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An Inverse Problem for the General Kinetic Equation and a Numerical Method

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Abstract: This paper has two purposes. The first is to prove existence and uniqueness theorems for the solution of an inverse problem for the general linear kinetic equation with a scattering term. The second one is to develop a numerical approximation method for the solution of this inverse problem for two dimensional case using finite difference method.

Keywords: Inverse Problem, Kinetic Equation, Solvability of the Problem, Finite Difference Method.

1 Introduction

Solvability of an inverse problem for the general kinetic equation with a scattering term is considered and an efficient approximation method is proposed to solve this problem. To demonstrate the feasibility of the given method, some numerical experiments are performed in the last section of the paper. The proofs of the solvability theorems pave a way for the development of a numerical method for the solution of the inverse problem. This is a new approach since nobody has solved such inverse problems for kinetic equations (KE) numerically in the past.

KE are widely used for qualitative and quantitative description of physical, chemical, biological, and other kinds of processes on a microscopic scale. They are often referred to as master equations since they play an important role in the theory of substance motion under the action of forces, in particular, irreversible processes.

We consider the linear kinetic equation

$$Lu \equiv \{u,H\} + I_1(u) = \lambda(x), \tag{1}$$

$$\{u,H\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial v_i}\frac{\partial u}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial u}{\partial v_i} \right), \; I_1(u) = \int\limits_G K\left(x,v,v'\right) u\left(x,v'\right) dv'$$

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in the domain $\Omega = \{(x,v) : x \in D \subset \mathbb{R}^n,\ v \in G \subset \mathbb{R}^n,\ n \geq 1\}$, where the boundaries $\partial D, \partial G \in C^3$, $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 = \partial D \times G$, $\Gamma_2 = D \times \partial G$ and $\overline{\Gamma}_1, \overline{\Gamma}_2$ are the closures of Γ_1, Γ_2, respectively. $H(x,v)$ is the Hamiltonian, $K(x,v,v')$ is a given function called scattering kernel and $\lambda(x)$ is a source function.

Eq. 1 is extensively used in plasma physics and astrophysics [Alexeev (1982); Liboff (1979)]. In applications, u represents the number (or the mass) of particles in the unit volume element of the phase space in the neighbourhood of the point (x,v), and $\nabla_x H$ is the force acting on a particle.

Problem 1. *Determine the functions $u(x,v)$ and $\lambda(x)$ that satisfy Eq. 1, assuming that the Hamiltonian $H(x,v) \in C^2\left(\overline{\Omega}\right)$, $K(x,v,v') \in C^1\left(\overline{\Omega}\right)$ are given and the trace of the solution of Eq. 1 on the boundary $\partial\Omega$ is known: $u|_{\partial\Omega} = u_0$.*

Inverse problems for KE and integral geometry problems are closely interrelated. In other words, many problems of integral geometry can be reduced to the corresponding inverse problems for KE, and vice versa [Amirov (2001)]. Problem 1 is also related to a problem of integral geometry when $K = 0$ [Amirov (2001)]. Investigation of the uniqueness of solution of the problem of integral geometry by reducing it to an equivalent inverse problem for KE was first carried out in [Lavrent'ev and Anikonov (1967)]. In many cases, in uniqueness theorems for inverse problems for various KE, unknown coefficients (or right hand side of the equation) depend only on space variable x [Anikonov, Kovtanyuk, and Prokhorov (2002); Anikonov and Amirov (1983); Lavrent'ev, Romanov, and Shishatskii (1980)]. Some other interesting results in this field can be found in [Amirov (2001); Anikonov (2001); Klibanov and Yamamoto (2007); Natterer (1986)]. However, the issue of existence of the solution of inverse problems for KE is basically unsolvable, as it is the case of all inverse (ill posed) problems. The main difficulty in studying the solvability of Problem 1, as in many classical cases of inverse problems, lies in their overdeterminacy. Therefore, the initial data for these problems can not be arbitrary; they should satisfy some "solvability conditions" which are difficult to establish [Amirov (2001)]. It should be noted that the set of functions u_0 for which Problem 1 is solvable is not everywhere dense in any of the spaces $L_2(\partial\Omega)$, $C^m(\partial\Omega)$ and $H^m(\partial\Omega)$. The standard spaces $C^m(\Omega)$, $L_2(\Omega)$ and $H^k(\Omega)$ are described in detail, for example, in [Lions and Magenes (1972); Mikhailov (1978)].

