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An Interval Optimization Method Considering the Dependence between Uncertain Parameters

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Abstract: In this paper, an interval optimization method is developed to deal with a class of problems that there exists dependence between the interval parameters. An ellipsoidal convex model is used to model the uncertainty domain, in which the parameter dependence can be well reflected through the shape of a multi-dimensional ellipsoid. Based on an order relation and a reliability-based possibility degree of interval, the uncertain optimization can be transformed to a deterministic nesting optimization. An efficient algorithm is then constructed to solve the created nesting optimization, in which a sequence of approximate interval optimizations are created and the optimal design is obtained through an iteration process. Two numerical examples are investigated to demonstrate the effectiveness of the present method.

Keywords: interval optimization; uncertain optimization; non-probability; convex model; multi-dimensional ellipsoid; dependence

1 Introduction

In conventional optimizations (Haftka and Gurdal, 1992; Mathur et al., 2003; Tapp et al., 2004; Fedelinski and Gorski, 2006; Amirante et al., 2007; Lamberti and Pappalettere, 2007) all the involved parameters can be given specific values, while in uncertain optimizations the imprecision of some important parameters needs to be considered. Traditional uncertain optimization methods (Kirjner-Neto et al., 1998; Royset et al., 2001; Cheng et al., 2006; Liang et al., 2007; Du and Chen, 2004; Doltsinis and Kang, 2006; Kang and Luo, 2010) are generally based on the *probability model*, in which the precise probability distributions of all the uncertain

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parameters need to be constructed through a large number of samples. However, in practical applications sufficient samples on the uncertainty are often unavailable or sometimes very expensive to obtain them, and hence the probability model will encounter difficulty. To conduct the uncertainty analysis for this class of complex problems, the *interval model* (Moore, 1979) has been developed to deal with the uncertain parameters without enough information, and based on it another kind of uncertain optimization methodology, namely *interval optimization* has come into being.

In interval optimization, only the bounds of the uncertain parameters are required rather than their precise probability distributions, and hence many complex engineering structures with limited information can be conveniently treated. An amount of investigation has been carried out for *linear interval optimization* (Tanaka et al., 1984; Ishibuchi and Tanaka, 1990; Rommelfanger, 1989; Tong, 1994; Zhang et al., 1999; Sengupta et al., 2001; Averbakh and Lebedev, 2005), in which the objective function and constraints are all linear functions of the interval parameters. Considering that nearly all of the practical engineering problems are nonlinear, thus in recent years the *nonlinear interval optimization* has been attracting more and more attentions, and some methods in this field have been well established. Ma (2002) considered the interval uncertainty in the objective function and transformed the interval optimization into a deterministic multi-objective optimization. Jiang et al. (2007) investigated a general optimization problem, in which the objective function and constraints both contain the intervals. Cheng et al. (2004) changed the interval optimization into a minimax problem and subsequently constructed a hybrid algorithm to solve this problem. Guo et al. (2009) considered the global optimality existing in the interval optimization and derived some interesting conclusions. Recently, Jiang et al. (2011) constructed a reliability-based *possibility degree of interval*, based on which the interval constraints can be changed to deterministic ones. In the above works, each uncertain parameter is treated as an isolated interval, and hence the whole uncertainty domain forms a multi-dimensional box. Thus actually the above analysis is based on a precondition that all the involved uncertain parameters are mutually independent. Nevertheless, in many practical engineering problems the involved uncertain parameters have some dependence, and if they are treated as independent variables large analysis errors are likely to be caused. Therefore, it seems absolutely necessary to develop some new interval optimization methods to accommodate the dependent uncertainty, and whereby significantly expand the applicability of the interval optimization techniques.

In the field of non-probabilistic structural analysis, the ellipsoidal convex model (Ben-Haim and Elishakoff, 1990; Qiu, 2003) is often used to deal with the uncertain parameters with certain correlativity, in which the parameters subjected to uncertainty are assumed to fall into a multi-dimensional ellipsoid instead of a multi-dimensional box. Through adjusting the shape of the ellipsoid, the dependence between the parameters can be well reflected. In this paper, by introducing the ellipsoidal convex model we aim to develop a new nonlinear interval optimization method, which can deal with the uncertain parameters with dependence. The remainder of this paper is organized as follows. Section 2 creates an interval optimization model based on the ellipsoidal convex model. Section 3 transforms the interval optimization to a conventional optimization problem. Section 4 formulates an efficient algorithm to solve the involved nesting optimization. Section 5 provides two numerical examples and some conclusions are summarized in Section 6.

