Solvability of an Inverse Problem for the Kinetic Equation and a Symbolic Algorithm

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Abstract: In this work, we derive the solvability conditions for an inverse problem for the kinetic equation and develop a new symbolic algorithm to obtain the approximate solution of the problem. The computational experiments show that proposed method provides highly accurate numerical solutions even subjecting to a large noise in the given data.

Keywords: Inverse Problem, Kinetic Equation, Galerkin Method, Symbolic Computation.

1 Introduction

Kinetic equations (KE) describe the evolution of many-body systems such as gases, plasmas and clusters of stars. They play a crucial role in many applications, ranging from gas dynamics to fusion plasma, from astrophysics to physical chemistry, from traffic flow to semiconductors [Cercignani (1975); Liboff (1979); Alexeev (1982); Lancellotti and Kiessling (2001)]. Inverse problems (IP) for KE appear to be important both from theoretical and practical points of view. The physical interpretation of these problems consists in finding particle interaction forces, scattering indicatrices, radiation sources and other physical parameters. Interesting results in this field are presented in [Amirov (2001), Anikonov (2001)].

In this paper, we deal with the IP of simultaneous determining of the solution u and the right-hand side λ of the following kinetic equation in the domain Ω:

$$Lu \equiv \sum_{i=1}^{n} \left(H_{p_i} u_{x_i} - H_{x_i} u_{p_i} \right) + \beta u + \int_{G} K\left(x, p, p'\right) u\left(x, p'\right) dp' = \lambda\left(x\right) + f, \qquad (1)$$

where Ω is a domain in the Euclidean space $\mathbb{R}^{2n}$ ($n \geq 1$) and for the variables $(x, p) \in \Omega$, it is assumed that $x \in D$, $p \in G$, where D, $G \subset \mathbb{R}^n$, ∂D, $\partial G \in C^3$,

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$\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 = \partial D \times G$, $\Gamma_2 = D \times \partial G$ and $\overline{\Gamma}_1$, $\overline{\Gamma}_2$ are the closures of Γ_1, Γ_2, respectively. Here $H(x,p)$ is the Hamiltonian, $\beta(x,p)$ is the absorption, $K(x,p,p')$ is a given function called scattering kernel and $\lambda(x)$ is a source function. In applications, u represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of the point (x,p), and $\nabla_x H$ is the force acting on a particle.

We present the solvability conditions for an overdetermined IP for equation (1) and develop a new symbolic algorithm to compute approximate solution of the problem. Some computational experiments are performed using noisy data with different noise levels which show that the proposed method gives efficient and reliable results. In [Amirov et al (2009)], an IP for the KE with a scattering term was considered and a numerical approximation method based on the finite difference method was developed. In this paper, we take into account the absorption term beside the collision process and propose a new and more effective approximation method based on Galerkin method to obtain approximate analytical solution of the IP.

In recent years, there has been a growing interest in developing fast, robust and efficient algorithms for solving inverse problems in science and engineering. In [Kabanikhin and Lorenzi (1999)], some numerical algorithms are studied and compared for the inverse problems related to wave propagation. Liu and Atluri (2008a) formulated the inverse Cauchy problem of Laplace equation in a rectangle as an optimization problem, and applied a fictitious time integration method to solve an algebraic equations system to obtain the data on an unspecified portion of boundary. In [Liu and Atluri (2008b)], a novel method was proposed for computing the unknown potential function, the unknown impedance function, or the unknown weighting function in the Sturm-Liouville operator, when the discrete eigenvalues are specified. They employed a $SL(2,\mathbb{R})$ Lie-group shooting method (LGSM), combined with the use of Fictitious Time Integration Method (FTIM), for solving the inverse Sturm-Liouville problems. Beilina and Klibanov (2008) developed a globally convergent numerical method for a multidimensional coefficient inverse problem for a hyperbolic PDE. On each iterative step, they solve a Dirichlet boundary value problem for a second-order elliptic equation.