As a rule, the data in problems of integral geometry and related inverse problems for KE are of quasianalytic character, i.e., their values specified in a domain of Lebesgue measure as small as desired determine their values in an essentially larger domain [Lavrent'ev, Romanov, and Shishatskii (1980); Courant and Hilbert (1962)]. In particular, this implies that it is not possible to avoid overdeterminacy of the problem by specifying the data on a part of the boundary rather than on the whole boundary. Even if it were possible to find the solvability conditions for the mentioned overdetermined problems, it seems that these conditions would not always be satisfactory for the practical point of view. The reason is that the real data in practice usually have some errors and thus fall out of the data class for which the existence of a solution is established.

It is worth to remember here that, in the theory of inverse problems, usually "overdeterminacy" means that the number of free variables in the data exceeds the number of free variables in the unknown coefficient or right hand side of the equation ($\lambda(x)$), and this is not the case for $n = 1$ here, whereas for dimension $n \geq 2$ Problem 1 (or the related IGP) is overdetermined in the last sense. But here, the underlying operator of the related IGP is compact and its inverse operator is unbounded. Therefore, it is impossible to prove general existence results. This is the true reason why for existence of solution to Problem 1 need such special conditions on the data u_0, so we use the term "overdeterminacy" in this sense here.

In [Amirov (2001)], a genereal scheme is presented for proving the solvability of these problems: using some extension of the class of unknown functions λ, overdetermined problem is replaced by a determined one. This is achieved by assuming the unknown function λ depends not only upon the space variables x (as in the case of the classical IGP), but also upon the direction v in a specific way, i.e., we consider $\lambda(x,v)$. It should be noted that $\lambda(x,v)$ cannot be arbitrarily dependent upon v, because in the opposite case the problem would be underdetermined and the nonuniqueness examples of a solution can be easily constructed. Herein the special dependence of $\lambda(x,v)$ upon the direction means that $\lambda(x,v)$ satisfies a certain differential equation, $\left(\widehat{L}\lambda = 0\right)$ with the following properties:

i) Problem 1 with the function $\lambda(x,v)$ becomes a determined one,

ii) The sufficiently smooth functions λ depending only on x satisfy this equation.

Suppose that, we have found a differential equation for $\lambda(x,v)$ satisfying the properties (i)-(ii), and that, a priori we know a function u_0^e to be the exact data of Problem 1 related to a function λ depending only on x. Then, utilizing u_0^e, we can construct a solution $\tilde{\lambda}$ to Problem 1. By uniqueness of a solution, $\tilde{\lambda}$ coincides with $\lambda(x)$. If we know the approximate data u_0^a with $\left\|u_0^e - u_0^a\right\|_{H^3(\partial\Omega)} \leq \varepsilon$, we can construct an approximate solution $\lambda^a(x,\varphi)$ such that $\|\lambda - \lambda^a\|_{L_2(\Omega)} \leq C\varepsilon$. Recall that, if λ depends only on x and u_0^a does not satisfy the "solvability conditions", the solution λ^a depending only x does not exist. Here the data are specified on $\partial\Omega$ and $C > 0$ is not dependent on u_0^e and u_0^a. In other words, we construct a regularising procedure for Problem 1. In general, the equation $\widehat{L}\lambda = 0$ with the properties (i)-(ii) for the same problem is not uniquely determined.

Consequently, for Problem 1 an equation ($\widehat{L}\lambda = 0$) possessing properties (i)-(ii) and some spaces (depends essentially on the problem) in which the problem is uniquely solvable are constructed. The proposed method for proving the solvability of inverse problems for KE and Problem 1 leads to a Dirichlet problem for the third order equation of the form $Au \equiv \widehat{L}Lu = \mathscr{F}$, where $\widehat{L}$ and L are second and first order differential expressions, respectively, defined in the domain Ω. Here the equation $Au = \mathscr{F}$ is satisfied in the sense of generalized functions and solution of the Dirichlet problem for this equation is sought in the appropriate classes of generalized functions.