2 Nonlinear interval optimization by using ellipsoidal convex model

A general interval optimization can be formulated as:

$$\min_{\mathbf{X}} f(\mathbf{X},\mathbf{U})$$

subject to

$$g_i(\mathbf{X},\mathbf{U}) \le b_i^I = \left[b_i^L, b_i^R\right],\ i=1,...,l$$

$$\mathbf{X}_l \le \mathbf{X} \le \mathbf{X}_r \tag{1}$$

where $\mathbf{X}$ is an n-dimensional design vector, and $\mathbf{X}_l$ and $\mathbf{X}_r$ denote its two searching bounds. f and g_i, $i=1,2,...,l$ are the objective function and constraints, respectively, and in our study they are required to be continuous and differentiable with respect to the design variables and the uncertain parameters. b_i^I is an allowable interval of the ith uncertain constraint. The superscripts I, L and R denote the interval, lower bound and upper bound of interval, respectively. $\mathbf{U}$ is a q-dimensional uncertain vector, and the possible values of each parameter of $\mathbf{U}$ belong to an interval $U_i^I, i=1,2,...,q$. Considering that there exists some dependence between the parameters, their uncertainty then can be assumed to fall into a following multi-dimensional ellipsoid (Ben-Haim and Elishakoff, 1990):

$$\mathbf{U} = \mathbf{U}^0 + \boldsymbol{\delta}$$

$$\boldsymbol{\delta} \in E(\boldsymbol{\delta},\theta) = \left\{\boldsymbol{\delta} : \boldsymbol{\delta}^T \Omega \boldsymbol{\delta} \le \theta^2\right\} \tag{2}$$

where $\mathbf{U}^0$ denotes the midpoint of the ellipsoid; Ω is a characteristic matrix which determines the shape of the ellipsoid ; θ is a parameter which determines the size of the ellipsoid. $\boldsymbol{\delta}$ is a q-dimensional vector introduced for convenience of analysis. Figure 1 gives an ellipsoidal convex model for a two-dimensional case.

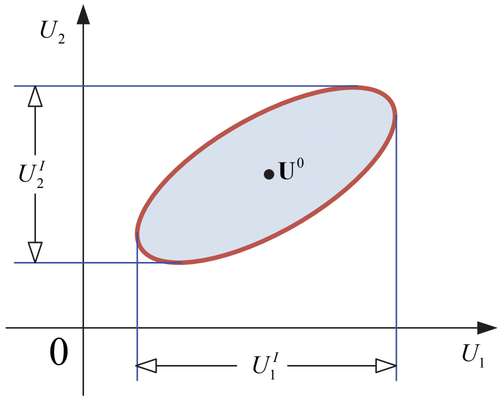

Figure 1: A two-dimensional ellipsoidal convex model

3 Creating a deterministic counterpart for the interval optimization

2. In this section, the above interval optimization will be transformed into a deterministic one. Firstly, based on an *order relation of interval* (Ishibuchi and Tanaka, 1990) the uncertain objective function in Eq. (1) can be changed to a deterministic multi-objective optimization problem:

$$\min_{\mathbf{X}} [f^c(\mathbf{X}), f^w(\mathbf{X})]$$

$$f^c(\mathbf{X}) = \frac{1}{2}(f^L(\mathbf{X}) + f^R(\mathbf{X})),\ f^w(\mathbf{X}) = \frac{1}{2}(f^R(\mathbf{X}) - f^L(\mathbf{X})) \tag{3}$$

where the superscripts c and w represent the midpoint and radius of interval, respectively. For each specific $\mathbf{X}$, the possible values of the objective function will form an interval $f^I(\mathbf{X})$, and its bounds can be expressed as:

$$f^L(\mathbf{X}) = \min_{\mathbf{U}\in\Gamma} f(\mathbf{X},\mathbf{U}),\ f^R(\mathbf{X}) = \max_{U\in\Gamma} f(\mathbf{X},\mathbf{U}) \tag{4}$$

where Γ denotes the uncertainty domain defined by Eq. (2).

The *possibility degree of interval* represents a quantitative extent that an interval is larger or smaller than another one. In the authors' previous work (Jiang et al., 2010; Jiang et al., 2011), a reliability-based possibility degree of interval (RPDI) was created, which can work not only for overlapped intervals but also completely separated intervals and furthermore can be used as a reliability index to deal with the interval constraints. Based on the RPDI, the constraints in Eq. (2) can be changed to:

$$p_r\left(g_j^I(\mathbf{X}) \le b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \ge \lambda_j,\ j = 1,2,...,l$$

$$g_j^w(\mathbf{X}) = \frac{g_j^R(\mathbf{X}) - g_j^L(\mathbf{X})}{2},\ b_j^w = \frac{b_j^R - b_j^L}{2} \tag{5}$$

where $p_r \in [-\infty, +\infty]$ denotes the RPDI and λ_j is a predetermined RPDI level for the jth constraint. A large RPDI level indicates a high requirement of reliability, and whereby a small feasible field of the constraints in Eq. (5). The constraint intervals $g_j^I(\mathbf{X}) = \left[g_j^L(\mathbf{X}), g_j^R(\mathbf{X})\right]$, $j = 1,2,...,l$ can be also computed through the optimization method like Eq. (4).