2   Formulation of the Problem

Problem 1 (Overdetermined Inverse Problem) *Find a pair of functions (u,λ) from kinetic equation (1), provided that the functions $H(x,p)$, $\beta(x,p)$, $K(x,p,p')$, $f(x,p)$ and the trace of the solution $u(x,p)$ of equation (1) on the boundary $\partial\Omega$ are known.*

In the theory of IP, if the number of free variables in the additional data exceeds the number of free variables in the unknown coefficient or unknown right hand side of the equation ($\lambda(x)$), then the problem is called overdetermined. Problem 1 is overdetermined in this sense for dimension $n \geq 2$. On the other hand, for $n = 1$, IP for KE and integral geometry problems (IGP) are closely connected, i.e., many problems of integral geometry can be reduced to the corresponding IP for KE, and vice versa. And here, the underlying operator of the related IGP for Problem 1 is compact and its inverse operator is unbounded. Therefore, it is impossible to prove general existence results. So, the initial data for these problems can not be arbitrary; they should satisfy some "solvability conditions" which are difficult to establish [Amirov (2001)]. The main difficulty in studying the solvability of such IP for KE is their overdeterminancy.

3 Existence, Uniqueness and Stability of the Solution

The method to be used here for proving the solvability of Problem 1 can be outlined as follows: using some extension of the class of unknown functions λ, overdetermined problem is replaced by a determined one. This is achieved by assuming the unknown function λ depends not only upon the space variables x, but also upon the direction p in a specific way, such that the sufficiently smooth functions λ, depending only on x, satisfy the equation $\widehat{L}(\lambda) = 0$. In other words, we immerse equation (1) into a system of equations (2) and (3) below in which a new unknown function $\widetilde{\lambda}$ is involved and $\widetilde{\lambda} = \widetilde{\lambda}(x, p)$. Here, p-dependence of the function $\widetilde{\lambda}(x, p)$ is via a nontrivial manner, because this function is assumed to satisfy the new equation (3).

Problem 2 (Determined Problem) *Find the functions $\widetilde{u}(x, p)$ and $\widetilde{\lambda}(x, p)$ defined in Ω that satisfy the equations*

$$L\widetilde{u} = \widetilde{\lambda}(x, p) + f, \tag{2}$$

$$\widehat{L}(\widetilde{\lambda}) = 0, \tag{3}$$

and the boundary condition $\widetilde{u}|_{\partial\Omega} = \widetilde{u}_0$, where $\widehat{L} = \sum_{i=1}^{n} \dfrac{\partial^2}{\partial x_i \partial p_i}$.

Equation (3) is satisfied in generalized functions sense, i.e., $\left\langle \widetilde{\lambda}, \left(\widehat{L}\right)^* \eta \right\rangle = 0$ for any $\eta \in C_0^\infty(\Omega)$, where $\left(\widehat{L}\right)^*$ is the conjugate operator to $\widehat{L}$ in the Lagrange sense, $\langle ., . \rangle$ is the scalar product in $L_2(\Omega)$, and $C_0^\infty(\Omega)$ denotes the set of all functions defined in Ω which have continuous partial derivatives of order up to all $k < \infty$, whose supports are compact subsets of Ω.

Suppose that, a priori we know a function u_0^e to be the exact data of Problem 1 related to a function λ depending only on x. Then, utilizing u_0^e, we can construct a solution $\tilde{\lambda}$ to Problem 1. By uniqueness of a solution, $\tilde{\lambda}$ coincides with $\lambda(x)$. If we know the approximate data $\widetilde{u}_0^a$ with $\left\| u_0^e - \widetilde{u}_0^a \right\|_{H^3(\partial\Omega)} \leq \varepsilon$, we can construct an approximate solution $\widetilde{\lambda}^a(x, \varphi)$ such that $\left\| \lambda - \widetilde{\lambda}^a \right\|_{L_2(\Omega)} \leq C\varepsilon$. Recall that, if λ depends only on x and $\widetilde{u}_0^a$ does not satisfy the "solvability conditions", the solution λ^a depending only x does not exist. Here the data are specified on $\partial\Omega$ and $C > 0$ is not dependent on u_0^e and $\widetilde{u}_0^a$. In other words, we construct a regularising procedure for Problem 1.

The proposed method of solvability of Problem 1 leads to a Dirichlet problem for the third order equation of the form $A\widetilde{u} \equiv \widehat{L}L\widetilde{u} = \mathscr{F}$. Here the equation $A\widetilde{u} = \mathscr{F}$ is satisfied in the sense of generalized functions and solution of this problem is sought in the appropriate classes of generalized functions. This method was firstly proposed by Amirov (1986) for the transport equation.