This new method of investigating the solvability of overdetermined inverse problems was firstly proposed by Amirov (1986) for transport equation. Some recent results on numerical methods for the inverse problems that occur in several branches of engineering and sciences are presented in [Ling and Atluri (2006), Liu (2006), Huang and Shih (2007), Wu, Al-Khoury, Kasbergen, Liu, and Scarpas (2007); Ling and Takeuchi (2008); Marin, Power, Bowtell, Sanchez, Becker, Glover and Jones (2008); Beilina and Klibanov (2008)].

2 Solvability of the Problem

Let us introduce some definitions and notations, which will be used throughout this paper. Let $\widetilde{C}_0^3 = \left\{\varphi : \varphi \in C^3(\Omega),\ \varphi = 0 \text{ on } \partial\Omega\right\}$ and select a set $\{w_1, w_2, ...\} \subset \widetilde{C}_0^3(\Omega)$, which is a complete and orthonormal set in $L_2(\Omega)$. We may assume here that the linear span of this set is everywhere dense in $H_{1,2}^0(\Omega)$, where $H_{1,2}^0(\Omega)$ is the set of all real-valued functions $u(x,v) \in L_2(\Omega)$ that have generalized derivatives $u_{x_i},\ u_{v_i},\ u_{x_i v_j},\ u_{v_i v_j}$ $(i,\ j = 1,2,...,n)$, which belong to $L_2(\Omega)$ and whose trace on $\partial\Omega$ is zero. Indeed, the space $H_{1,2}^0(\Omega)$ being separable, there exists a countable set $\{\varphi_i\}_{i=1}^\infty \subset \widetilde{C}_0^3(\Omega)$ which is everywhere dense in this space. If necessary, this set up can be extended to a set which is everywhere dense in $L_2(\Omega)$. Orthonormalizing the latter in $L_2(\Omega)$, we obtain $\{w_i\}_{i=1}^\infty$. We denote the orthogonal projector of $L_2(\Omega)$ onto $\mathscr{M}_n$ by $\mathscr{P}_n$, where $\mathscr{M}_n$ is the linear span of $\{w_1, w_2, ..., w_n\}$. The set of all functions u with the following two properties is denoted by $\Gamma(A)$:

i) For any $u \in L_2(\Omega)$ there exists a function $\mathscr{F} \in L_2(\Omega)$ such that for all $\varphi \in C_0^\infty(\Omega)$, $\langle u, A^*\varphi\rangle = \langle \mathscr{F}, \varphi\rangle$ and $Au = \mathscr{F}$, where $Au = \widehat{L}Lu$, $\widehat{L} = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i \partial v_i}$ and A^* is the operator which is conjugate to A in the sense of Lagrange. Here $\langle .,. \rangle$ is the scalar product in $L_2(\Omega)$ and $C_0^\infty(\Omega)$ is the set of all functions defined in Ω which have continuous partial derivatives of order up to all $k < \infty$, whose supports are compact subsets of Ω.

ii) There exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to \langle Au, u\rangle$ as $k \to \infty$.

Since the unknown function λ depends only on x, Problem 1 is overdetermined and this problem is replaced by the following determined one:

Problem 2. *Find a pair of functions (u,λ) defined in Ω that satisfies the following relations*

$$\begin{aligned} Lu &= \lambda(x,v), &(2)\\ u|_{\partial\Omega} &= u_0, &(3)\\ \widehat{L}\lambda &= 0, &(4)\end{aligned}$$

provided that the Hamiltonian $H(x,v) \in C^2\left(\overline{\Omega}\right)$, $K(x,v,v') \in C^1\left(\overline{\Omega}\right)$ are given.

Here Eq. 4 is satisfied in generalized functions sense, i.e., $\left\langle \lambda, \widehat{L}^*\eta \right\rangle = 0$ for any $\eta \in C_0^\infty(\Omega)$.

