 CMES, vol.66, no.1, pp.1-24, 2010

Unified Isoparametric 3D LagrangeFinite Elements

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Abstract: The paper presents unified approach to 3D isoparametric Lagrange brick, tetra, and prism finite elements. All shape functions, linear, quadratic and cubic, are depicted in one Cartesian orthogonal coordinate system x, y, z regardless of the type of element. This allows one to use a single transformation rule to calculate global derivatives and a second for integration. Proper numerical Gauss quadratures for these isoparametric elements in a unified approach are presented additionally.

Keywords: Finite element method Isoparametric elements Numerical integration

1 Introduction

The idea of mapped, isoparametric elements is now a standard part of any course of finite element method and is implemented in a majority of finite element codes. The isoparametric formulation of finite element stiffness matrices was presented originally in the papers of [Taig (1961),Irons (1966b),Irons and Zienkiewicz (1968),Irons (1966a)]. These works had an significant and immediate impact on the finite element research.

In brief, the concept of isoparametric elements bases on transforming basic elements with irregular shapes in a global, physical coordinate system into regular reference elements defined in a local system, and then operating on them. This allows us to calculate easily derivatives and integrals over complicated region of basic elements, polyhedrons often with curved edges, faces (for high order elements) [Bathe (2001), Bathe (1996), Hughes (1987), Kwon and Bang (1996), Zienkiewicz and Taylor (2005),Bäcklund (1978), Becker, Jox, and Meschke (2010)].

In the 3D isoparametric formulation the element displacements are interpolated in local Cartesian coordinates x, y, z in the same way as the geometry; i.e., we use for

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displacement

$$U(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)U_i, \tag{1}$$

$$V(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)V_i, \tag{2}$$

$$W(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)W_i \tag{3}$$

and similarly for geometry

$$X(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)X_i, \tag{4}$$

$$Y(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)Y_i, \tag{5}$$

$$Z(x,y,z) = \sum_{i=1}^{n} N_i(x,y,z)Z_i \tag{6}$$

We are looking for global derivatives of shape functions in respect of global coordinates X,Y,Z. Using the chain rule we get:

$$\frac{\partial N_i(x,y,z)}{\partial X} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial X} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial X}, \tag{7}$$

$$\frac{\partial N_i(x,y,z)}{\partial Y} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial Y} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial Y} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial Y}, \tag{8}$$

$$\frac{\partial N_i(x,y,z)}{\partial Z} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial Z} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial Z} + \frac{\partial N_i}{\partial z}\frac{\partial z}{\partial Z} \tag{9}$$

but these patterns are explicitly known only for some cases and we need general, but unfortunately, implicit ones. We can calculate local derivatives in local coordinate system x,y,z

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial x} & \frac{\partial Y}{\partial x} & \frac{\partial Z}{\partial x} \\ \frac{\partial X}{\partial y} & \frac{\partial Y}{\partial y} & \frac{\partial Z}{\partial y} \\ \frac{\partial X}{\partial z} & \frac{\partial Y}{\partial z} & \frac{\partial Z}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial X} \\ \frac{\partial N_i}{\partial Y} \\ \frac{\partial N_i}{\partial Z} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \frac{\partial N_i}{\partial X} \\ \frac{\partial N_i}{\partial Y} \\ \frac{\partial N_i}{\partial Z} \end{bmatrix} \tag{10}$$

and implicitly find global derivatives

$$\begin{bmatrix} \frac{\partial N_i}{\partial X} \\ \frac{\partial N_i}{\partial Y} \\ \frac{\partial N_i}{\partial Z} \end{bmatrix} = \mathbf{J}^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \end{bmatrix} \tag{11}$$

where $\mathbf{J}$ is *jacobian matrix* and is given by:

$$\mathbf{J} = \begin{bmatrix} \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} X_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} Y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial x} Z_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial y} X_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial y} Y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial y} Z_i \\ \sum_{i=1}^{n} \frac{\partial N_i}{\partial z} X_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial z} Y_i & \sum_{i=1}^{n} \frac{\partial N_i}{\partial z} Z_i \end{bmatrix} \tag{12}$$

The above formulae and transformations are valid for Cartesian orthogonal coordinate systems (global X,Y,Z and local x,y,z). In literature such orthogonal systems are used for brick family elements [Becker, Jox, and Meschke (2010)]. For tetrahedra family elements are used volume coordinates L_1, L_2, L_3, L_4 and for triangular prism family elements is used combination of area and Cartesian coordinates L_1, L_2, L_3, z [Lewis, Nithiarasu, and Seetharamu (2004) ,Bathe (1996), Zienkiewicz and Taylor (2005)]. It causes that each family of elements needs different rule for transformation (another coordinates) and for integration (another regions)[Bathe (1996)]. These nonorthogonal coordinates can be depicted in orthogonal ones [Zienkiewicz and Taylor (2005)] that allows one to use one unified general mapping and integration rule for all brick, tetrahedra and prism elements. Below are shown shape functions in local Cartesian orthogonal coordinate systems x,y,z for 3D *Lagrange* linear, quadratic and cubic brick, tetra and prism elements.

2 Brick elements- shape functions

2.1 linear elements

2.1.1 corner nodes

Node 1 (-1,-1,-1) $N_1 = \frac{1}{8}(1-x)(1-y)(1-z)$
Node 2 (1,-1,-1) $N_2 = \frac{1}{8}(1+x)(1-y)(1-z)$
Node 3 (1,1,-1) $N_3 = \frac{1}{8}(1+x)(1+y)(1-z)$
Node 4 (-1,1,-1) $N_4 = \frac{1}{8}(1-x)(1+y)(1-z)$
Node 5 (-1,-1,1) $N_5 = \frac{1}{8}(1-x)(1-y)(1+z)$
Node 6 (1,-1,1) $N_6 = \frac{1}{8}(1+x)(1-y)(1+z)$
Node 7 (1,1,1) $N_7 = \frac{1}{8}(1+x)(1+y)(1+z)$
Node 8 (-1,1,1) $N_8 = \frac{1}{8}(1-x)(1+y)(1+z)$

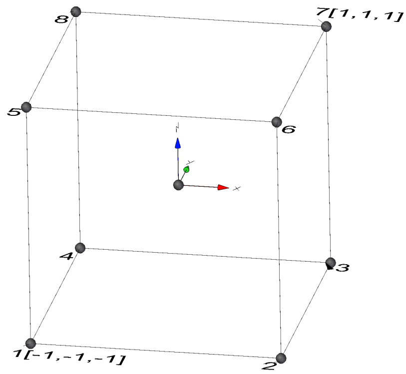

Figure 1: 8 node linear brick element

2.2 quadratic elements

2.2.1 corner nodes

Node 1 (-1,-1,-1) $N_1 = \frac{1}{8}x(x-1)y(y-1)z(z-1)$
Node 2 (1,-1,-1) $N_2 = \frac{1}{8}x(x+1)y(y-1)z(z-1)$
Node 3 (1,1,-1) $N_3 = \frac{1}{8}x(x+1)y(y+1)z(z-1)$
Node 4 (-1,1,-1) $N_4 = \frac{1}{8}x(x-1)y(y+1)z(z-1)$
Node 5 (-1,-1,1) $N_5 = \frac{1}{8}x(x-1)y(y-1)z(z+1)$
Node 6 (1,-1,1) $N_6 = \frac{1}{8}x(x+1)y(y-1)z(z+1)$
Node 7 (1,1,1) $N_7 = \frac{1}{8}x(x+1)y(y+1)z(z+1)$
Node 8 (-1,1,1) $N_8 = \frac{1}{8}x(x-1)y(y+1)z(z+1)$

