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Convergence of Electromagnetic Problems Modelled by Discrete Geometric Approach

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Abstract: This paper starts from the spatial discretization of an electromagnetic problem over pairs of oriented grids, one dual of the other, according to the so called *Discrete Geometric Approach* (DGA) to computational electromagnetism; the Cell Method or the Finite Integration Technique are examples of such an approach. The core of the work is providing for the first time a convergence analysis when the discrete counter-parts of constitutive relations are computed by means of an energetic framework.

Keywords: Cell Method; Finite Integration Technique; Discrete Constitutive Relations; Convergence Analysis

1 Introduction

The geometric structure behind Maxwell's equations allows to formulate them in a discrete way on a pair of oriented grids, one dual of the other, yielding to the so called Discrete Geometric Approach to computational electromagnetism, as shown by E. Tonti with its Cell Method [Tonti (2002); Tonti (1998); Tonti (2001); Heshmatzadeh and Bridges (2007); Cosmi (2001); Tonti and Zarantoello (2009); Codecasa, Specogna, and Trevisan (2008); Ferretti (2003); Ferretti (2004b); Ferretti (2004a); Ferretti (2005); Ferretti, Casadio, and Leo (2008)], by T. Weiland with its Finite Integration Technique [Weiland (1977); Weiland (1985)].

Integrals of the electromagnetic field over spatial geometric elements – nodes, edges, faces and volumes – of such a pair of oriented dual grids, are referred to as Degrees of Freedom (DoFs). *Exact* balance equations are written in terms of DoFs, discretizing in the space domain the physical laws of electromagnetism. Conversely, constitutive relations are discretized as approximate relations transforming the DoFs associated with geometric elements of one grid into the corresponding DoFs associated with geometric elements of the other grid.

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The spatially discretized physical laws and the constitutive relations, yield a final system of equations whose properties of *consistency* and *stability* depend solely on the way the constitutive relations are discretized [Schuhmann and Weiland (1998); Marrone (2004)]. Besides the natural use of pairs of oriented Cartesian orthogonal dual grids, the use of more general pairs of oriented dual grids have been proposed [Marrone (2001); Tonti (2002); Trevisan and Kettunen (2004)] without ensuring, however, consistency and stability simultaneously for the spatially discretized equations. It was only with the *energetic framework* proposed by the Authors in [Codecasa, Minerva, and Politi (2004);Codecasa and Trevisan (2006);Codecasa, Specogna, and Trevisan (2007);Codecasa and Trevisan (2007)] that consistency and stability were guaranteed. However, no theoretical results concerning the convergence of the discretized system towards the exact solution of the electromagnetic problem have been provided yet. Such a result would lead to a theoretical validation of the Discrete Geometric Approach which, as far as the Authors know, at this time is still lacking in literature.

This paper aims at providing for the first time a convergence analysis for electromagnetic problems spatially discretized by the Discrete Geometric Approach when constitutive relations are discretized within a the energetic framework. A numerical analysis confirms these theoretical predictions.

The remaining of this paper is organized as follows. In Sections 2, 3 the spatial discretization of an electromagnetic problem by the DGA is briefly recalled. The energetic framework, for discretizing constitutive relations is analysized in Section 4 and is propedeutic to the following Sections 5, 6, 7 which form the core of the paper. Mild regularity conditions on the electromagnetic problem and on the pair of dual grids are introduced in Section 5. Convergence analysis is performed in Sections 6, 7. Numerical results are given in Section 8. Definitions and ancillary results are derived in the Appendices.

2 Formulation of the electromagnetic problem

Hereafter a time-domain electromagnetic boundary value problem is considered in a *bounded* spatial region Ω and in a time interval $[0,T]$. The electromagnetic field is described by the electric field $e(r,t)$, the electric displacement $d(r,t)$, the magnetic induction $b(r,t)$ and the magnetic field $h(r,t)$. These quantities are functions of the position vector r and of time instant t and are ruled by:

1. Faraday equation for $r \in \Omega$ and $0 \leq t \leq T$

$$\nabla \times e(r,t) = -\frac{\partial b}{\partial t}(r,t).$$

2. Ampére-Maxwell equation for $r \in \Omega$ and $0 \leq t \leq T$

   $$\nabla \times h(r,t) = \frac{\partial d}{\partial t}(r,t) + j_s(r,t),$$

   in which $j_s(r,t)$ is the source current density.

3. Boundary conditions for $r \in \partial\Omega$ and $0 \leq t \leq T$, in which $\partial\Omega$ is the boundary of Ω. As motivated in Section 3, for the sake of simplicity, magnetic walls boundary conditions are considered, so that

   $$n(r) \times h(r,t) = 0$$

   being $n(r)$ the unit vector outward normal to $\partial\Omega$ at r.

4. Constitutive equations for $r \in \Omega$. Linear, non-dispersive, in general anisotropic electromagnetic media are considered. Thus the *electric* constitutive relation is

   $$d(r,t) = \varepsilon(r)e(r,t) \tag{1}$$

   in which $\varepsilon(r)$ is the permittivity double tensor, assumed to be symmetric, positive-definite or equivalently

   $$e(r,t) = \eta(r)d(r,t)$$

   in which $\eta(r)$ is the inverse of $\varepsilon(r)$. The *magnetic* constitutive relation is

   $$h(r,t) = \nu(r)b(r,t), \tag{2}$$

   in which $\nu(r)$ is the reluctivity double tensor, assumed to be symmetric, positive-definite, or equivalently

   $$b(r,t) = \mu(r)h(r,t),$$

   in which $\mu(r)$ is the inverse of $\nu(r)$.

5. Initial conditions for $d(r,t)$ and $b(r,t)$ for $r \in \Omega$ at $t = 0$.

   $$b(r,0) = b_0(r),$$
   $$d(r,0) = d_0(r).$$

Regularity conditions on material properties and electromagnetic field, assumed in this analysis, will be detailed in Section 5.

3 Spatial discretization of the electromagnetic problem by DGA

The electromagnetic problem in Section 2 is spatially discretized by DGA as follows. Firstly the Ω spatial region is covered by a pair of oriented dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ [Weiland (1996); Tonti (2002), Bossavit (1998)]. The primal grid $\mathscr{G}$ has n nodes, l edges, f faces and v volumes. Each of these geometrical elements is given an orientation. The dual grid $\tilde{\mathscr{G}}$ has $\tilde{n} = v$ nodes, $\tilde{l} = f$ edges, $\tilde{f} = l$ faces and $\tilde{v} = n$ volumes. Each of these geometrical elements has the orientation induced by the corresponding geometrical element of the primal grid $\mathscr{G}$ [Tonti (2002)]. Let $\mathbf{C}$ be the $f \times l$ face-edge incidence matrix for the primal grid $\mathscr{G}$ and let $\tilde{\mathbf{C}} = \mathbf{C}^T$ be the $\tilde{f} \times \tilde{l}$ face-edge incidence matrix for the dual grid $\tilde{\mathscr{G}}$.

Secondly, the electromagnetic field quantities in Section 2 are discretized into integral quantities associated with geometric elements of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ yielding the following arrays: the $l \times 1$ array $\mathbf{v}(t)$, of voltages $v_i(t)$, with $i = 1 \dots l$, along the edges of $\mathscr{G}$; the $f \times 1$ array $\varphi(t)$, of induction fluxes $\varphi_i(t)$ with $i = 1 \dots f$, associated with the faces of $\mathscr{G}$; the $\tilde{f} \times 1$ array $\tilde{\psi}(t)$, of electric fluxes $\tilde{\psi}_i(t)$ with $i = 1 \dots \tilde{f}$, associated with the faces of $\tilde{\mathscr{G}}$; the $\tilde{l} \times 1$ array $\tilde{\mathbf{f}}(t)$, of electro motive forces $\tilde{f}_i(t)$ with $i = 1 \dots \tilde{l}$, associated with the edges of $\tilde{\mathscr{G}}$.

Thirdly, such arrays of integral quantities are related by the following balance equations for the electromagnetic problem [Tonti (1998)]:

1. Faraday law is discretized into the matrix equation, for $0 \le t \le T$,

$$\mathbf{C}\mathbf{v}(t) = -\frac{d\varphi(t)}{dt}. \tag{3}$$

2, 3. Ampére-Maxwell law and magnetic wall boundary conditions are discretized together into the matrix equation, for $0 \le t \le T$,

$$\tilde{\mathbf{C}}\tilde{\mathbf{f}}(t) = \frac{d\tilde{\psi}(t)}{dt} + \rho_{\tilde{f}}\, j_s(r,t). \tag{4}$$

in which[1] $\rho_{\tilde{f}}\, j_s(r,t)$ is the known $\tilde{f} \times 1$ array of fluxes of the source current density $j_s(r,t)$ through the faces of $\tilde{\mathscr{G}}$.

4. Constitutive equations are then discretized. The *electric* constitutive relation (1) is discretized into a matrix equation

$$\tilde{\psi}(t) = \mathbf{E}\mathbf{v}(t), \tag{5}$$

[1] Symbol $\rho_{\tilde{f}}$ acts on a vector field yielding an array of fluxes through the faces of $\tilde{\mathscr{G}}$.

in which the $l \times l$ matrix **E** is a discrete counterpart of the $\varepsilon(r)$ tensor. This equation could be rewritten as

$$\mathbf{v}(t) = \mathbf{H}\tilde{\psi}(t)$$

in which the $l \times l$ matrix **H** is the inverse of **E** and is the discrete counterpart of the $\eta(r)$ tensor.

The *magnetic* constitutive relation (2) is discretized into a matrix equation

$$\tilde{\mathbf{f}}(t) = \mathbf{N}\varphi(t), \tag{6}$$

in which the $f \times f$ matrix **N** is a discrete counterpart of the $\nu(r)$ tensor. This equation could be rewritten as

$$\varphi(t) = \mathbf{M}\tilde{\mathbf{f}}(t),$$

in which the $f \times f$ matrix **M** is the inverse of **N** and is the discrete counterpart of the $\mu(r)$ tensor.
The problem of discretizing constitutive relations is crucial in DGA. It will be faced in Section 4.

5. Initial conditions are discretized as[2]

$$\varphi(0) = \rho_f b(r,0), \tag{7}$$
$$\tilde{\psi}(0) = \rho_{\tilde{f}} d(r,0), \tag{8}$$

Magnetic wall boundary conditions are considered, which are naturally represented in the proposed discretization (4). Nevertheless the convergence analysis hereafter proposed could be derived in a similar way also for different boundary conditions, such as mixed electric and magnetic boundary conditions.