Using the linear combination method (Miettinen, 1999) to deal with the multi-objective problem, a deterministic optimization problem can be finally created for Eq. (1):

$$\min_{\mathbf{X}} f_d = \beta f^c(\mathbf{X}) + (1-\beta) f^w(\mathbf{X})$$

subject to

$$p_r\left(g_j^I(\mathbf{X}) \le b_j^I\right) = \frac{b_j^R - g_j^L(\mathbf{X})}{2g_j^w(\mathbf{X}) + 2b_j^w} \ge \lambda_j,\ j = 1,2,...,l$$

$$\mathbf{X}_l \le \mathbf{X} \le \mathbf{X}_r \tag{6}$$

where $0.0 \le \beta \le 1.0$ is a weighting factor and f_d denotes a desirability function of the two objectives.

Obviously, Eq. (6) is a nesting optimization problem and it will in general lead to extremely low efficiency for a practical engineering problem based on the time-consuming simulation. To improve the practicability of the present interval optimization method, an efficient algorithm will be formulated to solve the above nesting optimization.

4 Solving the nesting optimization problem

In our algorithm, a sequence of approximate optimizations are created and an optimum is obtained through an iteration process. At the sth iterative step, based on the first-order Taylor expansion the approximate interval optimization has the following form:

$$\min_{\mathbf{X}} \tilde{f}(\mathbf{X},\mathbf{U}) \approx f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}(U_i-U_i^0)$$

subject to

$$\tilde{g}_j(\mathbf{X},\mathbf{U}) \approx g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}(U_i-U_i^0) \le b_j^I=\left[b_j^L,b_j^R\right],\ j=1,...,l$$

$$\max\left[\mathbf{X}_l,\mathbf{X}^{(s)}-\gamma^{(s)}\right]\le\mathbf{X}\le\min\left[\mathbf{X}_r,\mathbf{X}^{(s)}+\gamma^{(s)}\right] \tag{7}$$

where $\gamma^{(s)}$ denotes the move limit vector applied on the current design vector $\mathbf{X}^{(s)}$ to ensure the approximation accuracy.

Now, the approximate objective function $\tilde{f}$ and constraints $\tilde{g}_i,\ i=1,2,...,l$ are all linear functions with respect to $\mathbf{U}$, and hence their bounds at each specific $\mathbf{X}$ can be analytically computed. To obtain the interval $\tilde{f}^I(\mathbf{X})=\left[\tilde{f}^L(\mathbf{X}),\tilde{f}^R(\mathbf{X})\right]$ we need to solve the following optimization problems:

$$\tilde{f}^L(\mathbf{X})=\min_{\boldsymbol{\delta}\in E(\boldsymbol{\delta},\theta)} f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}\delta_i$$

$$\tilde{f}^R(\mathbf{X})=\max_{\boldsymbol{\delta}\in E(\boldsymbol{\delta},\theta)} f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})+\sum_{i=1}^{q}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}\delta_i \tag{8}$$

As two conventional optimization problems with linear objective function and quadratic inequality constraint, by using the Lagrange-multiplier-based method developed in (Qiu, 2003) their optima can be explicitly obtained:

$$\tilde{f}^L(\mathbf{X})=f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})-\theta\sqrt{\nabla f_U^T\Omega^{-1}\nabla f_U}$$

$$\tilde{f}^R(\mathbf{X})=f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)+\sum_{i=1}^{n}\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i-X_i^{(s)})+\theta\sqrt{\nabla f_U^T\Omega^{-1}\nabla f_U} \tag{9}$$

where ∇f_U denotes the gradient vector composed by $\frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}, i=1,2,...,q$.
Similarly, $\tilde{g}_j^I(\mathbf{X}) = \left[g_j^L(\mathbf{X}), g_j^R(\mathbf{X})\right],\ j=1,2,...,l$ can be also explicitly obtained:

$$g_j^L(\mathbf{X}) = g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right) + \sum_{i=1}^{n} \frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i - X_i^{(s)}) - \theta\sqrt{\nabla g_{Uj}^T \Omega^{-1} \nabla g_{Uj}},\quad j=1,2,...,l$$

$$g_j^R(\mathbf{X}) = g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right) + \sum_{i=1}^{n} \frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial X_i}(X_i - X_i^{(s)}) + \theta\sqrt{\nabla g_{Uj}^T \Omega^{-1} \nabla g_{Uj}},\quad j=1,2,...,l \tag{10}$$