2.2.2 mid-edge nodes

Node 9 (0,-1,-1) $N_9 = \frac{1}{4}(1-x^2)y(y-1)z(z-1)$
Node 10 (1,0,-1) $N_{10} = \frac{1}{4}x(x+1)(1-y^2)z(z-1)$
Node 11 (0,1,-1) $N_{11} = \frac{1}{4}(1-x^2)y(y+1)z(z-1)$
Node 12 (-1,0,-1) $N_{12} = \frac{1}{4}x(x-1)(1-y^2)z(z-1)$
Node 13 (0,-1,1) $N_{13} = \frac{1}{4}(1-x^2)y(y-1)z(z+1)$
Node 14 (1,0,1) $N_{14} = \frac{1}{4}x(x+1)(1-y^2)z(z+1)$
Node 15 (0,1,1) $N_{15} = \frac{1}{4}(1-x^2)y(y+1)z(z+1)$
Node 16 (-1,0,1) $N_{16} = \frac{1}{4}x(x-1)(1-y^2)z(z+1)$
Node 17 (-1,-1,0) $N_{17} = \frac{1}{4}x(x-1)y(y-1)(1-z^2)$
Node 18 (1,-1,0) $N_{18} = \frac{1}{4}x(x+1)y(y-1)(1-z^2)$

Node 19 (1,1,0) $N_{19} = \frac{1}{4}x(x+1)y(y+1)(1-z^2)$
Node 20 (-1,1,0) $N_{20} = \frac{1}{4}x(x-1)y(y+1)(1-z^2)$

2.2.3 mid-face nodes

Node 21 (-1,0,0) $N_{21} = \frac{1}{2}x(x-1)(1-y^2)(1-z^2)$
Node 22 (1,0,0) $N_{22} = \frac{1}{2}x(x+1)(1-y^2)(1-z^2)$
Node 23 (0,1,0) $N_{23} = \frac{1}{2}(1-x^2)y(y+1)(1-z^2)$
Node 24 (0,-1,0) $N_{24} = \frac{1}{2}(1-x^2)y(y-1)(1-z^2)$
Node 25 (0,0,-1) $N_{25} = \frac{1}{2}(1-x^2)(1-y^2)z(z-1)$
Node 26 (0,0,1) $N_{26} = \frac{1}{2}(1-x^2)(1-y^2)z(z+1)$

2.2.4 mid-volume nodes

Node 27 (0,0,0) $N_{27} = (1-x^2)(1-y^2)(1-z^2)$

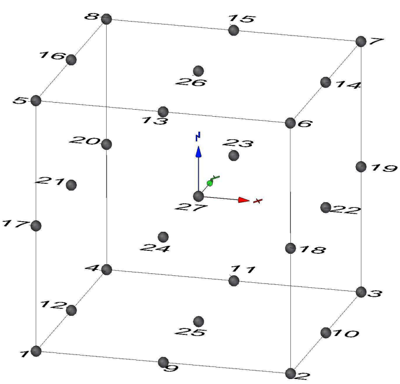

Figure 2: 27 node quadratic brick element

2.3 cubic elements

2.3.1 corner nodes

Node 1 (-1,-1,-1) $N_1 = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 2 (1,-1,-1) $N_2 = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 3 (1,1,-1) $N_3 = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 4 (-1,1,-1) $N_4 = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$

Node 5 (-1,-1,1) $N_5 = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 6 (1,-1,1) $N_6 = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 7 (1,1,1) $N_7 = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 8 (-1,1,1) $N_8 = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$

2.3.2 mid-edge nodes

Node 9 (-1,-1,-$\frac{1}{3}$) $N_9 = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 10 (1,-1,-$\frac{1}{3}$) $N_{10} = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 11 (1,1,-$\frac{1}{3}$) $N_{11} = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 12 (-1,1,-$\frac{1}{3}$) $N_{12} = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 13 (-1,-1,$\frac{1}{3}$) $N_{13} = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 14 (1,-1,$\frac{1}{3}$) $N_{14} = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 15 (1,1,$\frac{1}{3}$) $N_{15} = \frac{1}{16}(1+x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 16 (-1,1,$\frac{1}{3}$) $N_{16} = \frac{1}{16}(1-x)(9x^2-1)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 17 (-$\frac{1}{3}$,-1,-1) $N_{17} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 18 ($\frac{1}{3}$,-1,-1) $N_{18} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 19 (1,-$\frac{1}{3}$,-1) $N_{19} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 20 (1,$\frac{1}{3}$,-1) $N_{20} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 21 ($\frac{1}{3}$,1,-1) $N_{21} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 22 (-$\frac{1}{3}$,1,-1) $N_{22} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1-z)(9z^2-1)$
Node 23 (-1,$\frac{1}{3}$,-1) $N_{23} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 24 (-1,-$\frac{1}{3}$,-1) $N_{24} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 29 (-$\frac{1}{3}$,-1,1) $N_{29} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 30 ($\frac{1}{3}$,-1,1) $N_{30} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1-y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 31 (1,-$\frac{1}{3}$,1) $N_{31} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 32 (1,$\frac{1}{3}$,1) $N_{32} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 33 ($\frac{1}{3}$,1,1) $N_{33} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 34 (-$\frac{1}{3}$,1,1) $N_{34} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1+y)(9y^2-1)\frac{1}{16}(1+z)(9z^2-1)$
Node 35 (-1,$\frac{1}{3}$,1) $N_{35} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 36 (-1,-$\frac{1}{3}$,1) $N_{36} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1+z)(9z^2-1)$

2.3.3 mid-face nodes

Node 25 (-$\frac{1}{3}$,-$\frac{1}{3}$,-1) $N_{25} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 26 ($\frac{1}{3}$,-$\frac{1}{3}$,-1) $N_{26} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 27 ($\frac{1}{3}$,$\frac{1}{3}$,-1) $N_{27} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 28 (-$\frac{1}{3}$,$\frac{1}{3}$,-1) $N_{28} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1-z)(9z^2-1)$
Node 37 (-$\frac{1}{3}$,-$\frac{1}{3}$,1) $N_{37} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 38 ($\frac{1}{3}$,-$\frac{1}{3}$,1) $N_{38} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1-3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 39 ($\frac{1}{3}$,$\frac{1}{3}$,1) $N_{39} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1+z)(9z^2-1)$