With respect to the same pair of grids $\mathscr{G}$, $\tilde{\mathscr{G}}$, let us introduce the $l \times 1$ array[3] $\rho_e e(r,t)$ of the circulations of the electric field $e(r,t)$ along the edges of $\mathscr{G}$, the $f \times 1$ array $\rho_f b(r,t)$ of the fluxes of the magnetic induction $b(r,t)$ across the faces of $\mathscr{G}$, the $l \times 1$ array $\rho_{\tilde{f}} d(r,t)$ of the fluxes of the electric displacement $d(r,t)$ across the faces of $\tilde{\mathscr{G}}$ and the $f \times 1$ array[4] $\rho_{\tilde{e}} h(r,t)$ of the circulations of the magnetic field $h(r,t)$ along the edges of $\tilde{\mathscr{G}}$. It is noted that (3), (4) are *exactly* satisfied by $\rho_e e(r,t)$,

[2] Symbol ρ_f acts on a vector field yielding an array of fluxes through the faces of $\mathscr{G}$.
[3] Symbol ρ_e acts on a vector field yielding an array of circulations along the edges of $\mathscr{G}$.
[4] Symbol $\rho_{\tilde{e}}$ acts on the vector field yielding an array of circulations along the edges of $\tilde{\mathscr{G}}$.

$\rho_f b(r,t)$ and by $\rho_{\tilde{f}} d(r,t)$, $\rho_{\tilde{e}} h(r,t)$ respectively

$$\mathbf{C}\rho_e e(r,t) = -\frac{d}{dt}\rho_f b(r,t), \tag{9}$$

$$\tilde{\mathbf{C}}\rho_{\tilde{e}} h(r,t) = \frac{d}{dt}\rho_{\tilde{f}} d(r,t) + \rho_{\tilde{f}} j_s(r,t). \tag{10}$$

On the contrary, the equation obtained from (5) by substituting $\rho_e e(r,t)$ for $\mathbf{v}(t)$ and $\rho_{\tilde{f}} d(r,t)$ for $\tilde{\psi}(t)$, is only *approximate*. In a similar way, the equation obtained from (6) by substituting $\rho_f b(r,t)$ for $\varphi(t)$ and $\rho_{\tilde{e}} h(r,t)$ is only *approximate*. Thus, the discretized constitutive equations (5) and (6) cause $\mathbf{v}(t)$, $\varphi(t)$, $\tilde{\psi}(t)$ and $\tilde{\mathbf{f}}(t)$ to be *approximations* of $\rho_e e(r,t)$, $\rho_f b(r,t)$, $\rho_{\tilde{f}} d(r,t)$ and $\rho_{\tilde{e}} h(r,t)$ respectively.

Regularity conditions on the pair of dual grids, assumed in this analysis, will be introduced in Section 5.

4 Energetic framework for constructing discrete constitutive relations

In [Codecasa, Minerva, and Politi (2004)] the authors have proposed an energetic method for constructing discrete counter-parts of constitutive relations for electromagnetic problems spatially discretized by DGA.

Such discrete constitutive relations are here rederived in a novel way, useful for the convergence analysis here performed, by combining discrete constitutive relations *separately* constructed for the pairs of dual grids $\mathscr{G}^k$, $\tilde{\mathscr{G}}^k$ obtained by restricting the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ to the single volumes Ω^k of $\mathscr{G}$ with $k = 1,\dots,v$. Let Γ_i^k, $\tilde{\Sigma}_i^k$ with $i = 1,\dots,l^k$ be respectively the edges of $\mathscr{G}^k$ and the faces of $\tilde{\mathscr{G}}^k$, with $k = 1,\dots,v$. Let Σ_i^k, $\tilde{\Gamma}_i^k$ with $i = 1,\dots,f^k$ be respectively the faces of $\mathscr{G}^k$ and the edges of $\tilde{\mathscr{G}}^k$, with $k = 1,\dots,v$. Let

$$l_i^k = \int_{\Gamma_i^k} t(r)\, d\Gamma,$$

be the edge vector of the edge Γ_i^k, $t(r)$ being the unit vector tangent to and oriented as Γ_i^k with $i = 1,\dots,l^k$. Let

$$s_i^k = \int_{\Sigma_i^k} n(r)\, d\Sigma,$$

be the face vector of the face Σ_i^k, $n(r)$ being the unit vector normal to and oriented as Σ_i^k with $i = 1,\dots,f^k$. Similarly let $\tilde{l}_i^k$ be edge vector of edge $\tilde{\Gamma}_i^k$ with $i = 1,\dots,f^k$ and let $\tilde{s}_i^k$ be face vector of face $\tilde{\Sigma}_i^k$ with $i = 1,\dots,l^k$. Let r^k be the node of $\tilde{\mathscr{G}}^k$, with $k = 1,\dots,v$.

For the sake of clarity, in Fig. 1 the pair of restricted grids $\mathscr{G}^k$, $\tilde{\mathscr{G}}^k$ is shown assuming, as an example, an hexahedron as primal volume Ω^k.

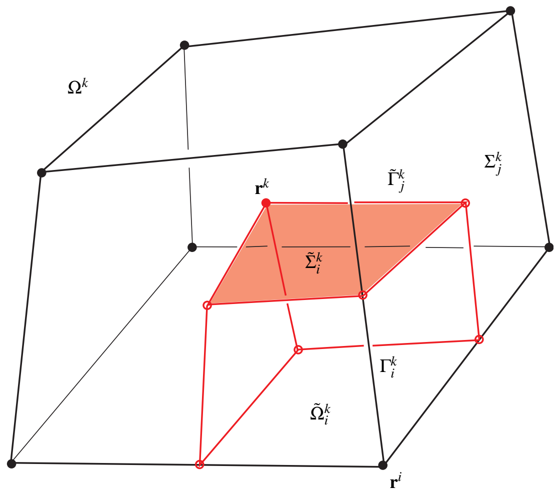

Figure 1: Visualization of the restriction of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ to a single volume when the primal volume Ω^k is an hexahedron. The geometric elements of the primal and dual grids $\mathscr{G}^k$, $\tilde{\mathscr{G}}^k$ are shown in addition.

4.1 Discrete counterpart of the permittivity tensor $\varepsilon(r)$

Let $\varepsilon^k(r)$ be the restriction of the $\varepsilon(r)$ permittivity tensor to the region Ω^k. Let $\mathbf{v}^k(t)$ and $\rho_e^k e(r,t)$ be the $l^k \times 1$ arrays respectively of the approximate and of the exact circulations[5] of the electric field along the edges of $\mathscr{G}^k$. Let $\tilde{\psi}^k(t)$ and $\rho_{\tilde{f}}^k d(r,t)$ be the $l^k \times 1$ arrays respectively of the approximate and of the exact[6] fluxes of the electric displacement through the faces of $\tilde{\mathscr{G}}^k$.

Let $w_i^{ek}(r)$, with $i = 1, \ldots, l^k$, be *bounded* vector functions satisfying the following *geometric* properties

$$\int_{\Gamma_j^k} w_i^{ek}(r) \cdot t(r)\, d\Gamma = \delta_{ij}, \quad i,j = 1,\ldots,l^k, \tag{11}$$

$$\sum_1^{l^k}{}_i\, w_i^{ek}(r) \otimes l_i^k = I, \tag{12}$$

$$\tilde{s}_i^k = \int_{\Omega^k} w_i^{ek}(r)\, d\Omega, \quad i = 1,\ldots,l^k, \tag{13}$$

[5] Symbol ρ_e^k acts on a vector field yielding an array of circulations along the edges of $\mathscr{G}^k$.
[6] Symbol $\rho_{\tilde{f}}^k$ acts on a vector field yielding an array of fluxes through the faces of $\tilde{\mathscr{G}}^k$.

in which δ_{ij} is the Kronecker's delta symbol and I is the identity double tensor. Vector functions satisfying such geometric properties have indeed been introduced by the Authors for various kinds of oriented dual grids [Codecasa and Trevisan (2006)]. From (11), (12) it follows

Lemma 1 *Functions $w_i^{ek}(r)$, with $i = 1,\dots,l^k$, are linearly independent. For the field*[7]

$$\pi_e^k(r)\mathbf{v}^k(t) = \sum_{i\,1}^{l^k} v_i^k(t) w_i^{ek}(r),$$

the degrees of freedom $v_i^k(t)$, with $i = 1,\dots,l^k$, are the circulations of $\pi_e^k(r)\mathbf{v}^k(t)$ along the edges of $\mathscr{G}^k$.

Proof. By computing the circulation of $\pi_e^k(r)\mathbf{v}^k(t)$ along the edges Γ_j^k, with $j = 1,\dots,l^k$, and by using (11) it results in

$$\begin{aligned}\int_{\Gamma_j^k} \pi_e^k(r)\mathbf{v}^k(t)\cdot t(r)\,d\Gamma &= \sum_{i\,1}^{l^k} v_i^k(t)\int_{\Gamma_j^k} w_i^{ek}(r)\cdot t(r)\,d\Gamma \\ &= \sum_{i\,1}^{l^k} v_i^k(t)\delta_{ij} = v_j^k(t), \quad j = 1,\dots,l^k. \qquad (14)\end{aligned}$$

Thus the degrees of freedom $v_j^k(t)$ are the circulations of $\pi_e^k(r)\mathbf{v}^k(t)$ along Γ_j^k, with $j = 1,\dots,l^k$. Besides, if $\pi_e^k(r)\mathbf{v}^k(t) = 0$ then from (14) it follows $v_j^k(t) = 0$, with $j = 1,\dots,l^k$. Thus vector functions $w_i^{ek}(r)$, with $i = 1,\dots,l^k$, are linearly independent. ■

Lemma 2 *Fields $\pi_e^k(r)\mathbf{v}^k(t)$ encompass all fields spatially uniform in Ω^k.*

Proof. By applying both members of (12) to a spatially uniform vector $e(\mathbf{r},t)$ it results in

$$e(r,t) = \left(\sum_{i\,1}^{l^k} w_i^{ek}(r)\otimes l_i^k\right) e(\mathbf{r},t) = \sum_{i\,1}^{l^k}(e(r,t)\cdot l_i^k) w_i^{ek}(r) = \sum_{i\,1}^{l^k} v_i^k w_i^{ek}(r)$$

and the thesis follows. ■

Thus, the $w_i^{ek}(r)$ vector functions, with $i = 1,\dots,l^k$, can be used as a basis for representing vector fields within Ω^k in particular uniform. They can also be used for constructing discrete constitutive matrices as follows.

Let matrix $\mathbf{E}^k$ be introduced, whose elements are

$$E_{ij}^k = \int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r^k) w_j^{ek}(r)\,d\Omega, \quad i,j = 1,\dots,l^k. \qquad (15)$$

[7] Symbol $\pi_e^k(r)$ acts on the circulations along the edges of $\mathscr{G}^k$ yielding a vector field.

in which the permeability tensor $\varepsilon^k(r)$ is evaluated at the node r^k of $\tilde{\mathscr{G}}^k$. The fact that elements (15) can be computed is ensured by the properties of vector functions $w_i^{ek}(r)$, with $i = 1,\ldots,l^k$.