Since $\widetilde{u}_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ then by Theorem 2, p. 130 in [Mikhailov (1978)], there is a function $\Phi \in C^3(\Omega)$ such that $\Phi|_{\partial\Omega} = \widetilde{u}_0$. Therefore, with the aid of substitution $\bar{u} = \widetilde{u} - \Phi$, Problem 2 can be reduced to the following one with homogenous data on $\partial\Omega$.

Problem 3 *Determine a pair of functions $(\bar{u}, \widetilde{\lambda})$ defined in Ω that satisfies*

$$L\bar{u} = \widetilde{\lambda}(x, p) + F, \tag{4}$$

provided that the functions F, H, β, K are known, the trace of the solution $\bar{u}$ on the boundary $\partial\Omega$ is zero and $\widetilde{\lambda}$ satisfies equation (3), where $F = -L\Phi + f$.

Here the function $\bar{u}$ depends on F (therefore on Φ). Since the corresponding homogeneous versions of Problem 2 and Problem 3 are the same, uniqueness of the solution to Problem 2 follows from Theorem 1 below. Hence, if $\left(\widetilde{u}, \widetilde{\lambda}\right)$ is a solution to Problem 2, then because of uniqueness of solution to Problem 2, the function $\widetilde{u} = \bar{u} + \Phi$ does not depend on choice of Φ (also on F) and it depends only on $\widetilde{u}_0$.

To formulate the solvability results for the problem, we need the following notations: the set of all functions $u \in L_2(\Omega)$ with the following properties is denoted by $\Gamma(A)$:

i) For any $u \in \Gamma(A)$ there exists a function $\mathscr{F} \in L_2(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$, $\langle u, A^*\varphi \rangle = \langle \mathscr{F}, \varphi \rangle$ and $Au = \mathscr{F}$, where $Au = \widehat{L}Lu$, A^* is the operator which is conjugate to A in the sense of Lagrange.

ii) There exists a sequence $\{u_k\} \subset C_0^3(\Omega)$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k \rangle \to \langle Au, u \rangle$ as $k \to \infty$, where

$$C_0^3(\Omega) = \left\{ \varphi : \varphi \in C^3(\Omega), \; \varphi|_{\partial\Omega} = 0 \right\}.$$

The standard spaces $L_2(\Omega)$, $C_0^\infty(\Omega)$, $H^k(\Omega)$ are described in [Lions and Magenes (1972), Mikhailov (1978)].

Theorem 1 *Let* $H(x,p) \in C^2(\overline{\Omega})$, $\beta(x,p) \in C^1(\overline{\Omega})$, $K(x,p,p') \in C^1(\overline{D} \times \overline{G} \times \overline{G})$ *and the inequalities*

$$\sum_{i,j=1}^n \frac{\partial^2 H}{\partial p_i \partial p_j} \xi^i \xi^j \geq \alpha_1 |\xi|^2, \quad \sum_{i,j=1}^n \frac{\partial^2 H}{\partial x_i \partial x_j} \xi^i \xi^j \leq -\alpha_2 |\xi|^2, \; (\alpha_1, \alpha_2 > 0), \tag{5}$$

$$(\alpha_1 - |\beta| - |b_0| - 1) \geq \varepsilon_0, \; (\alpha_2 - |\beta| - B_0 - L_0) \geq \varepsilon_1, \; (\varepsilon_0, \varepsilon_1 > 0), \tag{6}$$

hold for all $\xi \in \mathbb{R}^n$, $(x,p) \in \overline{\Omega}$, *where* $b_0 = \max\limits_{1 \leq j \leq n} \left\{ \max\limits_{(x,p) \in \overline{\Omega}} \left| \beta_{p_j} \right| \right\}$, $B_0 = b_0 C$, $l_0 = \max\limits_{1 \leq j \leq n} \left\{ \max\limits_{x \in \overline{D}} \int_G \int_G K_{p_j}^2(x,p,p') dp dp' \right\}$, $L_0 = l_0 C$, *and C is a constant depends on the domain* Ω. *Then Problem 3 has a unique solution* $\left(\bar{u}, \widetilde{\lambda} \right)$ *that satisfies the conditions* $\bar{u} \in \Gamma(A) \cap H^1(\Omega)$, $\widetilde{\lambda} \in L_2(\Omega)$.