Node 40 $(-\frac{1}{3},\frac{1}{3},1)$ $N_{40} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1+3y)\frac{1}{16}(1+z)(9z^2-1)$
Node 41 $(-\frac{1}{3},-1,-\frac{1}{3})$ $N_{41} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 42 $(\frac{1}{3},-1,-\frac{1}{3})$ $N_{42} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 43 $(1,-\frac{1}{3},-\frac{1}{3})$ $N_{43} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 44 $(1,\frac{1}{3},-\frac{1}{3})$ $N_{44} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 45 $(\frac{1}{3},1,-\frac{1}{3})$ $N_{45} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 46 $(-\frac{1}{3},1,-\frac{1}{3})$ $N_{46} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1-3z)$
Node 47 $(-1,\frac{1}{3},-\frac{1}{3})$ $N_{47} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 48 $(-1,-\frac{1}{3},-\frac{1}{3})$ $N_{48} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 53 $(-\frac{1}{3},-1,\frac{1}{3})$ $N_{53} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 54 $(\frac{1}{3},-1,\frac{1}{3})$ $N_{54} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1-y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 55 $(1,-\frac{1}{3},\frac{1}{3})$ $N_{55} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 56 $(1,\frac{1}{3},\frac{1}{3})$ $N_{56} = \frac{1}{16}(1+x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 57 $(\frac{1}{3},1,\frac{1}{3})$ $N_{57} = \frac{9}{16}(1-x^2)(1+3x)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 58 $(-\frac{1}{3},1,\frac{1}{3})$ $N_{58} = \frac{9}{16}(1-x^2)(1-3x)\frac{1}{16}(1+y)(9y^2-1)\frac{9}{16}(1-z^2)(1+3z)$
Node 59 $(-1,\frac{1}{3},\frac{1}{3})$ $N_{59} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 60 $(-1,-\frac{1}{3},\frac{1}{3})$ $N_{60} = \frac{1}{16}(1-x)(9x^2-1)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1+3z)$

2.3.4 mid-volume nodes

Node 49 $(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ $N_{49} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 50 $(\frac{1}{3},-\frac{1}{3},-\frac{1}{3})$ $N_{50} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 51 $(\frac{1}{3},\frac{1}{3},-\frac{1}{3})$ $N_{51} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 52 $(-\frac{1}{3},\frac{1}{3},-\frac{1}{3})$ $N_{52} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1-3z)$
Node 61 $(-\frac{1}{3},-\frac{1}{3},\frac{1}{3})$ $N_{61} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 62 $(\frac{1}{3},-\frac{1}{3},\frac{1}{3})$ $N_{62} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1-3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 63 $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ $N_{63} = \frac{9}{16}(1-x^2)(1+3x)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1+3z)$
Node 64 $(-\frac{1}{3},\frac{1}{3},\frac{1}{3})$ $N_{64} = \frac{9}{16}(1-x^2)(1-3x)\frac{9}{16}(1-y^2)(1+3y)\frac{9}{16}(1-z^2)(1+3z)$

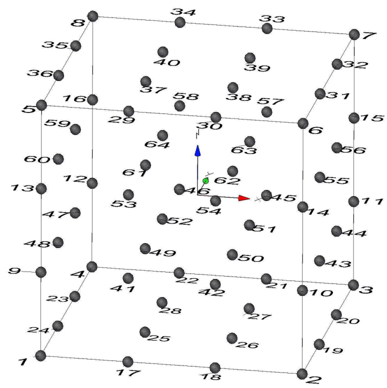

Figure 3: 64 node cubic brick element

3 Tetrahedron elements- shape functions

3.1 linear elements

3.1.1 corner nodes

Node 1 (0,0,0) $N_1 = 1 - x - y - z$
Node 2 (1,0,0) $N_2 = x$
Node 3 (0,1,0) $N_3 = y$
Node 4 (0,0,1) $N_4 = z$

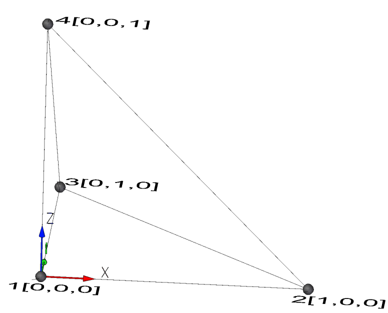

Figure 4: 4 node linear tetra element

3.2 quadratic elements

3.2.1 corner nodes

Node 1 (0,0,0) $N_1 = (1-x-y-z)(2(1-x-y-z)-1)$
Node 2 (1,0,0) $N_2 = x(2x-1)$
Node 3 (0,1,0) $N_3 = y(2y-1)$
Node 4 (0,0,1) $N_4 = z(2z-1)$

3.2.2 mid-edge nodes

Node 5 ($\frac{1}{2}$,0,0) $N_5 = 4x(1-x-y-z)$
Node 6 ($\frac{1}{2}$,$\frac{1}{2}$,0) $N_6 = 4yz$
Node 7 (0,$\frac{1}{2}$,0) $N_7 = 4y(1-x-y-z)$
Node 8 (0,0,$\frac{1}{2}$) $N_8 = 4z(1-x-y-z)$
Node 9 ($\frac{1}{2}$,0,$\frac{1}{2}$) $N_9 = 4xy$
Node 10 (0,$\frac{1}{2}$,$\frac{1}{2}$) $N_{10} = 4xz$

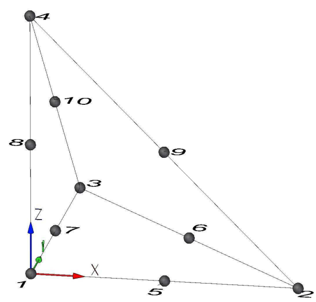

Figure 5: 10 node quadratic tetra element

3.3 cubic elements

3.3.1 corner nodes

Node 1 (0,0,0) $N_1 = \frac{1}{2}(3(1-x-y-z)-1)(3(1-x-y-z)-2)(1-x-y-z)$
Node 2 (1,0,0) $N_2 = \frac{1}{2}(3x-1)(3x-2)x$
Node 3 (0,1,0) $N_3 = \frac{1}{2}(3y-1)(3y-2)y$
Node 4 (0,0,1) $N_4 = \frac{1}{2}(3z-1)(3z-2)z$