where ∇g_{Uj} denotes the gradient vector of the jth constraint composed by $\frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial U_i}$, $i=1,2,...,q$.
Based on Eqs. (9) and (10), the approximate optimization in Eq. (7) can be changed to a deterministic optimization problem like Eq. (6):

$$\begin{aligned}\min_{\mathbf{X}} \tilde{f}_d = &\sum_{i=1}^{n} \beta \frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial Xi} Xi + \beta\left(f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right) - \frac{\partial f\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial Xi} X_i^{(s)}\right)\\ &+ (1-\beta)\,\theta\sqrt{\nabla f_U^T \Omega^{-1} \nabla f_U}\end{aligned}$$

subject to

$$\sum_{i=1}^{n} \frac{\partial g_j\left(\mathbf{X}^{(s)},\mathbf{U}^0\right)}{\partial Xi} Xi \le (1-\lambda j)\, b^R + \lambda j b^L + (1-2\lambda j)\,\theta\sqrt{\nabla g_{Uj}^T \Omega^{-1} \nabla g_{Uj}},\quad j=1,2,...,l$$

$$\max\left[\mathbf{X}_l, \mathbf{X}^{(s)} - \gamma^{(s)}\right] \le \mathbf{X} \le \min\left[\mathbf{X}_r, \mathbf{X}^{(s)} + \gamma^{(s)}\right] \tag{11}$$

As a linear programming problem, the well-established simplex method (Nocedal and Wright, 1999) can be used to solve the above problem. After obtaining the optimum $\bar{\mathbf{X}}$ of Eq. (11), it needs to be judged whether it is a feasible and descending solution of Eq. (6), and hence the intervals of the actual objective function and constraints at $\bar{\mathbf{X}}$ are required. Considering that the uncertainty levels in practical engineering problems are generally small, these intervals can be analytically obtained by also using the above linear approximation approach:

$$f^L\left(\bar{\mathbf{X}}\right) = f\left(\bar{\mathbf{X}},\mathbf{U}^0\right) - \theta\sqrt{\nabla f_{U\bar{X}}^T \Omega^{-1} \nabla f_{U\bar{X}}}$$

$$f^R(\bar{\mathbf{X}}) = f(\bar{\mathbf{X}}, \mathbf{U}^0) + \theta\sqrt{\nabla f_{U\bar{X}}^T \Omega^{-1} \nabla f_{U\bar{X}}}$$

$$g_j^L(\bar{\mathbf{X}}) = g_j(\bar{\mathbf{X}}, \mathbf{U}^0) - \theta\sqrt{\nabla g_{U\bar{X}j}^T \Omega^{-1} \nabla g_{U\bar{X}j}},\ j = 1,2,...,l$$

$$g_j^R(\bar{\mathbf{X}}) = g_j(\bar{\mathbf{X}}, \mathbf{U}^0) + \theta\sqrt{\nabla g_{U\bar{X}j}^T \Omega^{-1} \nabla g_{U\bar{X}j}},\ j = 1,2,...,l \tag{12}$$

where $\nabla f_{U\bar{X}}$ denotes the gradient vector composed by $\frac{\partial f(\bar{\mathbf{X}},\mathbf{U}^0)}{\partial U_i}, i = 1,2,...,q$, and $\nabla g_{U\bar{X}j}$denotes the gradient vector of the jth constraint composed by $\frac{\partial g_j(\bar{\mathbf{X}},\mathbf{U}^0)}{\partial U_i}$, $i = 1,2,...,q$.

Our algorithm can then be summarized below:

Step 1 Setting the following parameters: initial design $\mathbf{X}^{(1)}$, move limit $\gamma^{(1)}$, scaling factor $\alpha \in (0,1)$, allowable errors $\varepsilon_1 > 0, \varepsilon_2 > 0, \varepsilon_3 > 0$, RPDI levels $\lambda_j,\ j = 1,2,...,l$, and step index s=1.

Step 2 Solving the linear programming problem in Eq. (11) to obtain an optimum $\bar{\mathbf{X}}$.

Step 3 Calculating the intervals $f^I(\bar{\mathbf{X}})$ and $g_j^I(\bar{\mathbf{X}}),\ j = 1,2,...,l$ of the actual objective function and constraints at $\bar{\mathbf{X}}$ through Eq. (12).

Step 4 Calculating the desirability function $f_d(\bar{\mathbf{X}})$ and the constraint RPDIs $p_r(g_j^I(\bar{\mathbf{X}}) \le b_j^I)$, $j = 1,2,...,l$.