Proof. To prove the uniqueness part of the theorem, we will show that the corresponding homogeneous problem $(F = 0)$ has only trivial solution which satisfies the conditions of the theorem. Let $\left(\bar{u}, \widetilde{\lambda} \right)$ be a solution to Problem 3 such that $\bar{u} = 0$ on $\partial\Omega$ and $\bar{u} \in \Gamma(A) \cap H^1(\Omega)$. Equation (4) and condition 3 imply $A\bar{u} = 0$. Since $\bar{u} \in \Gamma(A)$, there exists a sequence $\{\bar{u}_k\} \subset C_0^3$ such that $\bar{u}_k \to \bar{u}$ in $L_2(\Omega)$ and $\langle A\bar{u}_k, \bar{u}_k \rangle \to 0$ as $k \to \infty$. It can be easily verified that

$$
\begin{aligned}
-\langle A\bar{u}_k, \bar{u}_k \rangle = \; & -\int_\Omega A\bar{u}_k \bar{u}_k d\Omega = -\int_\Omega \left(\widehat{L} L\bar{u}_k \right) \bar{u}_k d\Omega = \sum_{j=1}^n \int_\Omega \frac{\partial \bar{u}_k}{\partial x_j} \frac{\partial}{\partial p_j} (L\bar{u}_k) d\Omega \\
= \; & J(\bar{u}_k) + \frac{1}{2} \sum_{i,j=1}^n \int_\Omega \frac{\partial}{\partial p_j} \left[\frac{\partial \bar{u}_k}{\partial x_j} \left(\frac{\partial \bar{u}_k}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial \bar{u}_k}{\partial p_i} \frac{\partial H}{\partial x_i} \right) \right] d\Omega \\
& + \frac{1}{2} \sum_{i,j=1}^n \int_\Omega \left(\frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial p_i} \frac{\partial \bar{u}_k}{\partial x_j} \frac{\partial \bar{u}_k}{\partial p_j} \right) - \frac{\partial}{\partial p_i} \left(\frac{\partial H}{\partial x_i} \frac{\partial \bar{u}_k}{\partial x_j} \frac{\partial \bar{u}_k}{\partial p_j} \right) \right) d\Omega \\
& - \frac{1}{2} \sum_{i,j=1}^n \int_\Omega \frac{\partial}{\partial x_j} \left[\frac{\partial \bar{u}_k}{\partial p_j} \left(\frac{\partial \bar{u}_k}{\partial x_i} \frac{\partial H}{\partial p_i} - \frac{\partial \bar{u}_k}{\partial p_i} \frac{\partial H}{\partial x_i} \right) \right] d\Omega, \tag{7}
\end{aligned}
$$

where

$$J\left(\bar{u}_k\right) \equiv \frac{1}{2} \sum_{i,j=1}^{n} \int_{\Omega} \left(\frac{\partial^2 H}{\partial p_i \partial p_j} \frac{\partial \bar{u}_k}{\partial x_i} \frac{\partial \bar{u}_k}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j} \frac{\partial \bar{u}_k}{\partial p_i} \frac{\partial \bar{u}_k}{\partial p_j} \right) d\Omega$$

$$+ \sum_{j=1}^{n} \int_{\Omega} \left(\beta \frac{\partial \bar{u}_k}{\partial p_j} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial \beta}{\partial p_j} \frac{\partial \bar{u}_k}{\partial x_j} \bar{u}_k + \int_G K_{p_j}(x,p,p') \bar{u}_k\left(x,p'\right) dp' \frac{\partial \bar{u}_k}{\partial x_j} \right) d\Omega. \quad (8)$$