3.3.2 mid-edge nodes

Node 5 ($\frac{1}{3}$,0,0) $N_5 = \frac{9}{2}(1-x-y-z)x(3(1-x-y-z)-1)$
Node 6 ($\frac{2}{3}$,0,0) $N_6 = \frac{9}{2}(1-x-y-z)x(3x-1)$
Node 7 ($\frac{2}{3}$,$\frac{1}{3}$,0) $N_7 = \frac{9}{2}xy(3x-1)$
Node 8 ($\frac{1}{3}$,$\frac{2}{3}$,0) $N_8 = \frac{9}{2}xy(3y-1)$
Node 9 (0,$\frac{2}{3}$,0) $N_9 = \frac{9}{2}(1-x-y-z)y(3y-1)$
Node 10 (0,$\frac{1}{3}$,0) $N_{10} = \frac{9}{2}(1-x-y-z)y(3(1-x-y-z)-1)$
Node 11 (0,0,$\frac{1}{3}$) $N_{11} = \frac{9}{2}(1-x-y-z)z(3(1-x-y-z)-1)$
Node 12 ($\frac{2}{3}$,0,$\frac{1}{3}$) $N_{12} = \frac{9}{2}xz(3x-1)$
Node 13 (0,$\frac{2}{3}$,$\frac{1}{3}$) $N_{13} = \frac{9}{2}yz(3y-1)$
Node 14 (0,0,$\frac{2}{3}$) $N_{14} = \frac{9}{2}(1-x-y-z)z(3z-1)$
Node 15 ($\frac{1}{3}$,0,$\frac{2}{3}$) $N_{15} = \frac{9}{2}xz(3z-1)$
Node 16 (0,$\frac{1}{3}$,$\frac{2}{3}$) $N_{16} = \frac{9}{2}yz(3z-1)$

3.3.3 mid-face nodes

Node 17 ($\frac{1}{3}$,0,$\frac{1}{3}$) $N_{17} = 27(1-x-y-z)xz$
Node 18 ($\frac{1}{3}$,$\frac{1}{3}$,$\frac{1}{3}$) $N_{18} = 27xyz$
Node 19 (0,$\frac{1}{3}$,$\frac{1}{3}$) $N_{19} = 27(1-x-y-z)yz$
Node 20 ($\frac{1}{3}$,$\frac{1}{3}$,0) $N_{20} = 27(1-x-y-z)xy$

4 Prism elements- shape functions

4.1 linear elements

4.1.1 corner nodes

Node 1 (0,0,-1) $N_1 = \frac{1}{2}(1-x-y)(1-z)$
Node 2 (1,0,-1) $N_2 = \frac{1}{2}x(1-z)$
Node 3 (0,1,-1) $N_3 = \frac{1}{2}y(1-z)$
Node 4 (0,0,1) $N_4 = \frac{1}{2}(1-x-y)(1+z)$
Node 5 (1,0,1) $N_5 = \frac{1}{2}x(1+z)$
Node 6 (0,1,1) $N_6 = \frac{1}{2}y(1+z)$

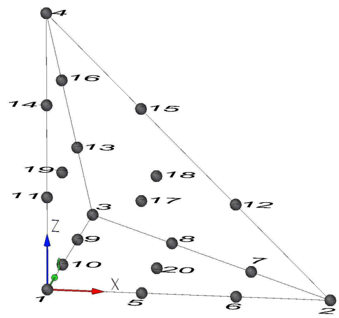

Figure 6: 20 node cubic tetra element

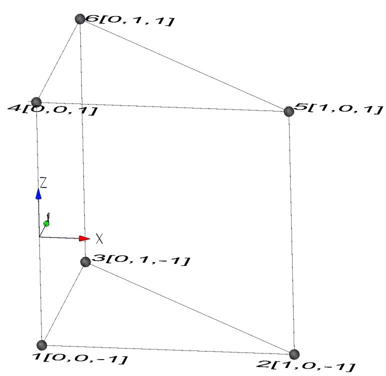

Figure 7: 6 node linear prism element

4.2 quadratic elements

4.2.1 corner nodes

Node 1 (0,0,-1) $N_1 = \frac{1}{2}(1-3x-3y+4xy+2x^2+2y^2)(z(z-1))$
Node 2 (1,0,-1) $N_2 = \frac{1}{2}(x(2x-1))(z(z-1))$
Node 3 (0,1,-1) $N_3 = \frac{1}{2}(y(2y-1))(z(z-1))$
Node 4 (0,0,1) $N_4 = \frac{1}{2}(1-3x-3y+4xy+2x^2+2y^2)(z(z+1))$
Node 5 (1,0,1) $N_5 = \frac{1}{2}(x(2x-1))(z(z+1))$
Node 6 (0,1,1) $N_6 = \frac{1}{2}(y(2y-1))(z(z+1))$

4.2.2 mid-edge nodes

Node 7 ($\frac{1}{2}$,0,-1) $N_7 = 2x(1-x-y)(z(z-1))$
Node 8 ($\frac{1}{2}$,$\frac{1}{2}$,-1) $N_8 = 2xy(z(z-1))$
Node 9 (0,$\frac{1}{2}$,-1) $N_9 = 2y(1-x-y)(z(z-1))$
Node 10 ($\frac{1}{2}$,0,1) $N_{10} = 2x(1-x-y)(z(z+1))$
Node 11 ($\frac{1}{2}$,$\frac{1}{2}$,1) $N_{11} = 2xy(z(z+1))$
Node 12 (0,$\frac{1}{2}$,1) $N_{12} = 2y(1-x-y)(z(z+1))$
Node 13 (0,0,0) $N_{13} = (1-3x-3y+4xy+2x^2+2y^2)(1-z^2)$
Node 14 (1,0,0) $N_{14} = x(2x-1)(1-z^2)$
Node 15 (0,1,0) $N_{15} = y(2y-1)(1-z^2)$

4.2.3 mid-face nodes

Node 16 ($\frac{1}{2}$,0,0) $N_{16} = 4x(1-x-y)(1-z^2)$
Node 17 ($\frac{1}{2}$,$\frac{1}{2}$,0) $N_{17} = 4xy(1-z^2)$
Node 18 (0,$\frac{1}{2}$,0) $N_{18} = 4y(1-x-y)(1-z^2)$

4.3 cubic elements

4.3.1 corner nodes

Node 1 (0,0,-1)
$N_1 = \frac{1}{32}(3(1-x-y)-1)(3(1-x-y)-2)(1-x-y)(1-z)(9z^2-1)$
Node 2 (1,0,-1) $N_2 = \frac{1}{32}(3x-1)(3x-2)x(1-z)(9z^2-1)$
Node 3 (0,1,-1) $N_3 = \frac{1}{32}(3y-1)(3y-2)y(1-z)(9z^2-1)$
Node 11 (0,0,1)
$N_{11} = \frac{1}{32}(3(1-x-y)-1)(3(1-x-y)-2)(1-x-y)(1+z)(9z^2-1)$
Node 12 (1,0,1) $N_{12} = \frac{1}{32}(3x-1)(3x-2)x(1+z)(9z^2-1)$
Node 13 (0,1,1) $N_{13} = \frac{1}{32}(3y-1)(3y-2)y(1+z)(9z^2-1)$