From (13) and from Lemmas 1, 2 it follows

Theorem 1 *Matrix* $\mathbf{E}^k$ *is symmetric, positive-definite.*

Proof. Since the permittivity tensor $\varepsilon^k(r^k)$ is symmetric, it follows

$$E_{ij}^k = \int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r^k)w_j^{ek}(r)\,d\Omega = \int_{\Omega^k} w_j^{ek}(r)\cdot\varepsilon^k(r^k)w_i^{ek}(r)\,d\Omega = E_{ji}^k \;\; i,j = 1,\ldots,l^k.$$

and $\mathbf{E}^k$ is symmetric. Since the permittivity tensor $\varepsilon^k(r)$ is positive-definite, it follows

$$\begin{aligned}
\mathbf{v}^k(t)^T\mathbf{E}^k\mathbf{v}^k(t) &= \sum_{1}^{l^k}{}_{ij}\, v_i^k(t)\left(\int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r^k)w_j^{ek}(r)\,d\Omega\right)v_j^k(t)\\
&= \int_{\Omega^k}\left(\sum_{1}^{l^k}{}_{i}\, v_i^k(t)w_i^{ek}(r)\right)\cdot\varepsilon^k(r^k)\left(\sum_{1}^{l}{}_{j}\, v_j^k(t)w_j^{ek}(r)\right)d\Omega\\
&= \int_{\Omega}\pi_e^k(r)\mathbf{v}^k(t)\cdot\varepsilon^k(r^k)\pi_e^k(r)\mathbf{v}^k(t)\,d\Omega \geq 0
\end{aligned}$$

and $\mathbf{E}^k$ is positive semi-definite. Besides $\mathbf{v}^k(t)^T\mathbf{E}^k\mathbf{v}^k(t) = 0$ implies $\pi_e^k(r)\mathbf{v}^k(t) = 0$ and, from Lemma 1, also $\mathbf{v}^k(t) = \mathbf{0}$. Thus $\mathbf{E}^k$ is positive definite. ■

Theorem 2 *Let* $\varepsilon^k(r)$ *be uniform and let* $e(r,t)$, $d(r,t)$ *be spatially uniform in* Ω^k. *Then it results in*

$$\rho_{\tilde{f}}^k d(r,t) = \mathbf{E}^k\rho_e^k e(r,t)$$

Proof. It results in

$$\begin{aligned}
\tilde{\psi}_i^k &= \int_{\Omega^k} w_i^{ek}(r)\,d\Omega\cdot d(r,t) && (16)\\
&= \int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r)e(r,t)\,d\Omega\\
&= \int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r)\left(\sum_{1}^{l^k}{}_{j}\, v_j^k w_j^{ek}(r)\right)d\Omega && (17)\\
&= \sum_{1}^{l^k}{}_{j}\left(\int_{\Omega^k} w_i^{ek}(r)\cdot\varepsilon^k(r)w_j^{ek}(r)\,d\Omega\right)v_j^k = \sum_{1}^{l^k}{}_{j}\, E_{ij}^k v_j^k.
\end{aligned}$$

Eq. (16) descends from (13), Eq. (17) descends from Lemma 2. ■

It is noted that Theorem 1 expresses in formal terms the consistency property of the discrete counter-part of the electric constitutive equation.

Matrix $\mathbf{E}$ in (5) is now generated from matrices $\mathbf{E}^k$, with $k = 1, \ldots, v$, as follows. Let $\mathbf{T}^k$ be the $l^k \times l$ matrix whose element t^k_{ij} is 1 if the i-th edge of $\mathscr{G}^k$ is the j-th edge of $\mathscr{G}$ and is 0 otherwise. Then matrix $\mathbf{E}$ is constructed as

$$\mathbf{E} = \sum_{1}^{v}{}_{k} \mathbf{T}^{k^T} \mathbf{E}^k \mathbf{T}^k$$

or equivalently

$$\mathbf{E} = \hat{\mathbf{T}}^T \hat{\mathbf{E}} \hat{\mathbf{T}}, \quad \hat{\mathbf{E}} = \begin{bmatrix} \mathbf{E}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{E}^v \end{bmatrix}, \quad \hat{\mathbf{T}} = \begin{bmatrix} \mathbf{T}^1 \\ \vdots \\ \mathbf{T}^v \end{bmatrix}. \tag{18}$$

It is noted from (18) that since $\hat{\mathbf{T}}$ is a full-rank matrix and since $\hat{\mathbf{E}}$ is a symmetric, positive-definite matrix, also matrix $\mathbf{E}$ is symmetric, positive-definite.

4.2 Discrete counterpart of the reluctivity tensor $\nu(r)$

Let $\nu^k(r)$ be the restriction of the $\nu(r)$ reluctivity tensor to the region Ω^k. Let $\varphi^k(t)$ and $\rho^k_f b(r,t)$ be the $f^k \times 1$ arrays respectively of the approximate and of the exact fluxes[8] of the magnetic induction through the faces of $\mathscr{G}^k$. Let $\tilde{\mathbf{f}}^k(t)$ and $\rho^k_{\tilde{e}} h(r,t)$ be the $f^k \times 1$ arrays respectively of the approximate and of the exact circulations[9] of the magnetic field along the edges of $\tilde{\mathscr{G}}^k$.

Let $w^{fk}_i(r)$, with $i = 1, \ldots, f^k$, be *bounded* vector functions satisfying the following geometric properties

$$\int_{\Sigma^k_j} w^{fk}_i(r) \cdot n(r)\, d\Sigma = \delta_{ij}, \;\; i, j = 1, \ldots, f^k, \tag{19}$$

$$\sum_{1}^{f^k}{}_{i} w^{fk}_i(r) \otimes s^k_i = I, \tag{20}$$

$$\tilde{l}^k_i = \int_{\Omega} w^{fk}_i(r)\, d\Omega, \;\; i = 1, \ldots, f^k. \tag{21}$$

Vector functions satisfying such geometric properties have indeed been introduced by the Authors for various kinds of oriented dual grids [Codecasa and Trevisan (2006)]. From (19), (20) it follows

[8] Symbol ρ^k_f acts on a vector field yielding an array of fluxes through the faces of $\mathscr{G}^k$.

[9] Symbol $\rho^k_{\tilde{e}}$ acts on a vector field yielding an array of circulations along the edges of $\tilde{\mathscr{G}}^k$.

Lemma 3 *The functions $w_i^{fk}(r)$ with $i = 1,\ldots,f^k$ are linearly independent. For the field*

$$\pi_f^k(r)\varphi^k(t) = \sum_{1}^{f^k}{}_i \varphi_i^k(t) w_i^{fk}(r).$$

the degrees of freedom $\varphi_i^k(t)$ with $i = 1,\ldots,f^k$ are the fluxes of $\pi_f^k(r)\varphi^k(t)$ through the faces of $\mathscr{G}^k$.

Proof. By computing the fluxes of $\pi_f^k(r)\varphi^k(t)$ through the faces Σ_j^k, with $j = 1,\ldots,f^k$, and by using (19) it results in

$$\begin{aligned}\int_{\Sigma_j^k} \pi_f^k(r)\varphi^k(t)\cdot n(r)\,d\Sigma &= \sum_{1}^{f^k}{}_i \varphi_i^k(t)\int_{\Sigma_j^k} w_i^{fk}(r)\cdot n(r)\,d\Sigma \\ &= \sum_{1}^{f^k}{}_i \varphi_i^k(t)\delta_{ij} = \varphi_j^k(t), \;\; j = 1,\ldots,f^k. \end{aligned} \tag{22}$$

Thus the degrees of freedom $\varphi_j^k(t)$ are the fluxes of $\pi_f^k(r)\varphi^k(t)$ through Σ_j^k, with $j = 1,\ldots,f^k$. Besides if $\pi_f^k(r)\varphi^k(t) = 0$ then from (22) it follows $\varphi_j^k(t) = 0$, with $j = 1,\ldots,f^k$. Thus vector functions $w_i^{fk}(r)$, with $i = 1,\ldots,f^k$, are linearly independents. ■

Lemma 4 *Fields $\pi_f^k(r)\varphi^k(t)$ encompass all fields spatially uniform in Ω^k.*

Proof. By applying both members of (20) to a spatially uniform vector $b(r,t)$, it results in

$$b(r,t) = \left(\sum_{1}^{f^k}{}_i w_i^{fk}(r)\otimes s_i^k\right) b(r,t) = \sum_{1}^{f^k}{}_i (b(r,t)\cdot s_i^k) w_i^{fk}(r) = \sum_{1}^{f^k}{}_i \varphi_i^k w_i^{fk}(r),$$

and the thesis follows. ■

Thus the $w_i^{fk}(r)$ vector functions, with $i = 1,\ldots,f^k$, can be used as a basis for representing vector fields within Ω^k in particular uniform. They can also be used for constructing discrete constitutive matrices as follows. Let matrix $\mathbf{N}^k$ have elements

$$N_{ij}^k = \int_{\Omega^k} w_i^{fk}(r)\cdot \nu^k(r^k) w_j^{fk}(r)\,d\Omega, \;\; i,j = 1,\ldots,f^k. \tag{23}$$

in which the reluctivity tensor $\nu^k(r)$ is evaluated at the node r^k of $\tilde{\mathscr{G}}^k$. The fact that elements (23) can be computed is ensured by the properties of vector functions $w_i^{fk}(r)$, with $i = 1,\ldots,f^k$. From (21) and from Lemmas 3, 4 it follows

Theorem 3 *Matrix $\mathbf{N}^k$ is symmetric, positive-definite.*

Proof. Since the reluctivity tensor $\nu^k(r^k)$ is symmetric, it follows

$$N_{ij}^k = \int_{\Omega^k} w_i^{fk}(r) \cdot \nu^k(r^k) w_j^{fk}(r)\, d\Omega = \int_{\Omega^k} w_j^{fk}(r) \cdot \nu^k(r^k) w_i^{fk}(r)\, d\Omega = N_{ji}^k \;\; i,j = 1,\ldots,f^k.$$

and $\mathbf{N}^k$ is symmetric. Since the reluctivity tensor $\nu^k(r^k)$ is positive-definite, it follows

$$\begin{aligned}
\varphi^k(t)^T \mathbf{N}^k \varphi^k(t) &= \sum_{1}^{f^k}{}_{ij} \varphi_i^k(t) \left(\int_{\Omega^k} w_i^{fk}(r) \cdot \nu^k(r^k) w_j^{fk}(r)\, d\Omega \right) \varphi_j^k(t) \\
&= \int_{\Omega^k} \left(\sum_{1}^{f^k}{}_{i} \varphi_i^k(t) w_i^{fk}(r) \right) \cdot \nu^k(r^k) \left(\sum_{1}^{f^k}{}_{j} \varphi_j^k(t) w_j^{fk}(r) \right) d\Omega \\
&= \int_{\Omega^k} \pi_f^k(r) \varphi^k(t) \cdot \nu^k(r^k) \pi_f^k(r) \varphi^k(t)\, d\Omega \geq 0
\end{aligned}$$

and $\mathbf{N}^k$ is positive semi-definite. Besides $\varphi^k(t)^T \mathbf{N}^k \varphi^k(t) = 0$ implies $\pi_f^k(r)\varphi^k(t) = 0$ and, from Lemma 3, also $\varphi^k(t) = \mathbf{0}$. Thus $\mathbf{N}^k$ is positive definite. ■

Theorem 4 *Let $\nu^k(r)$ be uniform and let $b(r,t)$, $h(r,t)$ be spatially uniform in Ω^k. Then it results in*

$$\rho_{\tilde{e}}^k h(r,t) = \mathbf{N}^k \rho_f^k b(r,t).$$

Proof. It results in

$$\begin{aligned}
\tilde{f}_i^k &= \int_{\Omega^k} w_i^{fk}(r)\, d\Omega \cdot h(r,t) && (24) \\
&= \int_{\Omega^k} w_i^{fk}(r) \cdot \nu(r) b(r,t)\, d\Omega \\
&= \int_{\Omega^k} w_i^{fk}(r) \cdot \nu(r) \left(\sum_{1}^{f^k}{}_{j} \varphi_j^k w_j^{fk}(r) \right) d\Omega && (25) \\
&= \sum_{1}^{f^k}{}_{j} \left(\int_{\Omega^k} w_i^{fk}(r) \cdot \nu(r) w_j^{fk}(r)\, d\Omega \right) \varphi_j^k = \sum_{1}^{f^k}{}_{j} N_{ij}^k \varphi_j^k.
\end{aligned}$$

Eq. (24) descends from (21), Eq. (25) descends from Lemma 4. Thus by taking $b(r,t) = a$, $\nu^k(r^k) = q$ and $h(r,t) = b$ the thesis follows. ■

It is noted that Theorem 4 expresses in formal terms the consistency property of the discrete counter-part of the magnetic constitutive equation.