If the geometry of the domain Ω and the condition $\bar{u}_k = 0$ on $\partial\Omega$ are taken into account, then $\frac{\partial}{\partial p_i} \bar{u} = 0$ on Γ_1 and $\frac{\partial}{\partial x_i} \bar{u} = 0$ on Γ_2, $i = \overline{1,n}$. Therefore the divergent terms will disappear in (7), so we obtain

$$-\langle A\bar{u}_k, \bar{u}_k \rangle = J\left(\bar{u}_k\right). \quad (9)$$

Using the Poincaré-Steklov inequality (since Ω is bounded and $\bar{u}_k = 0$ on $\partial\Omega$) and the Cauchy–Schwarz inequality, the right-hand side of (9) can be estimated as follows

$$2 \sum_{j=1}^{n} \int_{\Omega} \left(\beta \frac{\partial \bar{u}_k}{\partial p_j} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial \beta}{\partial p_j} \frac{\partial \bar{u}_k}{\partial x_j} \bar{u}_k \right) d\Omega$$

$$\geq \sum_{j=1}^{n} \int_{\Omega} \left(-|\beta| (\bar{u}_{k_{x_j}}^2 + \bar{u}_{k_{p_j}}^2) - \left|\beta_{p_j}\right| (\bar{u}_{k_{x_j}}^2 + \bar{u}_k^2) \right) d\Omega$$

$$\geq \int_{\Omega} \left((-|\beta| - |b_0|) |\nabla_x \bar{u}_k|^2 + (-|\beta| - B_0) |\nabla_p \bar{u}_k|^2 \right) d\Omega, \quad (10)$$

where $b_0 = \max_{1 \leq j \leq n} \left\{ \max_{(x,p) \in \overline{\Omega}} \left|\beta_{p_j}\right| \right\}$, $B_0 = b_0 C$ and C is a constant which depends on the domain Ω, and

$$2 \sum_{j=1}^{n} \int_{\Omega} \int_G K_{p_j}(x,p,p') \bar{u}_k\left(x,p'\right) dp' \bar{u}_{k_{x_j}} d\Omega$$

$$\geq -\sum_{j=1}^{n} \int_{\Omega} \left((\int_G K_{p_j}(x,p,p') \bar{u}_k\left(x,p'\right) dp')^2 + \bar{u}_{k_{x_j}}^2 \right) d\Omega$$

$$\geq -\sum_{j=1}^{n} \int_D \int_G \int_G (\int_G K_{p_j}^2(x,p,p') dp' \int_G \bar{u}_k^2\left(x,p'\right) dp') dp dx - \sum_{j=1}^{n} \int_{\Omega} \bar{u}_{k_{x_j}}^2 d\Omega$$

$$= -\sum_{j=1}^{n} \int_{D} \left(\int_{G} \bar{u}_k^2(x,p')\,dp' \int_{G}\int_{G} K_{p_j}^2(x,p,p')dp'dp \right) dx - \int_{\Omega} |\nabla_x \bar{u}_k|^2\,d\Omega$$

$$\geq -L_0 \int_{\Omega} |\nabla_p \bar{u}_k|^2\,d\Omega - \int_{\Omega} |\nabla_x \bar{u}_k|^2\,d\Omega, \tag{11}$$

where $l_0 = \max\limits_{1\leq j\leq n} \left\{ \max\limits_{x\in \overline{D}} \int_G \int_G K_{p_j}^2(x,p,p')dpdp' \right\}$, $L_0 = l_0 C$. Hence we obtain

$$
\begin{aligned}
2J(\bar{u}_k) &\geq (\alpha_1 - |\beta| - |b_0| - 1)\int_{\Omega} |\nabla_x \bar{u}_k|^2\,d\Omega \\
&\quad + (\alpha_2 - |\beta| - B_0 - L_0)\int_{\Omega} |\nabla_p \bar{u}_k|^2\,d\Omega \\
&\geq \varepsilon \int_{\Omega} |\nabla \bar{u}_k|^2\,d\Omega \geq C \int_{\Omega} \bar{u}_k^2\,d\Omega, \tag{12}
\end{aligned}
$$

where $\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}$, $\varepsilon_0, \varepsilon_1 > 0$. Since $-2\langle A\bar{u}_k, \bar{u}_k\rangle = 2J(\bar{u}_k)$, from the definition of $\Gamma(A)$, we have $C\int_{\Omega} \bar{u}^2\,d\Omega \leq 0$, i.e., $\bar{u} = 0$ and (4) implies $\widetilde{\lambda} = 0$. In the above expressions, C stands for different constants that depend only on the given functions and Lebesgue measure of the domain Ω.