Step 5 If $\min\left\{\left(p_r\left(g_j^I(\bar{\mathbf{X}}) \le b_j^I\right) - \lambda_j\right),\ j = 1,2,...,l\right\} > -\varepsilon_1$ and $f_d(\bar{\mathbf{X}}) < f_d(\mathbf{X}^{(s)})$, making $\mathbf{X}^{(s+1)} = \bar{\mathbf{X}}$ and go to Step 7; otherwise, reducing the move limit $\gamma^{(s)} := \alpha\gamma^{(s)}$ by a scaling factor α. (Judgement of a feasible and descending solution)

Step 6 If $\min\left\{\delta_i^{(s)},\ i = 1,2,...,n\right\} < \varepsilon_2$, $\mathbf{X}^{(s)}$ is obtained as an optimal design of the original problem and the iteration process terminates; otherwise, back to Step 2.

Step 7 Repeat the above steps until the distance of design vectors of the last two iterations is smaller than ε_3.

5 Numerical examples and discussions

5.1 A 10-bar truss

A well-known 10-bar aluminum truss (Au et al., 2003; Elishakoff et al., 1994) as shown in Fig. 2 is investigated. The length L of the horizontal and vertical bars is 360in. The joint 4 is subjected to a vertical load F_1, and the joint 2 is subjected to a vertical load F_2 and a horizontal load F_3. The density and Young's Modulus of the material are $\rho = 0.1\text{lb / in}^3$ and $E = 10^4$ksi, respectively. In order to obtain a minimum weight, the cross-sectional areas $A_j,\ j=1,2,...,10$ of the bars need to be optimized subjected to some stress and displacement constraints. The allowable stress in tension or compression is [55ksi, 55ksi] for bar 9, and the other bars are given a same allowable stress [30ksi, 40ksi]. The vertical displacement δ_2 of joint 2 should not exceed 5in. The loads $F_i,\ i=1,2,3$ are three uncertain parameters, and their midpoints are 100kips, 100kips and 400kips, respectively. The uncertainty domain can be represented by the following ellipsoidal convex model:

$$E(\boldsymbol{\delta},\theta)=\left\{\delta:\frac{\delta_{F_1}^2}{100}+\frac{\delta_{F_2}^2}{100}+\frac{\delta_{F_3}^2}{1600}\leq 1\right\} \tag{13}$$

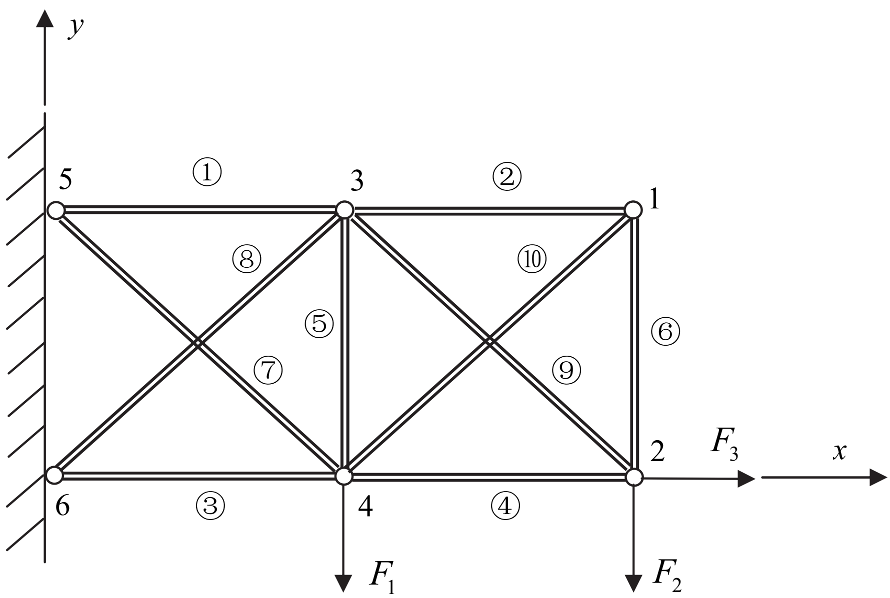

Figure 2: A 10-bar aluminium truss

Then a nonlinear interval optimization can be created:

$$\min_{\mathbf{A}} W(\mathbf{A})=\sum_{i=1}^{10}(\rho L_i A_i)=\rho L\left(\sum_{i=1}^{6}A_i+\sqrt{2}\sum_{i=7}^{10}A_i\right)$$

Subject to

$$\sigma_j(\mathbf{A},\mathbf{F})\leq\sigma_{i,allow}^{;}\ j=1,2,...,10$$

$\delta_2(\mathbf{A},\mathbf{F}) \leq 5\text{in}$

$$0.1 \leq A_i \leq 20,\ i = 1,2,...,10 \tag{14}$$

where W denotes the weight of the truss, and σ denotes the stress. In the above problem, all the stress and displacement constraints can be analytically obtained (Au et al., 2003; Elishakoff et al., 1994).