The matrix $\mathbf{N}$ is now generated from matrices $\mathbf{N}^k$, with $k = 1,\ldots,v$, as follows. Let $\mathbf{P}^k$ be the $f^k \times f$ matrix whose element p_{ij}^k is 1 if the i-th face of $\mathscr{G}^k$ is the j-th face of $\mathscr{G}$ and is 0 otherwise. Matrix $\mathbf{N}$ is constructed as

$$\mathbf{N} = \sum_{1}^{v}{}_{k} \mathbf{P}^{k^T} \mathbf{N}^k \mathbf{P}^k$$

or equivalently

$$\mathbf{N} = \hat{\mathbf{P}}^T \hat{\mathbf{N}} \hat{\mathbf{P}}, \quad \hat{\mathbf{N}} = \begin{bmatrix} \mathbf{N}^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{N}^v \end{bmatrix}, \quad \hat{\mathbf{P}} = \begin{bmatrix} \mathbf{P}^1 \\ \vdots \\ \mathbf{P}^v \end{bmatrix}. \tag{26}$$

It is noted from (26) that since $\hat{\mathbf{P}}$ is a full-rank matrix and since $\hat{\mathbf{N}}$ is a symmetric, positive-definite matrix, also matrix $\mathbf{N}$ is symmetric, positive-definite.

5 Regularity conditions on the electromagnetic problem and on the pair of dual grids

Consistency and stability analyses for a time domain electromagnetic boundary value problem spatially discretized by DGA, was substantially provided in [Schuhmann and Weiland (1998)] and is not repeated here. Instead here a convergence analysis is provided for DGA for the first time, assuming that constitutive relations are discretized by the energetic framework, under conditions of mild regularity for the electromagnetic problem and for the pair of oriented dual grids.

Firstly assumptions are made on the regularity of the material properties and of the solution to the electromagnetic problem. Precisely it is assumed that the spatial domain Ω can be partitioned in a finite set of subdomains Ω_i, with $i = 1, \ldots, s$ in each of which both the tensors $\varepsilon(r)$, $\nu(r)$ and their inverses $\eta(r)$, $\mu(r)$ are bounded and Lipschitz continuous. That is, if $A(r)$ is any of such tensors, constants M_A and L_A exist such that

$$||A(r)||_2 \leq M_A,$$
$$||A(r_1) - A(r_2)||_2 \leq L_A |r_1 - r_2|, \quad r_1, r_2 \in \Omega_i, \ i = 1, \ldots, s,$$

hold, in which $||\cdot||_2$ is the spectral norm, recalled in Appendix A. Similarly, it is assumed that for all time instants $0 \leq t \leq T$, the fields $e(r,t)$, $h(r,t)$, $b(r,t)$, $d(r,t)$, together with their time derivatives are bounded and Lipschitz continuous within each subdomain Ω_i, with $i = 1, \ldots, s$. That is, if $a(r,t)$ is any of such fields, constants M_a and L_a exist such that

$$|a(r,t)| \leq M_a, \tag{27}$$
$$|a(r_1,t) - a(r_2,t)| \leq L_a |r_1 - r_2|, \quad r_1, r_2 \in \Omega_i, \ i = 1, \ldots, s, \tag{28}$$

hold. It is noted that these assumptions exclude the case of unbounded electromagnetic field solutions. It is also noted that these assumptions ensure that quantities $\rho_e^k e(r,t)$, $\rho_{\tilde{f}}^k d(r,t)$, $\rho_f^k b(r,t)$ and $\rho_{\tilde{e}}^k h(r,t)$ can be computed with respect to a pair of oriented dual grids.

Secondly assumptions are made on the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$. Any chosen pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ is such that the following disequalities

$$||\rho_e^k e(r,t)||_{\mathbf{E}^k} \leq R_{\mathbf{E}} \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\varepsilon^k(r)||_2} \max_{r\in\Omega^k} |e(r,t)|, \quad k = 1,\dots,v, \tag{29}$$

$$||\rho_{\tilde{f}}^k d(r,t)||_{\mathbf{H}^k} \leq R_{\mathbf{H}} \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\eta^k(r)||_2} \max_{r\in\Omega^k} |d(r,t)|, \quad k = 1,\dots,v, \tag{30}$$

$$||\rho_f^k b(r,t)||_{\mathbf{N}^k} \leq R_{\mathbf{N}} \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\nu^k(r)||_2} \max_{r\in\Omega^k} |b(r,t)|, \quad k = 1,\dots,v, \tag{31}$$

$$||\rho_{\tilde{e}}^k h(r,t)||_{\mathbf{M}^k} \leq R_{\mathbf{M}} \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\mu^k(r)||_2} \max_{r\in\Omega^k} |h(r,t)|, \quad k = 1,\dots,v, \tag{32}$$

hold, in which the notation in Appendix A is used and the constants $R_{\mathbf{E}}$, $R_{\mathbf{H}}$, $R_{\mathbf{N}}$ and $R_{\mathbf{M}}$ are independent of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$. Further, as it is common in Finite Elements convergence analysis, it is assumed that the Ω_i subdomains, with $i = 1,\dots,s$, are exactly obtained as unions of volumes of the primal grid $\mathscr{G}$.

It is noted that a general condition for which (29)-(32) are satisfied is to choose the pairs of dual grids in such a way that each primal volume is geometrically similar to a volume in a finite set $\mathscr{S}$. This is proved by Lemmas 5, 6 in Appendix B.

The maximum diameter [Quarteroni and Valli (1994)] of the volumes of $\mathscr{G}$ is here denoted as h_M.

6 Error bounds for the approximation error of integral quantities

Hereafter error bounds are derived for the $\mathbf{v}(t)$ approximation of $\rho_e e(r,t)$, the $\varphi(t)$ approximation of $\rho_f b(r,t)$, the $\tilde{\psi}(t)$ approximation of $\rho_{\tilde{f}} d(r,t)$ and the $\tilde{\mathbf{f}}(t)$ approximation of $h(r,t)$.

Subtracting, member by member, (9) from (3) it results in

$$\mathbf{C}(\mathbf{v}(t) - \rho_e e(r,t)) = -\frac{d}{dt}(\varphi(t) - \rho_f b(r,t)), \tag{33}$$

Similarly from (4) and (10), it results in

$$\tilde{\mathbf{C}}(\tilde{\mathbf{f}}(t) - \rho_{\tilde{e}} h(r,t)) = \frac{d}{dt}(\tilde{\psi}(t) - \rho_{\tilde{f}} d(r,t)). \tag{34}$$

By multiplying both members of (33) by $(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))^T$, both members of (34) by $(\mathbf{v}(t)-\rho_e e(r,t))^T$ and by subtracting the two equations, member by member, it results in

$$(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))^T\frac{d}{dt}(\varphi(t)-\rho_f b(r,t))+ \\ +(\mathbf{v}(t)-\rho_e e(r,t))^T\frac{d}{dt}(\tilde{\psi}(t)-\rho_{\tilde{f}}d(r,t))=0. \quad (35)$$

The following two identities are now introduced

$$(\varphi(t)-\rho_f b(r,t))=\mathbf{M}\,(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))+(\mathbf{M}\rho_{\tilde{e}}-\rho_f\mu)\,h(r,t) \quad (36)$$

$$(\tilde{\psi}(t)-\rho_{\tilde{f}}d(r,t))=\mathbf{E}\,(\mathbf{v}(t)-\rho_e e(r,t))+(\mathbf{E}\rho_e-\rho_{\tilde{f}}\varepsilon)\,e(r,t) \quad (37)$$

By substituting such identities into (35), it follows

$$\frac{d}{dt}\left(\frac{1}{2}(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))^T\mathbf{M}(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))+\frac{1}{2}(\mathbf{v}(t)-\rho_e e(r,t))^T\mathbf{E}(\mathbf{v}(t)-\rho_e e(r,t))\right) \\ =(\tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t))^T(\mathbf{M}\rho_{\tilde{e}}-\rho_f\mu(r))\frac{\partial h}{\partial t}(r,t) \\ +(\mathbf{v}(t)-\rho_e e(r,t))^T(\mathbf{E}\rho_e-\rho_{\tilde{f}}\varepsilon(r))\frac{\partial e}{\partial t}(r,t). \quad (38)$$

Since matrices **E**, **N** and hence also their inverse matrices **H**, **M** are symmetric, positive-definite as from Section 4, this equation can be rewritten as follows. Let the following scalar product and its corresponding norm be introduced

$$(\mathbf{x}_1,\mathbf{x}_2)=(\mathbf{v}_1,\mathbf{v}_2)_{\mathbf{E}}+(\tilde{\mathbf{f}}_1,\tilde{\mathbf{f}}_2)_{\mathbf{M}}$$
$$||\mathbf{x}_1||=\sqrt{(\mathbf{x}_1,\mathbf{x}_1)}=\sqrt{||\mathbf{v}_1||^2_{\mathbf{E}}+||\tilde{\mathbf{f}}_1||^2_{\mathbf{M}}}$$

in which the notation in Appendix A has been used and

$$\mathbf{x}_1=\begin{bmatrix}\mathbf{v}_1\\ \tilde{\mathbf{f}}_1\end{bmatrix},\quad \mathbf{x}_2=\begin{bmatrix}\mathbf{v}_2\\ \tilde{\mathbf{f}}_2\end{bmatrix}.$$