Hence uniqueness of the solution of the problem is proved. The existence and stability of the solution of the problem can be proved in a similar way to that of Theorem 2 in [Amirov et al (2009)], under the assumptions of the theorem and the condition $F \in H_2(\Omega)$ using the Galerkin method. ■

4   A Symbolic Algorithm

In this section, we construct a symbolic algorithm for computing an approximate solution $\bar{u}_N$ of Problem 3. The approximate solution to the problem is sought in the following form:

$$\bar{u}_N = \sum_{i_1,i_2,...,i_n,j_1,j_2,...,j_n=0}^{N-1} \alpha_{N_{i_1,i_2,...,i_n,j_1,j_2,...,j_n}} w_{i_1,i_2,...,i_n,j_1,j_2,...,j_n} \eta(x)\mu(p). \tag{13}$$

The functions $\eta(x)$, $\mu(p)$ are selected such that they vanish on the boundary and outside of the corresponding domains. For example, if we consider the domains

$$D = \{x : |x| < 1\} \subset \mathbb{R}^n,\, G = \{p : |p| < 1\} \subset \mathbb{R}^n$$

then we should define the functions η and μ as follows

$$\eta(x) = \begin{cases} 1-|x|^2, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}, \mu(p) = \begin{cases} 1-|p|^2, & |p|<1 \\ 0, & |p|\geq 1 \end{cases}. \tag{14}$$

In (13), $w_{i_1,\ldots,i_n,j_1,\ldots,j_n} = x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} p_1^{j_1} p_2^{j_2} \ldots p_n^{j_n}$ and the systems $\left\{x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}\right\}_{i_1,\ldots,i_n=0}^{\infty}$, $\left\{p_1^{j_1} p_2^{j_2} \ldots p_n^{j_n}\right\}_{j_1,\ldots,j_n=0}^{\infty}$ are complete in $L_2(D)$ and $L_2(G)$, respectively. The unknown coefficients $\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n}}$, $(i_1,\ldots,i_n,j_1,\ldots,j_n=0,\ldots,N-1)$ are determined from the following system of linear algebraic equations (SLAE):

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n=0}^{N-1} \left(A\left(\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n}\right)\eta\mu, w_{i_1',\ldots,i_n',j_1',\ldots,j_n'}\eta\mu\right)_{L_2(\Omega)}$$
$$= \left(\mathscr{F}, w_{i_1',\ldots,i_n',j_1',\ldots,j_n'}\right)_{L_2(\Omega)}. \tag{15}$$

Algorithm 1.
INPUT: N, $F(x,p)$, $K(x,p,p')$, $H(x,p)$, $\beta(x,p)$
OUTPUT: $u_N(x,p)$, $\lambda_N(x,p)$
{Procedure *LeftSLAE* computes left side of each equation in (15)}
Procedure *LeftSLAE* $(i_1',\ldots,i_n',j_1',\ldots,j_n')$
$Left := 0$
for $i_1 = 0,\ldots,N-1$ do, for $i_2 = 0,\ldots,N-1$ do,..., for $i_n = 0,\ldots,N-1$ do
for $j_1 = 0,\ldots,N-1$ do, for $j_2 = 0,\ldots,N-1$ do,..., for $j_n = 0,\ldots,N-1$ do
begin
$Left := Left + \left(A\left(\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n}\right)\eta(x)\mu(p), w_{i_1',\ldots,i_n',j_1',\ldots,j_n'}\eta\mu\right)_{L_2(\Omega)}$
end;
{Procedure *SLAE* constructs the system of linear algebraic equations (15)}
Procedure *SLAE*
$Set := \{\}, \mathscr{F} := \widehat{L}F$
for $i_1' = 0,\ldots,N-1$ do, for $i_2' = 0,\ldots,N-1$ do,..., for $i_n' = 0,\ldots,N-1$ do
for $j_1' = 0,\ldots,N-1$ do, for $j_2' = 0,\ldots,N-1$ do,..., for $j_n' = 0,\ldots,N-1$ do
begin
$Set := Set \cup \left\{LeftSLAE(i_1',\ldots,i_n',j_1',..,j_n') = \left(\mathscr{F}, w_{i_1',\ldots,i_n',j_1',\ldots,j_n'}\eta\mu\right)_{L_2(\Omega)}\right\}$
end;