Theorem 1. *Suppose that $H \in C^2\left(\overline{\Omega}\right)$ and the inequalities:*

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial v_i \partial v_j}\xi^i\xi^j \geq \alpha_1 |\xi|^2, \ \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j}\xi^i\xi^j \leq 0, \ \left(\alpha_1 - \frac{1}{2}(1+L_0)\right) > 0, \tag{5}$$

hold for all $\xi \in \mathbb{R}^n$, $(x,v) \in \overline{\Omega}$. In (5), α_1 is a positive number, $L_0 = K_0 (mesG)^2 C_0$ where $mesG$ is Lebesgue measure of G, $K_0 = \max\limits_{(x,v)\in\overline{\Omega}} \left\{K_{v_j}^2\right\}$ and C_0 is a constant occurred by virtue of Steklov inequality. Then Problem 2 has at most one solution (u,λ) such that $u \in \Gamma(A)$ and $\lambda \in L_2(\Omega)$.

Proof. The proof of Theorem 1 is similar to Theorem 2.2.1 in [Amirov (2001)]. But, due to the scattered term, this proof requires non-trivial modifications. Let (u,λ) be a solution to Problem 2 such that $u = 0$ on $\partial\Omega$ and $u \in \Gamma(A)$. Eq. 2 and condition (4) imply $Au = 0$. Since $u \in \Gamma(A)$, there exists a sequence $\{u_k\} \subset \widetilde{C}_0^3$ such that $u_k \to u$ in $L_2(\Omega)$ and $\langle Au_k, u_k\rangle \to 0$ as $k \to \infty$. Observing that $u_k = 0$ on $\partial\Omega$, we have

$$-\langle Au_k, u_k\rangle = \sum_{i=1}^{n} \left\langle \frac{\partial}{\partial v_i}(Lu_k), u_{k_{x_i}} \right\rangle. \tag{6}$$

For $lu \equiv \{u, H\}$,

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial u_k}{\partial x_j} \frac{\partial}{\partial v_j} (lu_k) \\
= \; & \frac{1}{2} \sum_{i,j=1}^{n} \left(\frac{\partial^2 H}{\partial v_i \partial v_j} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j} \frac{\partial u_k}{\partial v_i} \frac{\partial u_k}{\partial v_j} \right) \\
& + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial v_j} \left[\frac{\partial u_k}{\partial x_j} \left(\frac{\partial u_k}{\partial x_i} \frac{\partial H}{\partial v_i} - \frac{\partial u_k}{\partial v_i} \frac{\partial H}{\partial x_i} \right) \right] + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial v_i} \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial v_j} \right) \\
& - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left[\frac{\partial u_k}{\partial v_j} \left(\frac{\partial u_k}{\partial x_i} \frac{\partial H}{\partial v_i} - \frac{\partial u_k}{\partial v_i} \frac{\partial H}{\partial x_i} \right) \right] - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial}{\partial v_i} \left(\frac{\partial H}{\partial x_i} \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial v_j} \right) \quad (7)
\end{aligned}$$

If the geometry of the domain Ω and the condition $u_k = 0$ on $\partial\Omega$ are taken into account, then from (6) and (7) we obtain

$$-\langle Au_k, u_k \rangle = J(u_k) + \sum_{j=1}^{n} \left\langle \frac{\partial}{\partial v_j} (I_1 u_k), u_{k_{x_j}} \right\rangle, \tag{8}$$

where

$$J(u_k) \equiv \frac{1}{2} \sum_{i,j=1}^{n} \int_{\Omega} \left(\frac{\partial^2 H}{\partial v_i \partial v_j} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j} \frac{\partial u_k}{\partial v_i} \frac{\partial u_k}{\partial v_j} \right) d\Omega. \tag{9}$$