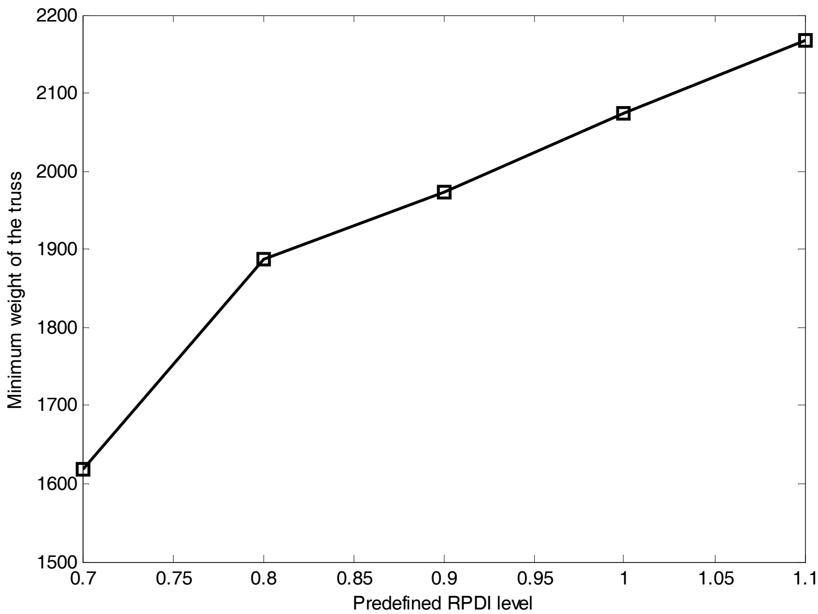

Figure 3: Relation of the predefined RPDI level and the truss weight

The involved parameters in our algorithm are set as: $\beta = 0.5$, $\alpha = 0.5$, $\varepsilon_1 = 0.01$, $\varepsilon_2 = 0.01$ and $\varepsilon_3 = 0.005$. For convenience of analysis, a same RPDI level is used for all the constraints, and the computational results under different RPDI levels are given in Tables 1-5. It can be found that the different RPDI levels for the interval constraints bring about different optimization results. For a relatively low RPDI level of 0.7, we can achieve a small truss weight of 1618.5lb, however, the constraints have relatively large possibilities to be violated. With increasing of the RPDI level, the optimized truss weight becomes larger while the reliability of all the constraints becomes better. For a relatively high RPDI level of 1.1, the truss weight increases to 2167.0lb, and under this case the stress and displacement intervals

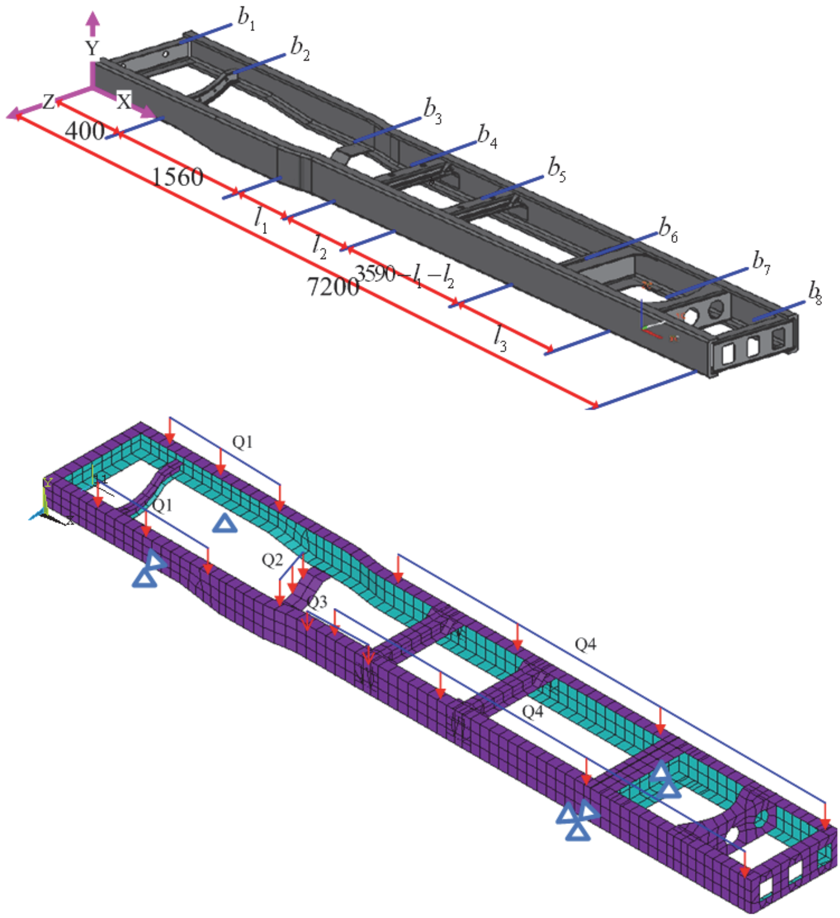

Figure 4: An automobile frame and its equivalent static model

caused by the uncertain forces have no possibilities to be beyond the allowable values. The relation between the RPDI level and the minimum truss weight is illustrated in Fig. 3. It can be found that for the above problem the truss weight behaves a nonlinear and monotonical relation with respect to the RPDI level of the constraints.