Then (38) can be rewritten as

$$\frac{d}{dt}||\mathbf{x}(t)||^2=2(\mathbf{x}(t),\omega(t)) \quad (39)$$

in which

$$\mathbf{x}(t)=\begin{bmatrix}\mathbf{v}(t)-\rho_e e(r,t)\\ \tilde{\mathbf{f}}(t)-\rho_{\tilde{e}}h(r,t)\end{bmatrix},\quad \omega(t)=\begin{bmatrix}(\mathbf{H}\rho_{\tilde{f}}-\rho_e\eta(r))\frac{\partial d}{\partial t}(r,t)\\ \\ (\mathbf{N}\rho_f-\rho_{\tilde{e}}\nu(r))\frac{\partial b}{\partial t}(r,t)\end{bmatrix}.$$

By integrating Eq. (39) with respect to time from 0 to t it results in

$$||\mathbf{x}(t)||^2 = ||\mathbf{x}(0)||^2 + 2\int_0^t (\mathbf{x}(\tau), \omega(\tau))\, d\tau$$

from which, by using the properties of the scalar product,

$$||\mathbf{x}(t)||^2 \leq ||\mathbf{x}(0)||^2 + 2\int_0^t ||\mathbf{x}(t)||\,||\omega(t)||\, d\tau.$$

Then, as it can be easily proven [Quarteroni and Valli (1994)],

$$||\mathbf{x}(t)|| \leq ||\mathbf{x}(0)|| + 2\int_0^t ||\omega(\tau)||\, d\tau.$$

Equivalently it is

$$\begin{aligned}
&\sqrt{||\mathbf{v}(t) - \rho_e e(r,t)||^2_{\mathbf{E}} + ||\tilde{\mathbf{f}}(t) - \rho_{\tilde{e}} h(r,t)||^2_{\mathbf{M}}} \\
&\leq \sqrt{||\mathbf{v}(0) - \rho_e e(r,0)||^2_{\mathbf{E}} + ||\tilde{\mathbf{f}}(0) - \rho_{\tilde{e}} h(r,0)||^2_{\mathbf{M}}} \\
&+ \int_0^t \sqrt{\left|\left|(\mathbf{H}\rho_{\tilde{f}} - \rho_e \eta(r))\, \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{E}} + \left|\left|(\mathbf{N}\rho_f - \rho_{\tilde{e}} \nu(r))\, \frac{\partial b}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{M}}}\, d\tau.
\end{aligned}$$

from which

$$\begin{aligned}
&\sqrt{||\mathbf{v}(t) - \rho_e e(r,t)||^2_{\mathbf{E}} + ||\tilde{\mathbf{f}}(t) - \rho_{\tilde{e}} h(r,t)||^2_{\mathbf{M}}} \\
&\leq \sqrt{\left|\left|(\mathbf{E}\rho_e - \rho_{\tilde{f}} \varepsilon(r))\, e(r,0)\right|\right|^2_{\mathbf{H}} + \left|\left|(\mathbf{M}\rho_{\tilde{e}} - \rho_f \mu(r)) h(r,0)\right|\right|^2_{\mathbf{N}}} \\
&+ \int_0^t \sqrt{\left|\left|(\mathbf{H}\rho_{\tilde{f}} - \rho_e \eta(r))\, \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{E}} + \left|\left|(\mathbf{N}\rho_f - \rho_{\tilde{e}} \nu(r))\, \frac{\partial b}{\partial \tau}(r,\tau))\right|\right|^2_{\mathbf{M}}}\, d\tau. \qquad (40)
\end{aligned}$$

By analyzing the single terms in the right hand side of (40) it results in

Theorem 5 *For $0 \leq t \leq T$, it is*

$$\sqrt{||\mathbf{v}(t) - \rho_e e(r,t)||^2_{\mathbf{E}} + ||\tilde{\mathbf{f}}(t) - \rho_{\tilde{e}} h(r,t)||^2_{\mathbf{M}}} \leq (\sqrt{S_e^2 + S_h^2} + T\sqrt{S_{\dot{d}}^2 + S_{\dot{b}}^2})\, h_M. \qquad (41)$$

in which

$$S_e = R_{\mathbf{H}} \sqrt{M_\eta |\Omega|}(L_\varepsilon M_e + M_\varepsilon L_e) + R_{\mathbf{E}} \sqrt{M_\varepsilon |\Omega|} L_e, \qquad (42)$$

$$S_h = R_{\mathbf{N}} \sqrt{M_\nu |\Omega|}(L_\mu M_h + M_\mu L_h) + R_{\mathbf{M}} \sqrt{M_\mu |\Omega|} L_h, \qquad (43)$$

$$S_{\dot{d}} = R_{\mathbf{E}} \sqrt{M_\varepsilon |\Omega|}(L_\eta M_{\dot{d}} + M_\eta L_{\dot{d}}) + R_{\mathbf{H}} \sqrt{M_\eta |\Omega|} L_{\dot{d}}, \qquad (44)$$

$$S_{\dot{b}} = R_{\mathbf{M}} \sqrt{M_\mu |\Omega|}(L_\nu M_{\dot{b}} + M_\nu L_{\dot{b}}) + R_{\mathbf{N}} \sqrt{M_\nu |\Omega|} L_{\dot{b}}. \qquad (45)$$

Proof. Since, from Theorem 2

$$(\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r^k))e(r^k,\tau) = \mathbf{0},$$

and since, by hypothesis, for each $r \in \Omega^k$ it is

$$|e(r,\tau) - e(r^k,\tau)| \leq M_e|r - r^k| \leq M_e h_M,$$

it results in

$$\begin{aligned}
&\left|\left|(\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r))e(r,\tau)\right|\right|_{\mathbf{H}^k} = \left|\left|\rho_{\tilde{f}}^k(\varepsilon(r^k) - \varepsilon(r))e(r,\tau) + (\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r^k))e(r,\tau)\right|\right|_{\mathbf{H}^k}\\
&\leq \left|\left|\rho_{\tilde{f}}^k(\varepsilon(r^k) - \varepsilon(r))e(r,\tau)\right|\right|_{\mathbf{H}^k} + \left|\left|(\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r^k))e(r,\tau)\right|\right|_{\mathbf{H}^k}\\
&\leq \left|\left|\rho_{\tilde{f}}^k(\varepsilon(r^k) - \varepsilon(r))e(r,\tau)\right|\right|_{\mathbf{H}^k} + \left|\left|(\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r^k))(e(r,\tau) - e(r^k,\tau))\right|\right|_{\mathbf{H}^k}\\
&\leq \left|\left|\rho_{\tilde{f}}^k(\varepsilon(r^k) - \varepsilon(r))e(r,\tau)\right|\right|_{\mathbf{H}^k} + \left|\left|\rho_{\tilde{f}}^k\varepsilon(r^k)(e(r,\tau) - e(r^k,\tau))\right|\right|_{\mathbf{H}^k}\\
&+ \left|\left|\rho_e^k(e(r,\tau) - e(r^k,\tau))\right|\right|_{\mathbf{E}^k} \leq \left(R_{\mathbf{H}}\sqrt{M_\eta|\Omega^k|}(L_\varepsilon M_e + M_\varepsilon L_e) + R_{\mathbf{E}}\sqrt{M_\varepsilon|\Omega^k|}L_e\right)h_M,
\end{aligned}$$

in which (29), (30) have been used. Thus by recalling (18) and by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}} = \hat{\mathbf{E}}$ and $\hat{\mathbf{Q}} = \hat{\mathbf{T}}$, it is

$$\begin{aligned}
&\left|\left|(\mathbf{E}\rho_e - \rho_{\tilde{f}}\varepsilon(r))\,e(r,\tau)\right|\right|_{\mathbf{H}}^2 \leq \sum_1^{v}{}_k \left|\left|(\mathbf{E}^k\rho_e^k - \rho_{\tilde{f}}^k\varepsilon(r))\,e(r,\tau)\right|\right|_{\mathbf{H}^k}^2\\
&\leq \left(R_{\mathbf{H}}\sqrt{M_\eta}(L_\varepsilon M_e + M_\varepsilon L_e) + R_{\mathbf{E}}\sqrt{M_\varepsilon}L_e\right)^2 h_M^2 \sum_1^{v}{}_k |\Omega^k| = S_e^2 h_M^2. \qquad (46)
\end{aligned}$$

Similarly, since from Theorem 4 it is

$$(\mathbf{M}^k\rho_{\tilde{e}}^k - \rho_f^k\mu(r^k))h(r^k,\tau) = \mathbf{0},$$

and since, by hypothesis, for each $r \in \Omega^k$ it is

$$\left|h(r,\tau) - h(r^k,\tau)\right| \leq M_h|r - r^k| \leq M_h h_M,$$

it results in

$$\begin{aligned}
&\left|\left|(\mathbf{M}^k\rho_{\tilde{e}}^k - \rho_f^k\mu(r))h(r,\tau)\right|\right|_{\mathbf{N}^k} = \left|\left|\rho_f^k(\mu(r^k) - \mu(r))h(r,\tau) + (\mathbf{M}^k\rho_{\tilde{e}}^k - \rho_f^k\mu(r^k))h(r,\tau)\right|\right|_{\mathbf{N}^k}\\
&\leq \left|\left|\rho_f^k(\mu(r^k) - \mu(r))h(r,\tau)\right|\right|_{\mathbf{N}^k} + \left|\left|(\mathbf{M}^k\rho_{\tilde{e}}^k - \rho_f^k\mu(r^k))h(r,\tau)\right|\right|_{\mathbf{N}^k}\\
&\leq \left|\left|\rho_f^k(\mu(r^k) - \mu(r))h(r,\tau)\right|\right|_{\mathbf{N}^k} + \left|\left|(\mathbf{M}^k\rho_{\tilde{e}}^k - \rho_f^k\mu(r^k))(h(r,\tau) - h(r^k,\tau))\right|\right|_{\mathbf{N}^k}\\
&\leq \left|\left|\rho_f^k(\mu(r^k) - \mu(r))h(r,\tau)\right|\right|_{\mathbf{N}^k} + \left|\left|\rho_f^k\mu(r^k)(h(r,\tau) - h(r^k,\tau))\right|\right|_{\mathbf{N}^k}\\
&+ \left|\left|\rho_{\tilde{e}}^k(h(r,\tau) - h(r^k,\tau))\right|\right|_{\mathbf{H}^k} \leq \left(R_{\mathbf{N}}\sqrt{M_\nu|\Omega^k|}(L_\mu M_h + M_\mu L_h) + R_{\mathbf{M}}\sqrt{M_\mu|\Omega^k|}L_h\right)h_M,
\end{aligned}$$

in which (31), (32) have been used. Thus by recalling (26) and by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}} = \hat{\mathbf{N}}$ and $\hat{\mathbf{Q}} = \hat{\mathbf{P}}$, it is

$$\begin{aligned}
&\left|\left|(\mathbf{M}\rho_{\tilde{e}} - \rho_f \mu(r))\, h(r,\tau)\right|\right|^2_{\mathbf{N}} = \left|\left|(\mathbf{N}\rho_f - \rho_{\tilde{e}} \nu(r))\, b(r,\tau)\right|\right|^2_{\mathbf{M}} \\
&\leq \sum_1^{v}{}_k \left|\left|(\mathbf{N}^k \rho_f^k - \rho_{\tilde{e}}^k \nu(r)) b(r,\tau)\right|\right|^2_{\mathbf{M}^k} = \sum_1^{v}{}_k \left|\left|(\mathbf{M}^k \rho_{\tilde{e}}^k - \rho_f^k \mu(r)) h(r,\tau)\right|\right|^2_{\mathbf{N}^k} \\
&\leq \left(R_{\mathbf{N}} \sqrt{M_\nu}(L_\mu M_h + M_\mu L_h) + R_{\mathbf{M}} \sqrt{M_\mu} L_h\right)^2 h_M^2 \sum_1^{v}{}_k |\Omega^k| = S_h^2 h_M^2.
\end{aligned} \tag{47}$$