We now estimate the second term on the right hand side of (8). Using the Cauchy-Schwarz inequality and the condition $u_k|_{\partial\Omega} = 0$, we have

$$\sum_{j=1}^{n} \left\langle \frac{\partial}{\partial v_j} (I_1 u_k), u_{k_{x_j}} \right\rangle \leq \frac{1}{2} \sum_{j=1}^{n} \int_{\Omega} u_{k_{x_j}}^2 d\Omega + \sum_{j=1}^{n} \frac{K_0}{2} (mesG)^2 C_0 \int_{\Omega} u_{k_{x_j}}^2 d\Omega, \tag{10}$$

where K_0, L_0, C_0 and $mesG$ are given in the statement of the theorem. Thus from (5), (8) and (10), we obtain the following inequality

$$\begin{aligned}
J(u_k) + \sum_{j=1}^{n} \left\langle \frac{\partial}{\partial v_j} (I_1 u_k), u_{k_{x_j}} \right\rangle &\geq \alpha_1 \sum_{j=1}^{n} \int_{\Omega} u_{k_{x_j}}^2 d\Omega + \sum_{j=1}^{n} \left\langle \frac{\partial}{\partial v_j} (I_1 u_k), u_{k_{x_j}} \right\rangle \\
&\geq \alpha_1 \sum_{j=1}^{n} \int_{\Omega} u_{k_{x_j}}^2 d\Omega - \frac{1}{2} (1 + L_0) \int_{\Omega} u_{k_{x_j}}^2 d\Omega \\
&= \left(\alpha_1 - \frac{1}{2} (1 + L_0) \right) \sum_{j=1}^{n} \int_{\Omega} u_{k_{x_j}}^2 d\Omega \qquad (11)
\end{aligned}$$

and using definition of $\Gamma(A)$ we have $\int\limits_{\Omega} |\nabla_x u|^2 \, d\Omega \leq 0$, where $\nabla_x u = (u_{x_1}, \ldots, u_{x_n})$. Since $u = 0$ on $\partial\Omega$, it follows that $u = 0$ in Ω. Then (2) implies $\lambda(x,v) = 0$. Hence uniqueness of the solution is proven. □

If $u_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ then Problem 2 can be reduced to the following problem [Amirov (2001)].

Problem 3. *Determine the pair* (u,λ) *from the equation*

$$Lu = \lambda(x,v) + F \tag{12}$$

provided that $F \in H^2(\Omega)$, $H \in C^2\left(\overline{\Omega}\right)$, $K \in C^1\left(\overline{\Omega}\right)$ *are given, the trace of the solution* u *on the boundary* $\partial\Omega$ *is zero and* λ *satisfies Eq. 4.*

Remark 1. *The solvability of Problem 1 depends essentially on the geometry of the domain* Ω*. More precisely, it is important that* Ω *can be represented in the form of the direct product of two domains D and G [Amirov (2001)].*

In this reduction, we simply consider a new unknown function $\overline{u} = u - \Phi$, where Φ is a function such that $\Phi|_{\partial\Omega} = u_0$ and $\Phi \in C^3(\Omega)$. Since $u_0 \in C^3(\partial\Omega)$ and $\partial D \in C^3$, $\partial G \in C^3$ the existence of the function Φ follows from Theorem 2, Sec. 4.2., Chapter III in [Mikhailov (1978)]. Finally, if we again denote $\overline{u}$ by u, we can obtain Eq. 12 and the condition $u|_{\partial\Omega} = 0$, where $F = -L\Phi$. Of course, the function $\overline{u}$ depends on F (therefore on Φ). But because of the uniqueness of the solution to Problem 2, a function $u = \overline{u} + \Phi$ does not depend on choice of Φ (also on F) and it depends only on u_0. This is a standart situation for the Galerkin method (see e.g. section 2.3., chapter 5, in [Mikhailov (1978)]).

Theorem 2. *Assume* $H \in C^2\left(\overline{\Omega}\right)$ *and the following inequalities hold for all* $(x,v) \in \overline{\Omega}$, $\xi \in \mathbb{R}^n$:

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial v_i \partial v_j} \xi^i \xi^j \geq \alpha_1 |\xi|^2, \ \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j} \xi^i \xi^j \leq -\alpha_2 |\xi|^2, \ \left(\alpha_1 - \frac{1}{2}(1 + L_0)\right) > 0, \tag{13}$$

where α_2 *is a positive number and* $F \in H^2(\Omega)$*. Then there exists a solution* (u,λ) *of Problem 3 such that* $u \in \Gamma(A) \cap H^1(\Omega)$, $\lambda \in L_2(\Omega)$.