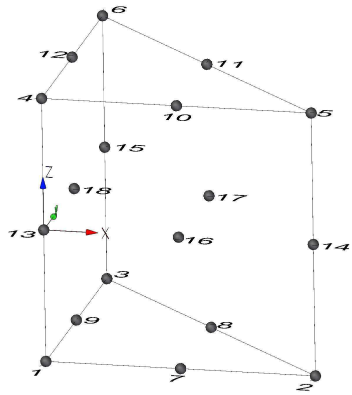

Figure 8: 18 node quadratic prism element

4.3.2 mid-edge nodes

Node 4 ($\frac{1}{3}$,0,-1) $N_4 = \frac{9}{32}(1-x-y)x(3(1-x-y)-1)(1-z)(9z^2-1)$
Node 5 ($\frac{2}{3}$,0,-1) $N_5 = \frac{9}{32}(1-x-y)x(3x-1)(1-z)(9z^2-1)$
Node 6 ($\frac{2}{3}$,$\frac{1}{3}$,-1) $N_6 = \frac{9}{32}xy(3x-1)(1-z)(9z^2-1)$
Node 7 ($\frac{1}{3}$,$\frac{2}{3}$,-1) $N_7 = \frac{9}{32}xy(3y-1)(1-z)(9z^2-1)$
Node 8 (0,$\frac{2}{3}$,-1) $N_8 = \frac{9}{32}(1-x-y)y(3y-1)(1-z)(9z^2-1)$
Node 9 (0,$\frac{1}{3}$,-1) $N_9 = \frac{9}{32}(1-x-y)y(3(1-x-y)-1)(1-z)(9z^2-1)$
Node 14 ($\frac{1}{3}$,0,1) $N_{14} = \frac{9}{32}(1-x-y)x(3(1-x-y)-1)(1+z)(9z^2-1)$
Node 15 ($\frac{2}{3}$,0,1) $N_{15} = \frac{9}{32}(1-x-y)x(3x-1)(1+z)(9z^2-1)$
Node 16 ($\frac{2}{3}$,$\frac{1}{3}$,1) $N_{16} = \frac{9}{32}xy(3x-1)(1+z)(9z^2-1)$
Node 17 ($\frac{1}{3}$,$\frac{2}{3}$,1) $N_{17} = \frac{9}{32}xy(3y-1)(1+z)(9z^2-1)$
Node 18 (0,$\frac{2}{3}$,1) $N_{18} = \frac{9}{32}(1-x-y)y(3y-1)(1+z)(9z^2-1)$
Node 19 (0,$\frac{1}{3}$,1) $N_{19} = \frac{9}{32}(1-x-y)y(3(1-x-y)-1)(1+z)(9z^2-1)$
Node 21 (0,0,-$\frac{1}{3}$)
$N_{21} = \frac{9}{32}(3(1-x-y)-1)(3(1-x-y)-2)(1-x-y)(1-z^2)(1-3z)$
Node 22 (1,0,-$\frac{1}{3}$) $N_{22} = \frac{9}{32}(3x-1)(3x-2)x(1-z^2)(1-3z)$
Node 23 (0,1,-$\frac{1}{3}$) $N_{23} = \frac{9}{32}(3y-1)(3y-2)y(1-z^2)(1-3z)$
Node 31 (0,0,$\frac{1}{3}$)
$N_{31} = \frac{9}{32}(3(1-x-y)-1)(3(1-x-y)-2)(1-x-y)(1-z^2)(1+3z)$
Node 32 (1,0,$\frac{1}{3}$) $N_{32} = \frac{9}{32}(3x-1)(3x-2)x(1-z^2)(1+3z)$
Node 33 (0,1,$\frac{1}{3}$) $N_{33} = \frac{9}{32}(3y-1)(3y-2)y(1-z^2)(1+3z)$

4.3.3 mid-face nodes

Node 10 $(\frac{1}{3},\frac{1}{3},\text{-}1)$ $N_{10}=\frac{27}{16}(1-x-y)xy(1-z)(9z^2-1)$
Node 20 $(\frac{1}{3},\frac{1}{3},1)$ $N_{20}=\frac{27}{16}(1-x-y)xy(1+z)(9z^2-1)$
Node 24 $(\frac{1}{3},0,\text{-}\frac{1}{3})$ $N_{24}=\frac{81}{32}(1-x-y)x(3(1-x-y)-1)(1-z^2)(1-3z)$
Node 25 $(\frac{2}{3},0,\text{-}\frac{1}{3})$ $N_{25}=\frac{81}{32}(1-x-y)x(3x-1)(1-z^2)(1-3z)$
Node 26 $(\frac{2}{3},\frac{1}{3},\text{-}\frac{1}{3})$ $N_{26}=\frac{81}{32}xy(3x-1)(1-z^2)(1-3z)$
Node 27 $(\frac{1}{3},\frac{2}{3},\text{-}\frac{1}{3})$ $N_{27}=\frac{81}{32}xy(3y-1)(1-z^2)(1-3z)$
Node 28 $(0,\frac{2}{3},\text{-}\frac{1}{3})$ $N_{28}=\frac{81}{32}(1-x-y)y(3y-1)(1-z^2)(1-3z)$
Node 29 $(0,\frac{1}{3},\text{-}\frac{1}{3})$ $N_{29}=\frac{81}{32}(1-x-y)y(3(1-x-y)-1)(1-z^2)(1-3z)$
Node 34 $(\frac{1}{3},0,\frac{1}{3})$ $N_{34}=\frac{81}{32}(1-x-y)x(3(1-x-y)-1)(1-z^2)(1+3z)$
Node 35 $(\frac{2}{3},0,\frac{1}{3})$ $N_{35}=\frac{81}{32}(1-x-y)x(3x-1)(1-z^2)(1+3z)$
Node 36 $(\frac{2}{3},\frac{1}{3},\frac{1}{3})$ $N_{36}=\frac{81}{32}xy(3x-1)(1-z^2)(1+3z)$
Node 37 $(\frac{1}{3},\frac{2}{3},\frac{1}{3})$ $N_{37}=\frac{81}{32}xy(3y-1)(1-z^2)(1+3z)$
Node 38 $(0,\frac{2}{3},\frac{1}{3})$ $N_{38}=\frac{81}{32}(1-x-y)y(3y-1)(1-z^2)(1+3z)$
Node 39 $(0,\frac{1}{3},\frac{1}{3})$ $N_{39}=\frac{81}{32}(1-x-y)y(3(1-x-y)-1)(1-z^2)(1+3z)$

4.3.4 mid-volume nodes

Node 30 $(\frac{1}{3},\frac{1}{3},\text{-}\frac{1}{3})$ $N_{30}=\frac{243}{16}(1-x-y)xy(1-z^2)(1-3z)$
Node 40 $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ $N_{40}=\frac{243}{16}(1-x-y)xy(1-z^2)(1+3z)$