Besides since from Theorem 2 it is

$$(\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r^k)) \frac{\partial d}{\partial \tau}(r^k,\tau) = \mathbf{0}$$

and since, by hypothesis, for each $r \in \Omega_k$ it is

$$\left|\frac{\partial d}{\partial \tau}(r,\tau) - \frac{\partial d}{\partial \tau}(r^k,\tau)\right| \leq M_{\dot{d}} |r - r^k| \leq M_{\dot{d}} h_M,$$

it results in

$$\begin{aligned}
&\left|\left|(\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} = \left|\left|\rho_e^k (\eta(r^k) - \eta(r)) \frac{\partial d}{\partial \tau} + (\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r^k)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} \\
&\leq \left|\left|\rho_e^k (\eta(r^k) - \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} + \left|\left|(\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r^k)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} \\
&\leq \left|\left|\rho_e^k (\eta(r^k) - \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} + \left|\left|(\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r^k)) \left(\frac{\partial d}{\partial \tau}(r,\tau) - \frac{\partial d}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{E}^k} \\
&\leq \left|\left|\rho_e^k (\eta(r^k) - \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{E}^k} + \left|\left|\rho_e^k \eta(r^k) \left(\frac{\partial d}{\partial \tau}(r,\tau) - \frac{\partial d}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{E}^k} \\
&+ \left|\left|\rho_{\tilde{f}}^k \left(\frac{\partial d}{\partial \tau}(r,\tau) - \frac{\partial d}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{H}^k} = \left(R_{\mathbf{E}} \sqrt{M_\varepsilon |\Omega^k|}(L_\eta M_{\dot{d}} + M_\eta L_d) + R_{\mathbf{H}} \sqrt{M_\eta |\Omega^k|} L_d\right) h_M,
\end{aligned}$$

in which (29), (30) have been used. Thus by recalling (18) and by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}} = \hat{\mathbf{E}}$ and $\hat{\mathbf{Q}} = \hat{\mathbf{T}}$, it is

$$\begin{aligned}
&\left|\left|(\mathbf{H}\rho_{\tilde{f}} - \rho_e \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{E}} = \left|\left|(\mathbf{E}\rho_e - \rho_{\tilde{f}} \varepsilon(r)) \frac{\partial e}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{H}} \\
&\leq \sum_1^{v}{}_k \left|\left|(\mathbf{E}^k \rho_e^k - \rho_{\tilde{f}}^k \varepsilon(r)) \frac{\partial e}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{H}^k} = \sum_1^{v}{}_k \left|\left|(\mathbf{H}^k \rho_{\tilde{f}}^k - \rho_e^k \eta(r)) \frac{\partial d}{\partial \tau}(r,\tau)\right|\right|^2_{\mathbf{E}^k} \\
&= \left(R_{\mathbf{E}} \sqrt{M_\varepsilon}(L_\eta M_{\dot{d}} + M_\eta L_d) + R_{\mathbf{H}} \sqrt{M_\eta} L_d\right)^2 h_M^2 \sum_1^{v}{}_k |\Omega^k| = S_{\dot{d}}^2 h_M^2.
\end{aligned} \tag{48}$$

Similarly, since from Theorem 4 it is

$$(\mathbf{N}^k \rho_f^k - \rho_{\tilde{e}}^k \nu(r^k)) \frac{\partial b}{\partial \tau}(r^k,\tau) = \mathbf{0},$$

and since, by hypothesis, it is

$$\left|\frac{\partial b}{\partial \tau}(r,\tau)-\frac{\partial b}{\partial \tau}(r^k,\tau)\right| \le M_{\dot b}|r-r^k| \le M_{\dot b} h_M,$$

it results in

$$\left|\left|(\mathbf{N}^k\rho_f^k-\rho_{\tilde e}^k\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k} = \left|\left|\rho_{\tilde e}^k(\nu(r^k)-\nu(r))\frac{\partial b}{\partial \tau}+(\mathbf{N}^k\rho_f^k-\rho_{\tilde e}^k\nu(r^k))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}$$
$$\le \left|\left|\rho_{\tilde e}^k(\nu(r^k)-\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}+\left|\left|(\mathbf{N}^k\rho_f^k-\rho_{\tilde e}^k\nu(r^k))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}$$
$$\le \left|\left|\rho_{\tilde e}^k(\nu(r^k)-\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}+\left|\left|(\mathbf{N}^k\rho_f^k-\rho_{\tilde e}^k\nu(r^k))\left(\frac{\partial b}{\partial \tau}(r,\tau)-\frac{\partial b}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{M}^k}$$
$$\le \left|\left|\rho_{\tilde e}^k(\nu(r^k)-\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}+\left|\left|\rho_{\tilde e}^k\nu(r^k)\left(\frac{\partial b}{\partial \tau}(r,\tau)-\frac{\partial b}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{M}^k}$$
$$+\left|\left|\rho_f^k\left(\frac{\partial b}{\partial \tau}(r,\tau)-\frac{\partial b}{\partial \tau}(r^k,\tau)\right)\right|\right|_{\mathbf{N}^k} = \left(R_{\mathbf{M}}\sqrt{M_\mu|\Omega^k|}(L_\nu M_{\dot b}+M_\nu L_b)+R_{\mathbf{N}}\sqrt{M_\nu|\Omega^k|}L_b\right)h_M,$$

in which (31), (32) have been used. Thus by recalling (26) and by applying Theorem 9 in Appendix A, with $\hat{\mathbf{A}}=\hat{\mathbf{N}}$ and $\hat{\mathbf{Q}}=\hat{\mathbf{P}}$, it is

$$\left|\left|(\mathbf{N}\rho_f-\rho_{\tilde e}\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}}^2 \le \sum_{1}^{v}{}_k\left|\left|(\mathbf{N}^k\rho_f^k-\rho_{\tilde e}^k\nu(r))\frac{\partial b}{\partial \tau}(r,\tau)\right|\right|_{\mathbf{M}^k}^2$$
$$= \left(R_{\mathbf{M}}\sqrt{M_\mu}(L_\nu M_{\dot b}+M_\nu L_{\dot b})+R_{\mathbf{N}}\sqrt{M_\nu}L_{\dot b}\right)^2 h_M^2\sum_{1}^{v}{}_k|\Omega^k| = S_{\dot b}^2 h_M^2. \tag{49}$$

By combining (40) with (46), (47), (48) and (49) the thesis follows. ∎

Theorem 6 *For* $0\le t\le T$ *it is*

$$\sqrt{||\tilde\psi(t)-\rho_{\tilde f}d(r,t)||_{\mathbf{H}}^2+\left|\left|\varphi(t)-\rho_f b(r,t)\right|\right|_{\mathbf{N}}^2} \le \left(2\sqrt{S_e^2+S_h^2}+T\sqrt{S_{\dot d}^2+S_{\dot b}^2}\right)h_M. \tag{50}$$

Proof. From (36), (37), it straightforwardly results in

$$\sqrt{||\tilde\psi(t)-\rho_{\tilde f}d(r,t)||_{\mathbf{H}}^2+\left|\left|\varphi(t)-\rho_f b(r,t)\right|\right|_{\mathbf{N}}^2} \le \sqrt{||\mathbf{v}(t)-\rho_e e(r,t)||_{\mathbf{E}}^2+||\tilde{\mathbf{f}}(t)-\rho_{\tilde e}h(r,t)||_{\mathbf{M}}^2}$$
$$+\sqrt{||(\mathbf{E}\rho_e-\rho_{\tilde f}\varepsilon)\,e(r,t)||_{\mathbf{E}}^2+||(\mathbf{M}\rho_{\tilde e}-\rho_f\mu)\,h(r,t)||_{\mathbf{N}}^2} \tag{51}$$

The first term in the right hand side of (51), is bounded by (41). The second term in the right hand side is bounded by (46) and (47) and the thesis follows. ∎

Eqs. (41) and (50) establish bounds for the approximation error of the integral quantities in DGA. These relations are now exploited for deriving bounds for the approximation error of the electromagnetic field.

7 Error estimations for the electromagnetic field

The fields $\pi_e^k(r)\mathbf{v}^k(t)$ introduced in each Ω^k with $k=1,\ldots,\nu$ can be used to construct the field $\pi_e(r)\mathbf{v}(t)$ over the whole Ω as

$$\pi_e(r)\mathbf{v}(t)=\pi_e^k(r)\mathbf{v}^k(t),\quad r\in\Omega^k,\ k=1,\ldots,\nu.$$

Hereafter it is shown that the field $\pi_e(r)\mathbf{v}(t)$ is an approximation of $e(r,t)$. To this aim the approximation error $||\pi_e(r)\mathbf{v}(t)-e(r,t)||_\varepsilon$ is estimated, in which the notation of Appendix A is used. As a result it is also shown that the field $\varepsilon(r)\pi_e(r)\mathbf{v}(t)$ is an approximation of the electric displacement $d(r,t)$ and the approximation error $||\varepsilon(r)\pi_e(r)\mathbf{v}(t)-d(r,t)||_\eta$ is estimated. It is noted that such norms can be computed because of the assumed properties of both $e(r,t)$, $d(r,t)$, $\varepsilon(r)$ and $\pi_e(r)\mathbf{v}(t)$.