Proof. Let us consider the following auxiliary problem

$$Au = \mathscr{F}, \tag{14}$$

$$u|_{\partial\Omega} = 0, \tag{15}$$

where $\mathscr{F} = \widehat{L}F$. An approximate solution to the Problem (14)-(15) is sought in the form

$$u_N = \sum_{i=1}^{N} \alpha_{N_i} w_i, \quad \alpha_N = (\alpha_{N_1}, \alpha_{N_2}, ..., \alpha_{N_N}) \in \mathbb{R}^N, \tag{16}$$

with the help of the following relations:

$$\langle Au_N - \mathscr{F}, w_i \rangle = 0, \qquad i = 1, 2, ..., N. \tag{17}$$

Equalities (17) form a system of linear algebraic equations for the vector α_N. Let's multiply ith equation of the homogeneous system ($\mathscr{F} = 0$) by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i, then $-2\langle Au_N, u_N\rangle = 0$ is obtained. If identity (8) is considered then the assumptions of the theorem imply $\nabla u_N = 0$, $\nabla u_N = \left(u_{N_{x_1}}, ..., u_{N_{x_n}}, u_{N_{v_1}}, ..., u_{N_{v_n}}\right)$ and due to the conditions $u_N = 0$ on $\partial\Omega$ and $u_N \in \widetilde{C}_0^3(\Omega)$, we have $u_N = 0$ in Ω. Since the system $\{w_i\}$ is linearly independent, we obtain $\alpha_{N_i} = 0$, $i = 1, 2, ..., N$. The homogeneous version of system (17) has only trivial solution and thus, system (17) has a unique solution $\alpha_N = (\alpha_{N_i})$, $i = 1, ..., N$ for any function $F \in H^2(\Omega)$.

Now we estimate the solution u_N in terms of F. We multiply the ith equation of the system by $-2\alpha_{N_i}$ and sum from 1 to N with respect to i. Since $\mathscr{F} = \widehat{L}F$,

$$-2\langle Au_N, u_N\rangle = -2\left\langle \widehat{L}F, u_N \right\rangle, \tag{18}$$

is obtaned. Observing that $u_N = 0$ on $\partial\Omega$ and transferring the derivatives with respect to x_i on the function u_N, the right-hand side of (18) can be estimated as

$$-2\left\langle \widehat{L}F, u_N \right\rangle \le \beta \int_{\Omega} |\nabla_v F|^2 d\Omega + \beta^{-1} \int_{\Omega} |\nabla_x u_N|^2 d\Omega, \tag{19}$$

where $0 < \beta^{-1} < \alpha_3 = \alpha_1 - \frac{1}{2}(1 + L_0)$ and $\nabla_v F = (F_{v_1}, ..., F_{v_n})$. In the proof of Theorem 1, we showed that $-\langle Au_N, u_N\rangle$ is equal to

$$J(u_N) + \sum_{j=1}^{n} \int_{\Omega} u_{N_{x_j}} \int_{G} K_{v_j} u_N dv' d\Omega. \tag{20}$$

Then using (10), (13) and (18) we have

$$\alpha_2 \int_{\Omega} |\nabla_v u_N|^2 d\Omega + \alpha_3 \int_{\Omega} |\nabla_x u_N|^2 d\Omega \le \beta \int_{\Omega} |\nabla_v F|^2 d\Omega + \beta^{-1} \int_{\Omega} |\nabla_x u_N|^2 d\Omega. \tag{21}$$

Recalling that Ω is bounded and $u_N = 0$ on $\partial\Omega$, the last inequality implies

$$\|u_N\|_{\overset{\circ}{H}^1(\Omega)} \leq C \, \| |\nabla_v F| \|_{L_2(\Omega)}, \tag{22}$$

where the constant $C > 0$ does not depend on N. Since $\overset{\circ}{H}^1(\Omega)$ is a Hilbert space, there exists a subsequence in this set, denoted again by $\{u_N\}$, converges weakly to a certain function $u \in \overset{\circ}{H}^1(\Omega)$ and $\|u\|_{\overset{\circ}{H}^1(\Omega)} \leq \underline{\lim}_{N\to\infty} \|u_N\|_{\overset{\circ}{H}^1(\Omega)} \leq C \, \| |\nabla_v F| \|_{L_2(\Omega)}$ holds. Transferring the operator $\widehat{L}$ to w_i in (17) and passing to the limit as $N \to \infty$ yield to