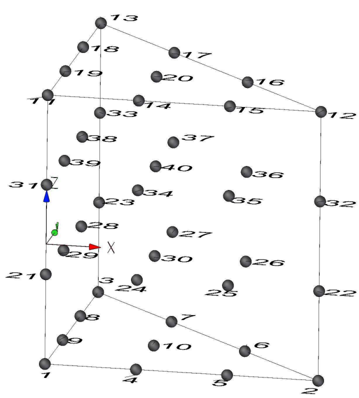

Figure 9: 40 node cubic prism element

5 Gauss numerical quadratures

We would like to calculate integral over the reference element i.e. brick, tetra or prism element. For brick we have:

$$\begin{aligned} I &= \int_{Brick} f(x,y,z)dxdydz \\ &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(x,y,z)dxdydz = \sum_{l=1}^{m} f(x_l,y_l,z_l)W_l \end{aligned} \tag{13}$$

for tetra:

$$\begin{aligned} I &= \int_{Tetra} f(x,y,z)dxdydz \\ &= \int_{0}^{1}\int_{0}^{1-x}\int_{0}^{1-x-y} f(x,y,z)dxdydz \approx \sum_{l=1}^{m} f(x_l,y_l,z_l)W_l \end{aligned} \tag{14}$$

for prism:

$$\begin{aligned} I &= \int_{Prism} f(x,y,z)dxdydz \\ &= \int_{-1}^{1}\int_{0}^{1}\int_{0}^{1-x} f(x,y,z)dxdydz \approx \sum_{l=1}^{m} f(x_l,y_l,z_l)W_l \end{aligned} \tag{15}$$

where m denotes the number of integration points. Order of numerical integration must preserve exact results for integrating polynomials, the same as from analytical and assure nonsingularity of stiffness matrix [Zienkiewicz and Taylor (2005)]. Such proper quadratures for brick, tetra and prism elements are presented below. A rich study of numerical integration of finite elements can be found in [Solin, Segeth, and Dolezel (2004)].

Table 1: Gauss quadrature on brick, order p = 0

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.0000000000	0.0000000000	0.0000000000	8.0000000000

Table 2: Gauss quadrature on brick, adequate for linear element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	-0.5773502692	-0.5773502692	0.5773502692	1.0000000000
2	0.5773502692	-0.5773502692	0.5773502692	1.0000000000
3	0.5773502692	0.5773502692	0.5773502692	1.0000000000
4	-0.5773502692	0.5773502692	0.5773502692	1.0000000000
5	-0.5773502692	-0.5773502692	-0.5773502692	1.0000000000
6	0.5773502692	-0.5773502692	-0.5773502692	1.0000000000
7	0.5773502692	0.5773502692	-0.5773502692	1.0000000000
8	-0.5773502692	0.5773502692	-0.5773502692	1.0000000000

Table 3: Gauss quadrature on brick, adequate for quadratic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	-0.7745966692	-0.7745966692	-0.7745966692	0.1714677641
2	0.7745966692	-0.7745966692	-0.7745966692	0.1714677641
3	-0.7745966692	0.7745966692	-0.7745966692	0.1714677641
4	0.7745966692	0.7745966692	-0.7745966692	0.1714677641
5	-0.7745966692	-0.7745966692	0.7745966692	0.1714677641
6	0.7745966692	-0.7745966692	0.7745966692	0.1714677641
7	-0.7745966692	0.7745966692	0.7745966692	0.1714677641
8	0.7745966692	0.7745966692	0.7745966692	0.1714677641
9	0.0000000000	-0.7745966692	-0.7745966692	0.2743484225
10	-0.7745966692	0.0000000000	-0.7745966692	0.2743484225
11	0.7745966692	0.0000000000	-0.7745966692	0.2743484225
12	0.0000000000	0.7745966692	-0.7745966692	0.2743484225
13	-0.7745966692	-0.7745966692	0.0000000000	0.2743484225
14	0.7745966692	-0.7745966692	0.0000000000	0.2743484225
15	-0.7745966692	0.7745966692	0.0000000000	0.2743484225
16	0.7745966692	0.7745966692	0.0000000000	0.2743484225
17	0.0000000000	-0.7745966692	0.7745966692	0.2743484225
18	-0.7745966692	0.0000000000	0.7745966692	0.2743484225
19	0.7745966692	0.0000000000	0.7745966692	0.2743484225
20	0.0000000000	0.7745966692	0.7745966692	0.2743484225
21	0.0000000000	0.0000000000	-0.7745966692	0.4389574760
22	0.0000000000	-0.7745966692	0.0000000000	0.4389574760
23	-0.7745966692	0.0000000000	0.0000000000	0.4389574760
24	0.7745966692	0.0000000000	0.0000000000	0.4389574760

25	0.0000000000	0.7745966692	0.0000000000	0.4389574760
26	0.0000000000	0.0000000000	0.7745966692	0.4389574760
27	0.0000000000	0.0000000000	0.0000000000	0.7023319616

Table 4: Gauss quadrature on brick, adequate for cubic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	-0.8611363116	-0.8611363116	-0.8611363116	0.0420914775
2	0.8611363116	-0.8611363116	-0.8611363116	0.0420914775
3	-0.8611363116	0.8611363116	-0.8611363116	0.0420914775
4	0.8611363116	0.8611363116	-0.8611363116	0.0420914775
5	-0.8611363116	-0.8611363116	0.8611363116	0.0420914775
6	0.8611363116	-0.8611363116	0.8611363116	0.0420914775
7	-0.8611363116	0.8611363116	0.8611363116	0.0420914775
8	0.8611363116	0.8611363116	0.8611363116	0.0420914775
9	-0.3399810436	-0.8611363116	-0.8611363116	0.0789115158
10	0.3399810436	-0.8611363116	-0.8611363116	0.0789115158
11	-0.8611363116	-0.3399810436	-0.8611363116	0.0789115158
12	0.8611363116	-0.3399810436	-0.8611363116	0.0789115158
13	-0.8611363116	0.3399810436	-0.8611363116	0.0789115158
14	0.8611363116	0.3399810436	-0.8611363116	0.0789115158
15	-0.3399810436	0.8611363116	-0.8611363116	0.0789115158
16	0.3399810436	0.8611363116	-0.8611363116	0.0789115158
17	-0.8611363116	-0.8611363116	-0.3399810436	0.0789115158
18	0.8611363116	-0.8611363116	-0.3399810436	0.0789115158
19	-0.8611363116	0.8611363116	-0.3399810436	0.0789115158
20	0.8611363116	0.8611363116	-0.3399810436	0.0789115158
21	-0.8611363116	-0.8611363116	0.3399810436	0.0789115158
22	0.8611363116	-0.8611363116	0.3399810436	0.0789115158
23	-0.8611363116	0.8611363116	0.3399810436	0.0789115158
24	0.8611363116	0.8611363116	0.3399810436	0.0789115158
25	-0.3399810436	-0.8611363116	0.8611363116	0.0789115158
26	0.3399810436	-0.8611363116	0.8611363116	0.0789115158
27	-0.8611363116	-0.3399810436	0.8611363116	0.0789115158
28	0.8611363116	-0.3399810436	0.8611363116	0.0789115158