5.2 Design of an automobile frame

A practical automobile frame (Jiang et al., 2010) is investigated, as shown in Fig. 4. The frame is composed of two side beams and eight cross beams denoted by $b_i,\ i = 1,2,...,8$. $Q1$, $Q2$, $Q3$, and $Q4$ are four equivalent forces applied on the frame, and

Table 1: Optimization results under the RPDI level 1.1

Bar's number	Optimal design vector (in^2)	Interval of stress constraint (ksi)	RPDI of constraint	Interval of objective function (lb)
1	18.5625	[5.9241, 7.3551]	2.9810	[2166.9644, 2166.9644]
2	0.2751	[21.7359, 28.3475]	1.0995	
3	5.9975	[13.5121, 27.5874]	1.1002	
4	12.2712	[21.6793, 28.3383]	1.0997	
5	3.3047	[19.0844, 23.1976]	1.4820	
6	0.2751	[21.7359, 28.3475]	1.0995	
7	9.2725	[25.2562, 28.6594]	1.1000	
8	1.3736	[19.7124, 28.1548]	1.1000	
9	2.7004	[43.9916, 53.5323]	1.1538	
10	0.4472	[18.9103, 24.6624]	1.3388	
Interval of the displacement constraint is [3.0074in, 4.8193in], RPDI is 1.0997.				

Table 2: Optimization results under the RPDI level 1.0

Bar's number	Optimal design vector (in^2)	Interval of stress constraint (ksi)	RPDI of constraint	Interval of objective function (lb)
1	18.4844	[5.9434, 7.3873]	2.9759	[2073.1279, 2073.1279]
2	0.1936	[22.9359, 30.0060]	0.9997	
3	5.5120	[14.7074, 29.9967]	1.0001	
4	11.5344	[22.9036, 30.0032]	0.9998	
5	3.0583	[21.1615, 25.7078]	1.2951	
6	0.1936	[22.9359, 30.0060]	0.9997	
7	8.8593	[26.4441, 29.9998]	1.0000	
8	1.2904	[20.8610, 30.0002]	1.0000	
9	2.6815	[45.1170, 54.9584]	1.0042	
10	0.3285	[19.1117, 25.0030]	1.3145	
Interval of the displacement constraint is [3.0611in, 5.0007in], RPDI is 0.9996.				

Table 3: Optimization results under the RPDI level 0.9

Bar's number	Optimal design vector (in^2)	Interval of stress constraint (ksi)	RPDI of constraint	Interval of objective function (lb)
1	17.8438	[5.6300, 6.9780]	3.0287	[1973.8965, 1973.8965]
2	0.1000	[18.1221, 23.9624]	1.3812	
3	4.7473	[14.5615, 32.8280]	0.9000	
4	10.8059	[24.1542, 31.7607]	0.9000	
5	3.5675	[21.6706, 26.2108]	1.2606	
6	0.1000	[18.1221, 23.9624]	1.3812	
7	9.0000	[27.5447, 31.3847]	0.8999	
8	0.6680	[20.7260, 32.1455]	0.8998	
9	2.7171	[45.8970, 56.0115]	0.9000	
10	0.1066	[24.0313, 31.7760]	0.8999	
Interval of the displacement constraint is [2.9688in, 5.2046in], RPDI is 0.9085.				

Table 4: Optimization results under the RPDI level 0.8

Bar's number	Optimal design vector (in^2)	Interval of stress constraint (ksi)	RPDI of constraint	Interval of objective function (lb)
1	17.7656	[5.9944, 7.4626]	2.9652	[1887.1790, 1887.1790]
2	0.1000	[18.7686, 24.7406]	1.3293	
3	4.5291	[17.0370, 35.7483]	0.7998	
4	10.2133	[25.5623, 33.6106]	0.7999	
5	2.7259	[25.9852, 31.4553]	0.9059	
6	0.1000	[18.7686, 24.7406]	1.3293	
7	8.2785	[28.8753, 32.7816]	0.8000	
8	0.9651	[22.9957, 34.2602]	0.7997	
9	2.6650	[46.7602, 57.0643]	0.7997	
10	0.1035	[25.6446, 33.8044]	0.7905	
Interval of the displacement constraint is [3.0452in, 5.3144in], RPDI is 0.8614				