{Principle part}

Solve $\left(SLAE, \left\{\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n}}\right\}\right)$

for $i_1 = 0, \ldots, N-1$ do, for $i_2 = 0, \ldots, N-1$ do,..., for $i_n = 0, \ldots, N-1$ do

for $j_1 = 0, \ldots, N-1$ do, for $j_2 = 0, \ldots, N-1$ do,..., for $j_n = 0, \ldots, N-1$ do

begin

$\bar{u}_N = \bar{u}_N + \left(\alpha_{N_{i_1,\ldots,i_n,j_1,\ldots,j_n}} w_{i_1,\ldots,i_n,j_1,\ldots,j_n}\right) \eta(x) \mu(p)$

end

$\lambda_N(x,p) = L(\bar{u}_N) - F(x,p)$

end of the algorithm.

5   Computational Experiments

Algorithm 1 has been implemented in the computer algebra system Maple and tested for several IP. In order to show the stability of the proposed method, computational experiments have been carried out using noisy data F_σ, which is obtained by adding a random perturbation to the exact data F using the expression $F_\sigma = F\left(1 + \dfrac{\alpha\sigma}{100}\right)$, where α is a random number in the interval [-1,1] and σ is the noise level in percents. Two examples are presented below.

Example 1 *Let us consider Problem 3 on the domain*

$\Omega = \{(x,p) \mid x \in (2,3), \ p \in (-1,1)\},$

where the functions

$F(x,p) = xp(p^2(11x - 45 - p^2(7x - 25)) - 2x + 10 + p(x - 5 - p^2(x - 5))),$

$H(x,p) = p^2 + \ln x, \ \beta(x,p) = \dfrac{(p+1)}{x}, \ K(x,p,p') = 2xp$

are given. Then Algorithm 1 gives the result:

$U_3 = \left(x^3 - 5x^2 + 6x\right)\left(p^2 - p^4\right),$

$\lambda_3 = -12p + 42p^3 - 18p^5 + 6p^2 - 6p^4$

at $N = 3$, which is also the exact solution of the problem. On Fig. 1 below, a comparison between the exact solution $u(x,p)$ and the approximate solution of the IP for different noise levels ($\sigma = 0\%, 5\%, 10\%, 15\%$) is presented by one dimensional cross sections ($x = 2.9$). The unknown right-hand side λ can be obtained from the algorithm with a similar accuracy.
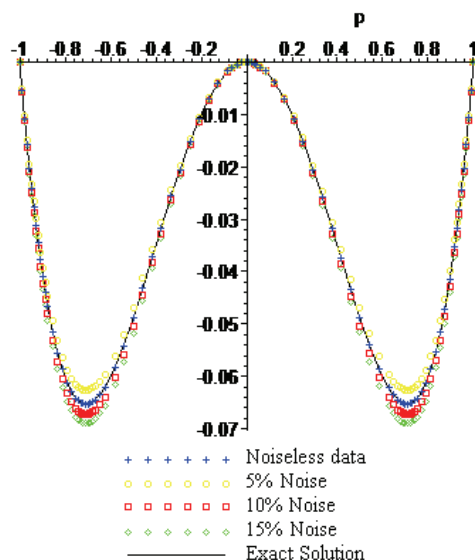
Figure 1: Approximate solutions for different noise levels and the exact solution $u(x, p)$ of the problem.

Example 2 *In the domain* $\Omega = \{(x, p) | x \in (-1, 1), p \in (-1, 1)\}$, *according to the given functions*

$$
\begin{aligned}
F(x, p) &= p(10x^2 + xp(e^x(x^2 + 3 + 2x) + 1 - 8x^2 - 4p^2 - x^4) \\
&\quad + x^2 p^2(7xp - 10 + x^3 p) + 2x^4(1 - p^2) - 2pe^x),
\end{aligned}
$$