29	-0.8611363116	0.3399810436	0.8611363116	0.0789115158
30	0.8611363116	0.3399810436	0.8611363116	0.0789115158
31	-0.3399810436	0.8611363116	0.8611363116	0.0789115158
32	0.3399810436	0.8611363116	0.8611363116	0.0789115158
33	-0.3399810436	-0.3399810436	-0.8611363116	0.1479403361
34	0.3399810436	-0.3399810436	-0.8611363116	0.1479403361
35	-0.3399810436	0.3399810436	-0.8611363116	0.1479403361
36	0.3399810436	0.3399810436	-0.8611363116	0.1479403361
37	-0.3399810436	-0.8611363116	-0.3399810436	0.1479403361
38	0.3399810436	-0.8611363116	-0.3399810436	0.1479403361
39	-0.8611363116	-0.3399810436	-0.3399810436	0.1479403361
40	0.8611363116	-0.3399810436	-0.3399810436	0.1479403361
41	-0.8611363116	0.3399810436	-0.3399810436	0.1479403361
42	0.8611363116	0.3399810436	-0.3399810436	0.1479403361
43	-0.3399810436	0.8611363116	-0.3399810436	0.1479403361
44	0.3399810436	0.8611363116	-0.3399810436	0.1479403361
45	-0.3399810436	-0.8611363116	0.3399810436	0.1479403361
46	0.3399810436	-0.8611363116	0.3399810436	0.1479403361
47	-0.8611363116	-0.3399810436	0.3399810436	0.1479403361
48	0.8611363116	-0.3399810436	0.3399810436	0.1479403361
49	-0.8611363116	0.3399810436	0.3399810436	0.1479403361
50	0.8611363116	0.3399810436	0.3399810436	0.1479403361
51	-0.3399810436	0.8611363116	0.3399810436	0.1479403361
52	0.3399810436	0.8611363116	0.3399810436	0.1479403361
53	-0.3399810436	-0.3399810436	0.8611363116	0.1479403361
54	0.3399810436	-0.3399810436	0.8611363116	0.1479403361
55	-0.3399810436	0.3399810436	0.8611363116	0.1479403361
56	0.3399810436	0.3399810436	0.8611363116	0.1479403361
57	-0.3399810436	-0.3399810436	-0.3399810436	0.2773529670
58	0.3399810436	-0.3399810436	-0.3399810436	0.2773529670
59	-0.3399810436	0.3399810436	-0.3399810436	0.2773529670
60	0.3399810436	0.3399810436	-0.3399810436	0.2773529670
61	-0.3399810436	-0.3399810436	0.3399810436	0.2773529670
62	0.3399810436	-0.3399810436	0.3399810436	0.2773529670
63	-0.3399810436	0.3399810436	0.3399810436	0.2773529670
64	0.3399810436	0.3399810436	0.3399810436	0.2773529670

Table 5: Gauss quadrature on tetra, order p = 0

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.25000000000	0.2500000000	0.2500000000	0.1666666666

Table 6: Gauss quadrature on tetra, proper to linear element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.5854101966	0.1381966011	0.1381966011	0.0416666667
2	0.1381966011	0.5854101966	0.1381966011	0.0416666667
3	0.1381966011	0.1381966011	0.5854101966	0.0416666667
4	0.1381966011	0.1381966011	0.1381966011	0.0416666667

Table 7: Gauss quadrature on tetra, proper to quadratic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	1.0000000000	0.0000000000	0.0000000000	0.0027777778
2	0.0000000000	1.0000000000	0.0000000000	0.0027777778
3	0.0000000000	0.0000000000	1.0000000000	0.0027777778
4	0.0000000000	0.0000000000	0.0000000000	0.0027777778
5	0.5000000000	0.5000000000	0.0000000000	0.0111111111
6	0.0000000000	0.5000000000	0.5000000000	0.0111111111
7	0.0000000000	0.0000000000	0.5000000000	0.0111111111
8	0.5000000000	0.0000000000	0.0000000000	0.0111111111
9	0.0000000000	0.5000000000	0.0000000000	0.0111111111
10	0.5000000000	0.0000000000	0.5000000000	0.0111111111
11	0.2500000000	0.2500000000	0.2500000000	0.0888888889

Table 8: Gauss quadrature on tetra, proper to cubic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.7716429021	0.0761190326	0.0761190326	0.0083956325
2	0.0761190326	0.7716429021	0.0761190326	0.0083956325
3	0.0761190326	0.0761190326	0.7716429021	0.0083956325
4	0.0761190326	0.0761190326	0.0761190326	0.0083956325
5	0.1197005278	0.0718316453	0.4042339135	0.0110903448
6	0.4042339135	0.1197005278	0.0718316453	0.0110903448
7	0.4042339135	0.4042339135	0.1197005278	0.0110903448
8	0.0718316453	0.4042339135	0.4042339135	0.0110903448
9	0.1197005278	0.4042339135	0.0718316453	0.0110903448
10	0.4042339135	0.1197005278	0.4042339135	0.0110903448
11	0.0718316453	0.4042339135	0.1197005278	0.0110903448
12	0.4042339135	0.0718316453	0.4042339135	0.0110903448
13	0.1197005278	0.4042339135	0.4042339135	0.0110903448
14	0.0718316453	0.1197005278	0.4042339135	0.0110903448
15	0.4042339135	0.0718316453	0.1197005278	0.0110903448
16	0.4042339135	0.4042339135	0.0718316453	0.0110903448

Table 9: Gauss quadrature on prism, order p = 0

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.3333333333	0.3333333333	0.0000000000	2.0000000000

Table 10: Gauss quadrature on prism, proper to linear element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.5000000000	0.0000000000	-0.5773502692	0.1666666667
2	0.0000000000	0.5000000000	-0.5773502692	0.1666666667
3	0.5000000000	0.5000000000	-0.5773502692	0.1666666667
4	0.5000000000	0.0000000000	0.5773502692	0.1666666667
5	0.0000000000	0.5000000000	0.5773502692	0.1666666667
6	0.5000000000	0.5000000000	0.5773502692	0.1666666667

Table 11: Gauss quadrature on prism, proper to quadratic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.3333333333	0.3333333333	-0.7745696692	0.0625000000
2	0.0597158718	0.4701420641	-0.7745696692	0.0367761536
3	0.4701420641	0.0597158718	-0.7745696692	0.0367761536
4	0.4701420641	0.4701420641	-0.7745696692	0.0367761536
5	0.7974269854	0.1012865073	-0.7745696692	0.0349831057
6	0.1012865073	0.7974269854	-0.7745696692	0.0349831057
7	0.1012865073	0.1012865073	-0.7745696692	0.0349831057
8	0.3333333333	0.3333333333	0.0000000000	0.1000000000
9	0.0597158718	0.4701420641	0.0000000000	0.0588418457
10	0.4701420641	0.0597158718	0.0000000000	0.0588418457
11	0.4701420641	0.4701420641	0.0000000000	0.0588418457
12	0.7974269854	0.1012865073	0.0000000000	0.0559729691
13	0.1012865073	0.7974269854	0.0000000000	0.0559729691
14	0.1012865073	0.1012865073	0.0000000000	0.0559729691
15	0.3333333333	0.3333333333	0.7745696692	0.0625000000
16	0.0597158718	0.4701420641	0.7745696692	0.0367761536
17	0.4701420641	0.0597158718	0.7745696692	0.0367761536
18	0.4701420641	0.4701420641	0.7745696692	0.0367761536
19	0.7974269854	0.1012865073	0.7745696692	0.0349831057
20	0.1012865073	0.7974269854	0.7745696692	0.0349831057
21	0.1012865073	0.1012865073	0.7745696692	0.0349831057