$$\left\langle Lu - F, \widehat{L}\eta \right\rangle = 0, \tag{23}$$

for any $\eta \in \overset{\circ}{H}_{1,2}(\Omega)$. Setting $\lambda = Lu - F$ and taking into account $C_0^\infty(\Omega) \subset \overset{\circ}{H}_{1,2}(\Omega)$, we see that λ satisfies the condition (4) and using (22) we obtain

$$\|\lambda\|_{L_2(\Omega)} \leq C \, \|\nabla_v F\|_{L_2(\Omega)} + \|F\|_{L_2(\Omega)}. \tag{24}$$

In expression (24), C stands for different constants that depend only on the given functions and the measure of the domain. Thus we have found a solution (u, λ) to Problem 3, where $u \in \overset{\circ}{H}^1(\Omega)$ and $\lambda \in L_2(\Omega)$. Now it will be proven that $u \in \Gamma(A)$. Since $u \in L_2(\Omega)$, $F \in H^2(\Omega)$ and $\mathscr{F} = \widehat{L}F$, from (23) it follows that $\mathscr{F} = Au \in L_2(\Omega)$ in the generalized functions sense. Indeed, for any $\eta \in C_0^\infty(\Omega)$ we have

$$\langle u, A^*\eta \rangle = \left\langle u, \left(\widehat{L}L\right)^* \eta \right\rangle = \left\langle Lu, \widehat{L}\eta \right\rangle = \left\langle F, \widehat{L}\eta \right\rangle = \langle \mathscr{F}, \eta \rangle. \tag{25}$$

To complete the proof, it remains to show the convergence $\langle Au_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$. From (17), it follows that $\mathscr{P}_N A u_N = \mathscr{P}_N \mathscr{F}$. Since $\mathscr{P}_N$ is an orthogonal projector onto $\mathscr{M}_n$, $\mathscr{P}_N \mathscr{F}$ strongly converges to $\mathscr{F}$ in $L_2(\Omega)$ as $N \to \infty$, i.e., $\mathscr{P}_N A u_N \to \mathscr{F} = Au$ strongly in $L_2(\Omega)$ as $N \to \infty$. Then, $\langle \mathscr{P}_N A u_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$ because $\{u_N\}$ weakly converges to u in $L_2(\Omega)$ as $N \to \infty$. By the definition of $\mathscr{P}_N$ and u_N (since the operator $\mathscr{P}_N$ is self adjoint in $L_2(\Omega)$), $\langle Au_N, u_N \rangle = \langle Au_N, \mathscr{P}_N u_N \rangle = \langle \mathscr{P}_N A u_N, u_N \rangle$. Hence $\langle Au_N, u_N \rangle \to \langle Au, u \rangle$ as $N \to \infty$, which completes the proof. □

3 The Finite Difference Method (FDM)

Now we concern with the construction of finite difference approximation for the following 2-dimensional inverse problem: Find (u,λ) from the relations

$$H_v(x,v)\,u_x(x,v) - H_x(x,v)\,u_v(x,v) = \lambda(x,v), \tag{26}$$

$$u(x,v)|_{\partial\Omega} = u_0(x,v), \tag{27}$$

$$\widehat{L}\lambda = 0, \tag{28}$$

where $\Omega = \{(x,v)|\ x\in(a,b)\subset\mathbb{R},\ v\in(c,d)\subset\mathbb{R}\}$. By applying operator $\widehat{L}$ to both sides of the Eq. 26, the following auxiliary Dirichlet problem for third order partial differential equation is obtained:

$$Au \equiv u_{xvx}H_v - u_{vvx}H_x + u_{xx}H_{vv} - u_{vv}H_{xx} + u_{xv}H_{vx} - u_{vx}H_{xv} + u_xH_{vvx} - u_vH_{xvx} = 0, \tag{29}$$