$\beta(x, p) = xp + 2x,\ H(x, p) = (p + 2)^2,\ K(x, p, p') = p \ln(1 + x^2),$

computed solutions (U_1, U_2, U_6) of the problem at $N = 1$, $N = 2$ and $N = 6$ are shown in Figure 2, where the exact solution is

$$
u(x, p) = \frac{1}{p+2}\left(x^2 p^2 - 2xp + e^x - 1\right)(1 - x^2)(1 - p^2),
$$

$$
\lambda(x, p) = 4p(p^2 - 1) + 3x + \ln(x^2 + 1)\left(e^x(\ln 3(6 - 6x^2) - \frac{20}{3} + \frac{20x^2}{3})\right.
$$

$$
+ \ln 3(-6 - 24x^4 + 30x^2 - 24x^3 + 24x) + \frac{20}{3} + \frac{80x^3}{3} - \frac{80x}{3} - \frac{496x^2}{15} + \frac{132x^4}{5})
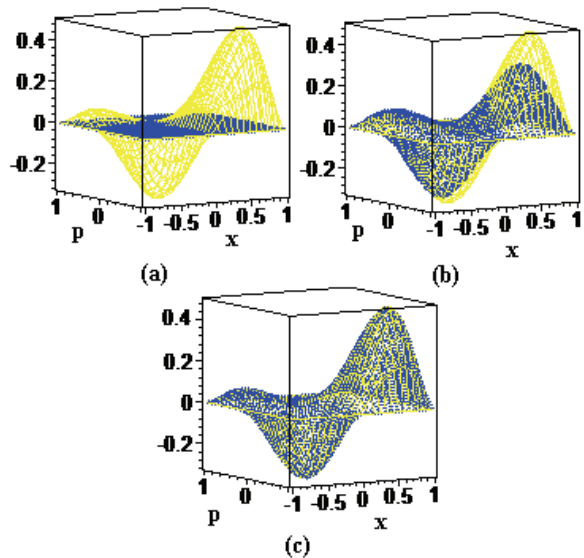$$

$$
+ e^x(2 - 3x - x^3 - 2x^2) + x^3.
$$

Figure 2: A comparison between the approximate (blue graph) and exact solution $u(x, p)$ (yellow graph) of the problem (a) $N = 1$, (b) $N = 2$, (c) $N = 6$.
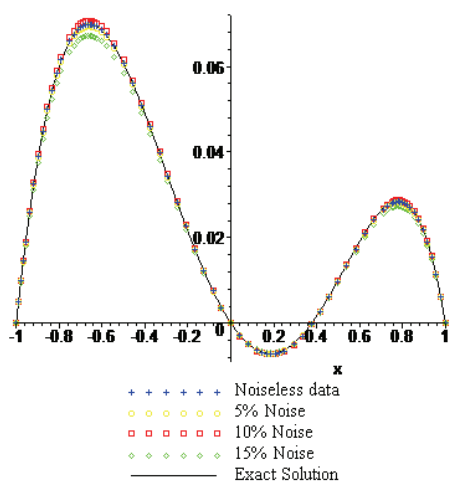

Figure 3: Approximate solutions for different noise levels and the exact solution $u(x, p)$ of the problem.
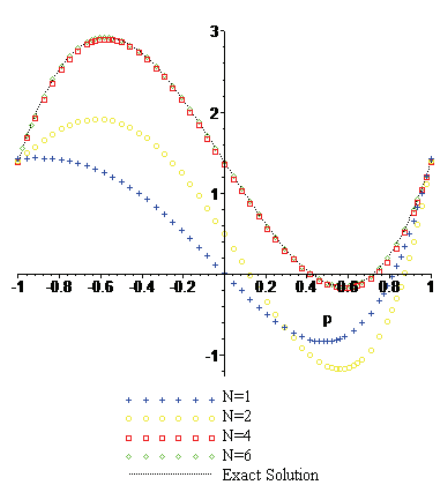
Figure 4: A comparison of exact $\lambda(x, p)$ and approximate solutions for different computation levels.

Figure 3 displays the one dimensional cross sections ($p = 0.7$) of computed approximate solutions at $N = 6$ for different noise levels ($\sigma = 0\%$, 5%, 10%, 15%) superimposed with the exact solution $u(x, p)$ of the inverse problem. And on Figure 4, we present a 1-d cross-section comparison of exact solution $\lambda(x, p)$ and approximate solutions at $x = 0.5$ for different computation levels.

As it can be seen from the figures, approximate solution at $N = 6$ is very closed to the exact solution of the problem. Consequently, the computational experiments show that proposed method provides highly accurate numerical solutions and it is robust against the data noises.

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