Table 12: Gauss quadrature on prism, proper to cubic element

Point #	x-Coordinate	y-Coordinate	z-Coordinate	Weight
1	0.0630890131	0.0630890131	-0.8611363116	0.0088433238
2	0.0630890131	0.8738219738	-0.8611363116	0.0088433238
3	0.8738219738	0.0630890131	-0.8611363116	0.0088433238
4	0.0630890131	0.0630890131	0.8611363116	0.0088433238
5	0.0630890131	0.8738219738	0.8611363116	0.0088433238
6	0.8738219738	0.0630890131	0.8611363116	0.0088433238
7	0.3103524446	0.6365025043	-0.8611363116	0.0144100739
8	0.6365025043	0.0531450510	-0.8611363116	0.0144100739
9	0.0531450510	0.3103524446	-0.8611363116	0.0144100739
10	0.3103524446	0.0531450510	-0.8611363116	0.0144100739
11	0.6365025043	0.3103524446	-0.8611363116	0.0144100739
12	0.0531450510	0.6365025043	-0.8611363116	0.0144100739
13	0.3103524446	0.6365025043	0.8611363116	0.0144100739
14	0.6365025043	0.0531450510	0.8611363116	0.0144100739
15	0.0531450510	0.3103524446	0.8611363116	0.0144100739
16	0.3103524446	0.0531450510	0.8611363116	0.0144100739
17	0.6365025043	0.3103524446	0.8611363116	0.0144100739
18	0.0531450510	0.6365025043	0.8611363116	0.0144100739
19	0.0630890131	0.0630890131	-0.3399810436	0.0165791301
20	0.0630890131	0.8738219738	-0.3399810436	0.0165791301
21	0.8738219738	0.0630890131	-0.3399810436	0.0165791301
22	0.0630890131	0.0630890131	0.3399810436	0.0165791301

23	0.0630890131	0.8738219738	0.3399810436	0.0165791301
24	0.8738219738	0.0630890131	0.3399810436	0.0165791301
25	0.2492867410	0.2492867410	-0.8611363116	0.0203123365
26	0.2492867410	0.5014265180	-0.8611363116	0.0203123365
27	0.5014265180	0.2492867410	-0.8611363116	0.0203123365
28	0.2492867410	0.2492867410	0.8611363116	0.0203123365
29	0.2492867410	0.5014265180	0.8611363116	0.0203123365
30	0.5014265180	0.2492867410	0.8611363116	0.0203123365
31	0.3103524446	0.6365025043	-0.3399810436	0.0270154634
32	0.6365025043	0.0531450510	-0.3399810436	0.0270154634
33	0.0531450510	0.3103524446	-0.3399810436	0.0270154634
34	0.3103524446	0.0531450510	-0.3399810436	0.0270154634
35	0.6365025043	0.3103524446	-0.3399810436	0.0270154634
36	0.0531450510	0.6365025043	-0.3399810436	0.0270154634
37	0.3103524446	0.6365025043	0.3399810436	0.0270154634
38	0.6365025043	0.0531450510	0.3399810436	0.0270154634
39	0.0531450510	0.3103524446	0.3399810436	0.0270154634
40	0.3103524446	0.0531450510	0.3399810436	0.0270154634
41	0.6365025043	0.3103524446	0.3399810436	0.0270154634
42	0.0531450510	0.6365025043	0.3399810436	0.0270154634
43	0.2492867410	0.2492867410	-0.3399810436	0.0380808029
44	0.2492867410	0.5014265180	-0.3399810436	0.0380808029
45	0.5014265180	0.2492867410	-0.3399810436	0.0380808029
46	0.2492867410	0.2492867410	0.3399810436	0.0380808029
47	0.2492867410	0.5014265180	0.3399810436	0.0380808029
48	0.5014265180	0.2492867410	0.3399810436	0.0380808029

6 Singular isoparametric elements

Using finite element method for modelling of cracks in fracture mechanics was the subject of many review publications [Sakakibara (2008)], books [Atluri (1983)] and monographs [Atluri and Nakagaki (1986),Seweryn (2003)] and literature cited therein. The use of classical isoparametric elements of arbitrary order in the enriched element approach and in Extended Finite Element Method (XFEM) is the natural and obvious [Sakakibara (2008)] and we concentrate on the so-called singular isoparametric elements. Consider at a glance, without loss of generality, a classical one-dimensional element [Atluri and Nakagaki (1986),Horvath (1994)]. Nodes in such element are equidistant and their positions can be described by the relation $x_i = x_0 + (x_M - x_0)(i/r)$ or $(x_i - x_0) = (x_M - x_0)(i/r)$, where i is nodal index, x_0, x_M are positions of the first and last node and r is the order of polynomial.We can require that the nodes will be spaced by a more general relation

$$(x_i - x_0) = (x_M - x_0)(i/r)^t \tag{16}$$

where $t \leq r$ is positive integer and hence for the displacement $\frac{\partial u(x)}{\partial x}$ varies and has a singularity of the type $R^{\frac{1-t}{t}}$. For $t = 1$ we have classical elements, for other t we have tth-order singular elements[Seweryn (2003)].

By analogy with paragraphs 2-4, using the shape functions for brick, tetra and prism elements, and the general transformation relation 16, we can create a whole family of isoparametric tth-order singular elements with $R^{\frac{1-t}{t}}$ type singularity. For tetra elements, bearing in mind that ($t \leq r$), we get rth-order interpolation tth-order singularity tetra elements, useful in representing point one. For line singularities we can transform prism elements analogically to get rth-order interpolation tth-order singularity prisms. Another approach might be to apply the so-called collapsed elements where corresponding pairs of nodes are grouped together as it transforms prism elements to the tetras, and bricks to prisms [Grummitt and Baker (1999),Seweryn (2003)].Explicit representation of shape functions for the entire family of singular elements is the material of a separate paper.

7 Summary

Representing shape functions for brick, tetra and prism isoparametric elements in one Cartesian orthogonal coordinate systems x, y, z allows to utilize one general transformation rule (10-11) to determine global derivatives and one similar for integration (14-16). Gauss quadratures for all types of elements can have the same structure, three coordinates in local systems x, y, z and weight. One transformation rule, integration rule, and the same structure of quadratures simplifies and unifies the structure of finite element codes and facilitate adding new elements.

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