Table 5: Optimization results under the RPDI level 0.7

Bar's number	Optimal design vector (in^2)	Interval of stress constraint (ksi)	RPDI of constraint	Interval of objective function (lb)
1	12.1413	[7.5541, 9.2654]	2.7705	[1618.5467, 1618.5467]
2	0.1000	[18.8811, 25.1358]	1.2993	
3	3.6431	[15.5956, 40.4590]	0.7000	
4	9.6555	[27.0436, 35.5528]	0.7000	
5	3.0129	[28.6641, 34.8582]	0.7000	
6	0.1000	[18.8811, 25.1358]	1.2993	
7	8.7162	[29.8802, 34.3371]	0.7000	
8	0.1000	[21.6973, 37.8440]	0.7000	
9	2.6145	[47.6557, 58.1476]	0.7000	
10	0.1000	[26.7020, 35.5474]	0.7056	
Interval of the displacement constraint is [2.9065in, 5.7573in], RPDI is 0.7344				

the small triangle denotes the fixed displacement constraint. The density of the material is $\rho = 7.8 \times 10^{-3}\text{Kg/mm}^3$. The cross beams $b1$, $b2$, $b3$ and $b6$ are fixed, and the spans $l_i,\ i = 1,2,3$ need to be optimized to achieve a maximum stiffness of the frame in Y direction. In this problem, the Young's Modulus E and Poisson's ratio ν are both treated as uncertain parameters because of the manufacturing and measuring deviations, and their midpoints are 2.0×10^5Mpa and 0.3, respectively. Their uncertainty domain can be represented by the following convex model:

$$E(\boldsymbol{\delta}, \theta) = \left\{ \delta : \frac{\delta_E^2}{4 \times 10^8} + \frac{\delta_\upsilon^2}{9 \times 10^{-4}} \leq 1 \right\} \tag{15}$$

An interval optimization problem then can be created:

$$\min_{\mathbf{l}} d_{\max}(\mathbf{l}, E, \nu)$$

Subject to

$$\sigma_{\max}(\mathbf{l}, E, \nu) \leq [85\text{Mpa}, 88\text{Mpa}]$$

$$500\text{mm} \leq l_i \leq 1200\text{mm},\ i = 1,2,3 \tag{16}$$

where $d_{\max}$ denotes the maximum displacement in Y direction and it is used to describe the vertical stiffness of the frame. The constraint $\sigma_{\max}$ represents the maximum stress in the frame.

The involved parameters in our algorithm are set as: $\beta = 0.5$, $\alpha = 0.5$, $\varepsilon_1 = 0.05$, $\varepsilon_2 = 0.005$ and $\varepsilon_3 = 0.001$. The initial design vector and move limit vector are specified as [750, 750, 750] and [40, 40, 40], respectively. A RPDI level of 1.3 is applied to the stress constraint, and the optimization results are given in Table 6. An optimal design of [743.75mm, 783.75mm, 786.25mm] is obtained for the spans **l**, and the corresponding interval of the maximum vertical displacement is only $[1.582\text{mm}, 1.606\text{mm}]$ which implies a fine vertical stiffness of the automobile frame. The maximum stress interval in the frame at the optimal design is $[82.019\text{Mpa}, 83.454\text{Mpa}]$, which indicates a high reliability of the constraint because it has no possibility to exceed the allowable interval $[85\text{Mpa}, 88\text{Mpa}]$. On the other hand, for this practical engineering problem our algorithm converges only after 5 iterative steps.

Table 6: Optimization results of the automobile frame

RPDI level	Optimal design vector (mm)	Interval of objective function (mm)	Interval of constraint (Mpa)	f_d	RPDI of stress constraint	Number of steps
1.30	743.750 783.750 786.250	[1.582, 1.606]	[82.019, 83.454]	0.803	1.349	5

6 Conclusions

In this paper, a new method is developed to deal with the interval optimization problems existing dependence between the uncertain parameters. By using the ellipsoidal convex model to model the uncertainty domain, the possible combinations of the interval parameters will fall into a multi-dimensional ellipsoid rather than a multi-dimensional box. An algorithm is formulated to solve the involved nesting optimization, in which an optimal design can be obtained through creating and solving a sequence of approximate interval optimizations. Each approximate interval optimization can be changed to a conventional linear programming problem which we can easily deal with. An iterative mechanism is also constructed to update the design space and whereby ensure the optimization accuracy. In the first numerical example, opposite trends are observed between the reliability of the interval constraints and the objective function as the RPDI level varies. In the second numerical example, a practical automobile frame is optimized by using our method, and a fine design is obtained only after a small number of iterations.

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