Theorem 7 *For $0\le t\le T$, it results in*

$$\begin{aligned}||\pi_e(r)\mathbf{v}(t)-e(r,t)||_\varepsilon&=||\varepsilon(r)\pi_e(r)\mathbf{v}(t)-d(r,t)||_\eta\\&\le\left(I_e+\sqrt{S_e^2+S_h^2}+T\sqrt{S_d^2+S_b^2}\right)h_M\end{aligned}$$

in which

$$I_e=(R_\mathbf{E}+1)\sqrt{M_\varepsilon|\Omega|}L_e.$$

Proof. It results in

$$\begin{aligned}||\pi_e(r)\mathbf{v}(t)-e(r,t)||_\varepsilon&=||\pi_e(r)(\mathbf{v}(t)-\rho_e e(r,t))+(\pi_e(r)\rho_e-1)e(r,t)||_\varepsilon\\&\le||\pi_e(r)(\mathbf{v}(t)-\rho_e e(r,t))||_\varepsilon+||(\pi_e(r)\rho_e-1)e(r,t)||_\varepsilon.\end{aligned}$$

From (41), it results in

$$||\pi_e(r)(\mathbf{v}(t)-\rho_e e(r,t))||_\varepsilon=||\mathbf{v}(t)-\rho_e e(r,t)||_\mathbf{E}\le\left(\sqrt{S_e^2+S_h^2}+T\sqrt{S_d^2+S_b^2}\right)h_M.\tag{52}$$

Besides it is

$$||(\pi_e(r)\rho_e-1)e(r,t)||_\varepsilon^2=\sum_1^\nu{}_k||(\pi_e^k(r)\rho_e^k-1)e(r,t)||_{\varepsilon^k}^2.\tag{53}$$

Since

$$e(r^k,t)=\pi_e^k(r)\rho_e^k e(r^k,t),\quad r\in\Omega^k,\quad k=1,\ldots,\nu,$$

it follows

$$\begin{aligned}||(\pi_e^k(r)\rho_e^k-1)e(r,t)||_{\varepsilon^k}&=||(\pi_e^k(r)\rho_e^k-1)(e(r,t)-e(r^k,t))||_{\varepsilon^k}\\&\le||e(r,t)-e(r^k,t)||_{\varepsilon^k}+||\pi_e^k(r)\rho_e^k(e(r,t)-e(r^k,t))||_{\varepsilon^k}\\&=||e(r,t)-e(r^k,t)||_{\varepsilon^k}+||\rho_e^k(e(r,t)-e(r^k,t))||_{\mathbf{E}^k}.\end{aligned}$$

From (27) and (29) respectively, it results in

$$||e(r,t) - e(r^k,t)||_{\varepsilon^k} \leq \sqrt{M_\varepsilon |\Omega^k|} L_e h_M, \quad ||\rho_e^k(e(r,t) - e(r^k,t))||_{\mathbf{E}^k} \leq R_{\mathbf{E}} \sqrt{M_\varepsilon |\Omega^k|} L_e h_M.$$

Thus it is

$$||(\pi_e^k(r)\rho_e^k - 1)e(r,t)||_{\varepsilon^k} \leq (R_E + 1)\sqrt{M_\varepsilon |\Omega^k|} L_e h_M, \quad k = 1, \dots, \nu$$

and, from (53),

$$||(\pi_e(r)\rho_e - 1)e(r,t)||_\varepsilon \leq (R_{\mathbf{E}} + 1)\sqrt{M_\varepsilon |\Omega|} L_e h_M. \tag{54}$$

From (52), (54) the thesis follows. ■

In a similar way the fields $\pi_f^k(r)\varphi^k(t)$ introduced in each Ω^k with $k = 1, \dots, \nu$ can be used to construct the field $\pi_f(r)\varphi(t)$ over the whole Ω as

$$\pi_f(r)\varphi(t) = \pi_f^k(r)\varphi^k(t), \quad r \in \Omega^k, \; k = 1, \dots, \nu.$$

It is now shown that the field $\pi_f(r)\varphi(t)$ is an approximation of $b(r,t)$. To this aim the approximation error $||\pi_f(r)\varphi(t) - b(r,t)||_\nu$ is estimated. As a result it is also shown that the field $\nu(r)\pi_f(r)\varphi(t)$ is an approximation of $h(r,t)$ and the approximation error $||\nu(r)\pi_f(r)\varphi(t) - h(r,t)||_\mu$ is estimated. It is noted that such norms can be computed because of the assumed properties of both $b(r,t)$, $h(r,t)$, $\nu(r)$ and $\pi_f(r)\varphi(t)$.

Theorem 8 *For $0 \leq t \leq T$ it results in*

$$\begin{aligned} ||\pi_f(r)\varphi(t) - b(r,t)||_\nu &= ||\nu(r)\pi_f(r)\varphi(t) - h(r,t)||_\mu \\ &\leq \left(I_b + 2\sqrt{S_e^2 + S_h^2} + T\sqrt{S_d^2 + S_b^2} \right) h_M \end{aligned}$$

in which

$$I_b = (R_{\mathbf{N}} + 1)\sqrt{M_\nu |\Omega|} L_b.$$

Proof. It results in

$$\begin{aligned} ||\pi_f(r)\varphi(t) - b(r,t)||_\nu &= ||\pi_f(r)(\varphi - \rho_f b(r,t)) + (\pi_f(r)\rho_f - 1)b(r,t)||_\nu \\ &\leq ||\pi_f(r)(\varphi(t) - \rho_f b(r,t))||_\nu + ||(\pi_f(r)\rho_f - 1)b(r,t)||_\nu. \end{aligned}$$

From (50), it results in

$$||\pi_f(r)(\varphi(t) - \rho_f b(r,t))||_\nu = ||\varphi(t) - \rho_f b(r,t)||_{\mathbf{N}} \leq \left(2\sqrt{S_e^2 + S_h^2} + T\sqrt{S_d^2 + S_b^2} \right) h_M. \tag{55}$$

Besides it is

$$||(\pi_f(r)\rho_f - 1)b(r,t)||_v^2 = \sum_{1}^{v}{}_k ||(\pi_f^k(r)\rho_f^k - 1)b(r,t)||_{v^k}^2 \tag{56}$$

Since

$$b(r^k,t) = \pi_f^k(r)\rho_f^k b(r^k,t), \quad r \in \Omega^k,\ k = 1,\dots,v,$$

it follows

$$\begin{aligned}||(\pi_f^k(r)\rho_f^k - 1)b(r,t)||_{v^k} &= ||(\pi_f^k(r)\rho_f^k - 1)(b(r,t) - b(r^k,t))||_{v^k}\\ &\le ||b(r,t) - b(r^k,t)||_{v^k} + ||\pi_f^k(r)\rho_f^k(b(r,t) - b(r^k,t))||_{v^k}\\ &= ||b(r,t) - b(r^k,t)||_{v^k} + ||\rho_f^k(b(r,t) - b(r^k,t))||_{\mathbf{N}^k}.\end{aligned}$$

From (28) and (31) respectively it results in

$$||b(r,t) - b(r^k,t)||_{v^k} \le \sqrt{M_v|\Omega^k|}L_b h_M, \quad ||\rho_f^k(b(r,t) - b(r^k,t))||_{\mathbf{N}^k} \le R_{\mathbf{N}}\sqrt{M_v|\Omega^k|}L_b h_M.$$

Thus it is

$$||(\pi_f^k(r)\rho_f^k - 1)b(r,t)||_{v^k} \le (R_{\mathbf{N}} + 1)\sqrt{M_v|\Omega^k|}L_b h_M, k = 1,\dots,v$$

and, from (56),

$$||(\pi_f(r)\rho_f - 1)b(r,t)||_v \le (R_{\mathbf{N}} + 1)\sqrt{M_\mu|\Omega|}L_h h_M. \tag{57}$$

From (55), (57) the thesis follows. ■

It is thus proved that the approximate electromagnetic field provided by DGA with constitutive relations discretized by the proposed energetic framework converges almost everywhere to the exact electromagnetic field with at least a first order of convergence.

It is noted that such convergence results regard the semi-discrete equations obtained by spatially discretizing the time domain electromagnetic boundary value problem by DGA. Convergence results for the discrete equations obtained by discretizing such semi-discrete equations also with respect to time are not reported here but can be derived from the convergence results of the semi-discrete equations with standard approach [Quarteroni and Valli (1994)].

8 Numerical results

A rectangular waveguide of section 5cm $\times$ 2.5cm and length 10cm is considered. At one end a TE_{10} electric field is applied. At the other end a PEC termination is

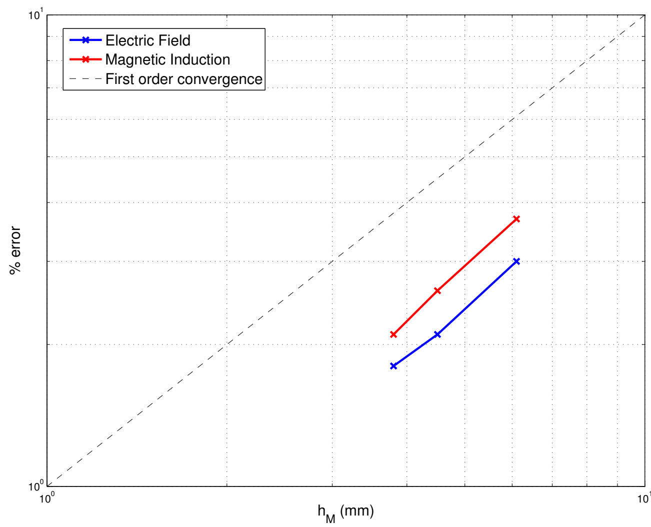

Figure 2: Percent error of the electric and magnetic fields in the energy norm versus maximum grid diameter.

applied. The corresponding time domain electromagnetic boundary value problem has been spatially discretized by means of DGA, the oriented primal grid being tetrahedral, the oriented dual grid being its barycentric subdivision and constitutive relations being discretized as in [Codecasa, Minerva, and Politi (2004)]. The resulting semi-discrete equations have been discretized with respect to time by means of the FD-TD scheme, in the time interval $0\text{ns} \leq t \leq 0.95\text{ns}$. The time step has been chosen in such a way that its effect on the approximate electromagnetic field is negligible. The approximation error in the energy norm for the electromagnetic field has been evaluated at $t = 0.95\text{ns}$ for different grids. The evaluated percent errors, for the electric and magnetic induction fields, are reported in Fig. 2, and exhibit a first order convergence, in accordance with the theoretical predictions.

9 Conclusions

In this paper a convergence analysis has been provided for the first time, for electromagnetic problems spatially discretized by the Discrete Geometric Approach when constitutive relations are discretized by an energetic framework. Bounds for the approximation error of the electromagnetic field have been given under mild regularity conditions on the electromagnetic field and on the pairs of dual grids.

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Appendix A

We recall that the spectral norm $||\mathbf{A}||_2$ of a square matrix $\mathbf{A}$ of order n is the maximum modulus of the eigenvalues of $\mathbf{A}$.

If $\mathbf{A}$ is a symmetric, positive definite matrix of order n and $\mathbf{x}_1$, $\mathbf{x}_2$ are a pair of column arrays of n rows, then a scalar product and its corresponding norm in the space of column arrays of n rows are defined by

$$(\mathbf{x}_1,\mathbf{x}_2)_{\mathbf{A}} = \mathbf{x}_1^T\mathbf{A}\mathbf{x}_2$$
$$||\mathbf{x}_1||_{\mathbf{A}} = \sqrt{(\mathbf{x}_1,\mathbf{x}_1)_{\mathbf{A}}} = \sqrt{\mathbf{x}_1^T\mathbf{A}\mathbf{x}_1}.$$

Theorem 9 *Let $\hat{\mathbf{A}}$ be a symmetric, positive definite matrix of order m and let $\hat{\mathbf{Q}}$ be a real, full rank, $m \times n$ matrix with $m \geq n$. Let it be $\mathbf{A} = \hat{\mathbf{Q}}^T\hat{\mathbf{A}}\hat{\mathbf{Q}}$. Then for each real column vector $\hat{\mathbf{x}}$ of m rows*

$$||\mathbf{x}||_{\mathbf{A}^{-1}} \leq ||\hat{\mathbf{x}}||_{\hat{\mathbf{A}}^{-1}} \tag{58}$$

holds, being $\mathbf{x} = \hat{\mathbf{Q}}^T\hat{\mathbf{x}}$.

Proof. For each real column vector $\mathbf{c}$ of n rows it results in

$$\mathscr{H} = (\hat{\mathbf{A}}^{-\frac{1}{2}}\hat{\mathbf{x}} - \hat{\mathbf{A}}^{\frac{1}{2}}\hat{\mathbf{Q}}\mathbf{c})^T(\hat{\mathbf{A}}^{-\frac{1}{2}}\hat{\mathbf{x}} - \hat{\mathbf{A}}^{\frac{1}{2}}\hat{\mathbf{Q}}\mathbf{c}) \geq 0. \tag{59}$$

By expanding the terms in (59) it results in

$$\mathscr{H} = \hat{\mathbf{x}}^T\hat{\mathbf{A}}^{-1}\hat{\mathbf{x}} - 2\hat{\mathbf{x}}^T\hat{\mathbf{Q}}\mathbf{c} + \mathbf{c}^T\hat{\mathbf{Q}}^T\hat{\mathbf{A}}\mathbf{Q}\mathbf{c} \geq 0.$$

In particular, by choosing

$$\mathbf{c} = \mathbf{A}^{-1}\hat{\mathbf{Q}}^T\hat{\mathbf{x}}.$$

it results in

$$\mathscr{H} = \hat{\mathbf{x}}^T\hat{\mathbf{A}}^{-1}\hat{\mathbf{x}} - \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x} \geq 0$$

from which (58) descends. ■

We recall that the spectral norm $||A||_2$ of a double tensor A is the spectral norm of the corresponding square matrix.

Let now $A(r)$ be a symmetric, positive definite double tensor defined in a spatial region Ω. If both $||A(r)||_2$ and $||A^{-1}(r)||_2$ are bounded in Ω, then in the space of vector functions square integrable in Ω, a scalar product and its corresponding norm are defined as

$$(x_1(r),x_2(r))_{A(r)} = \int_\Omega x_1(r)\cdot A(r)x_2(r)\,d\Omega$$
$$||x_1||_{A(r)} = \sqrt{(x_1,x_1)_{A(r)}} = \sqrt{\int_\Omega x_1(r)\cdot A(r)x_1(r)\,d\Omega}.$$

Appendix B

Let $\mathscr{S}$ be a *finite* set of volumes.

Lemma 5 *If the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ is such that each volume of $\mathscr{G}$ is geometrically similar to a volume in the finite set $\mathscr{S}$ then constants $R_{\mathbf{E}}$, $R_{\mathbf{H}}$ independent of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$, exist, such that (29), (30) hold.*

Proof. For each Ω^k, it is

$$||\rho_e^k e(r,t)||_{\mathbf{E}^k} \leq \sqrt{||\mathbf{E}^k||_2}||\rho_e^k e(r,t)||_2.$$

Thus, since

$$||\rho_e^k e(r,t)||_2 \leq \sqrt{\sum_{1}^{l^k}{}_i |l_i^k|^2} \max_{r\in\Omega^k} |e(r,t)|.$$

and since, from (15),

$$||\mathbf{E}^k||_2 \leq \max_{r\in\Omega^k} ||\varepsilon^k(r)||_2||\mathbf{E}_1^k||_2,$$

in which $||\cdot||_2$ is the spectral norm, the elements of $\mathbf{E}_1^k$ being defined by (15) with $\varepsilon^k(r) = I$, it follows

$$||\rho_e^k e(r,t)||_{\mathbf{E}^k} \leq R_{\mathbf{E}}^k \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\varepsilon^k(r)||_2} \max_{r\in\Omega^k} |e(r,t)|,$$

with

$$R_{\mathbf{E}}^k = \sqrt{\frac{\sum_{1}^{l^k}{}_i |l_i^k|^2}{|\Omega^k|}||\mathbf{E}_1^k||_2}.$$

Since it can be assumed that the basis functions scale with Ω^k so that $R_{\mathbf{E}}^k$ does not change if the Ω^k is scaled, and since each Ω^k is geometrically similar to a volume in the finite set $\mathscr{S}$, then (29) holds with

$$R_{\mathbf{E}} = \max_{\Omega^k\in\mathscr{S}} R_{\mathbf{E}}^k.$$

For each Ω^k, it is

$$||\rho_{\tilde{f}}^k d(r,t)||_{\mathbf{H}^k} \leq \sqrt{||\mathbf{H}^k||_2}||\rho_{\tilde{f}}^k d(r,t)||_2.$$

Thus, since

$$||\rho_{\tilde{f}}^k d(r,t)||_2 \leq \sqrt{\sum_{i\,1}^{l^k} |\tilde{s}_i^k|^2} \max_{r\in\Omega^k} |d(r,t)|.$$

and since, from (15) and from the notion of spectral norm of a symmetric positive-definite matrix, it is

$$\frac{1}{||\mathbf{H}^k||_2} = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{E}^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{E}^k\mathbf{x}}{\mathbf{x}^T\mathbf{E}_1^k\mathbf{x}} \frac{\mathbf{x}^T\mathbf{E}_1^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{E}^k\mathbf{x}}{\mathbf{x}^T\mathbf{E}_1^k\mathbf{x}} \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{E}_1^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{1}{\max_{r\in\Omega^k}||\eta^k(r)||_2} \frac{1}{||\mathbf{H}_1^k||_2},$$

being $\mathbf{H}_1^k$ the inverse of $\mathbf{E}_1^k$, so that

$$||\mathbf{H}^k||_2 \leq \max_{r\in\Omega^k} ||\eta^k(r)||_2 ||\mathbf{H}_1^k||_2,$$

it follows

$$||\rho_{\tilde{f}}^k d(r,t)||_{\mathbf{H}^k} \leq R_{\mathbf{H}}^k \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\eta^k(r)||_2} \max_{r\in\Omega^k} |d(r,t)|,$$

with

$$R_{\mathbf{H}}^k = \sqrt{\frac{\sum_{i\,1}^{l^k} |\tilde{s}_i^k|^2}{|\Omega^k|} ||\mathbf{H}_1^k||_2}.$$

Since it can be assumed that the basis functions scale with Ω^k, $R_{\mathbf{H}}^k$ does not change if the Ω^k is scaled, and since each Ω^k is geometrically similar to a volume in the finite set $\mathscr{S}$, then (30) holds with

$$R_{\mathbf{H}} = \max_{\Omega^k\in\mathscr{S}} R_{\mathbf{H}}^k$$

and the thesis follows. ■

Lemma 6 *If the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ is such that each volume of $\mathscr{G}$ is geometrically similar to a volume in the finite set $\mathscr{S}$ then constants $R_{\mathbf{N}}$, $R_{\mathbf{M}}$, independent of the pair of dual grids $\mathscr{G}$, $\tilde{\mathscr{G}}$ exist, such that (31), (32) hold.*

Proof. For each Ω^k, it is

$$||\rho_f^k b(r,t)||_{\mathbf{N}^k} \leq \sqrt{||\mathbf{N}^k||_2} ||\rho_f^k b(r,t)||_2.$$

Thus, since

$$||\rho_f^k b(r,t)||_2 \leq \sqrt{\sum_{i\,1}^{f^k} |s_i^k|^2} \max_{r\in\Omega^k} |b(r,t)|.$$

and since, from (23),

$$||\mathbf{N}^k||_2 \le \max_{r\in\Omega^k} ||\nu^k(r)||_2 ||\mathbf{N}_1^k||_2,$$

the elements of $\mathbf{N}_1^k$ being defined by (23) with $\nu^k(r) = I$, it follows

$$||\rho_f^k b(r,t)||_{\mathbf{N}^k} \le R_{\mathbf{N}}^k \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\nu^k(r)||_2} \max_{r\in\Omega^k} |b(r,t)|,$$

with

$$R_{\mathbf{N}}^k = \sqrt{\frac{\sum_1^{f^k}{}_i |s_i^k|^2}{|\Omega^k|} ||\mathbf{N}_1^k||_2}.$$

Since it can be assumed that the basis functions scale with Ω^k, $R_{\mathbf{N}}^k$ does not change if the Ω^k is scaled, and since each Ω^k is similar to a volume in the finite set $\mathscr{S}$, then (31) holds with

$$R_{\mathbf{N}} = \max_{\Omega^k \in \mathscr{S}} R_{\mathbf{N}}^k.$$

For each Ω^k, it is

$$||\rho_{\tilde{e}}^k h(r,t)||_{\mathbf{M}^k} \le \sqrt{||\mathbf{M}^k||_2} ||\rho_{\tilde{e}}^k h(r,t)||_2.$$

Thus, since

$$||\rho_{\tilde{e}}^k h(r,t)||_2 \le \sqrt{\sum_1^{f^k}{}_i |\tilde{l}_i^k|^2} \max_{r\in\Omega^k} |h(r,t)|.$$

and since, from (23) and from the notion of spectral norm of a symmetric positive-definite matrix, it is

$$\frac{1}{||\mathbf{M}^k||_2} = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{N}^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{N}^k\mathbf{x}}{\mathbf{x}^T\mathbf{N}_1^k\mathbf{x}} \frac{\mathbf{x}^T\mathbf{N}_1^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \ge \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{N}^k\mathbf{x}}{\mathbf{x}^T\mathbf{N}_1^k\mathbf{x}} \min_{\mathbf{x}\neq\mathbf{0}} \frac{\mathbf{x}^T\mathbf{N}_1^k\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{1}{\max\limits_{r\in\Omega^k} ||\mu^k(r)||_2} \frac{1}{||\mathbf{M}_1^k||_2},$$

being $\mathbf{M}_1^k$ the inverse of $\mathbf{N}_1^k$, so that

$$||\mathbf{M}^k||_2 \le \max_{r\in\Omega^k} ||\mu^k(r)||_2 ||\mathbf{M}_1^k||_2,$$

it follows

$$||\rho_{\tilde{e}}^k h(r,t)||_{\mathbf{M}^k} \le R_{\mathbf{M}}^k \sqrt{|\Omega^k|} \max_{r\in\Omega^k} \sqrt{||\mu^k(r)||_2} \max_{r\in\Omega^k} |h(r,t)|,$$

with

$$R_{\mathbf{M}}^{k} = \sqrt{\frac{\sum_{i}^{f^k}\limits_{1} |\bar{l}_i^k|^2}{|\Omega^k|}} ||\mathbf{M}_1^k||_2.$$

Since it can be assumed that the basis functions scale with Ω^k so that $R_{\mathbf{M}}^k$ does not change if the Ω^k is scaled, and since each Ω^k is similar to a volume in the finite set $\mathscr{S}$, then (32) holds with

$$R_{\mathbf{M}} = \max_{\Omega^k \in \mathscr{S}} R_{\mathbf{M}}^k.$$

and the thesis follows ∎