$$u|_{\partial\Omega} = u_0. \tag{30}$$

Using the central finite difference formulas in (29), we obtain the following system of simultaneous algebraic nodal equations:

$$\begin{aligned}
&(-k_1+k_2)\,\tilde{u}_{i-1,j-1} + (2k_1-k_4+k_6)\,\tilde{u}_{i,j-1} + (-k_1-k_2)\,\tilde{u}_{i+1,j-1}\\
&+(-2k_2+k_3-k_5)\,\tilde{u}_{i-1,j} + (-2k_3+2k_4)\,\tilde{u}_{i,j} + (2k_2+k_3+k_5)\,\tilde{u}_{i+1,j}\\
&+(k_1+k_2)\,\tilde{u}_{i-1,j+1} + (-2k_1-k_4-k_6)\,\tilde{u}_{i,j+1} + (k_1-k_2)\,\tilde{u}_{i+1,j+1}\\
=\ &0,\ i=1,\ldots,I,\ j=1,\ldots,J,
\end{aligned} \tag{31}$$

where $I,\ J$ are positive integers, $\Delta x = (b-a)/(I+1)$ and $\Delta v = (d-c)/(J+1)$ are step sizes in the directions $x,\ v$, respectively and $\tilde{u}_{i,j}$ is the finite difference approximation for the solution $u(x_i,v_j) = u(a+i\Delta x, c+j\Delta v)$,

$$k_1 = \frac{h_{i,j+1}-h_{i,j-1}}{4(\Delta x)^2(\Delta v)^2},\ k_2 = \frac{h_{i+1,j}-h_{i-1,j}}{4(\Delta x)^2(\Delta v)^2}, \tag{32}$$

$$k_3 = \frac{h_{i,j+1}-2h_{i,j}+h_{i,j-1}}{(\Delta x)^2(\Delta v)^2},\ k_4 = \frac{h_{i+1,j}-2h_{i,j}+h_{i-1,j}}{(\Delta x)^2(\Delta v)^2}, \tag{33}$$

$$k_5 = \frac{h_{i+1,j+1}-2h_{i+1,j}+h_{i+1,j-1}-h_{i-1,j+1}+2h_{i-1,j}-h_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2}, \tag{34}$$

$$k_6 = \frac{h_{i+1,j+1}-2h_{i,j+1}+h_{i-1,j+1}-h_{i+1,j-1}+2h_{i,j-1}-h_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2}. \tag{35}$$

Taking into account (30), we have the following discrete boundary conditions

$$\tilde{u}_{0,j} = u(a, v_j),\ \tilde{u}_{I+1,j} = u(b, v_j),\ j = 0, 1, ..., J+1, \tag{36}$$

$$\tilde{u}_{i,0} = u(x_i, c),\ \tilde{u}_{i,J+1} = u(v_i, d),\ i = 0, 1, ..., I+1. \tag{37}$$

The approximate solution $\tilde{u}_{i,j}$ is obtained at $I \times J$ mesh points of Ω by solving the matrix equation

$$\mathbf{T}\,\widetilde{\mathbf{U}} = \mathbf{V}, \tag{38}$$

where $\mathbf{T}$ is a block tridiagonal matrix

$$\mathbf{T} = \begin{bmatrix} A^{(1)} & B^{(1)} & 0 & \cdots & 0 \\ C^{(2)} & A^{(2)} & B^{(2)} & \ddots & \vdots \\ 0 & C^{(3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B^{(J-1)} \\ 0 & \cdots & 0 & C^{(J)} & A^{(J)} \end{bmatrix}_{IJ\times IJ} \tag{39}$$

and $A^{(j)}$, $B^{(j)}$, $C^{(j)}$ are given by

$$A^{(j)} = \begin{bmatrix} a_1^{(1,j)} & a_2^{(1,j)} & 0 & \cdots & 0 \\ a_3^{(2,j)} & a_1^{(2,j)} & a_2^{(2,j)} & \ddots & \vdots \\ 0 & a_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_2^{(I-1,j)} \\ 0 & \cdots & 0 & a_3^{(I,j)} & a_1^{(I,j)} \end{bmatrix}_{I\times I}, \tag{40}$$

$$B^{(j)} = \begin{bmatrix} b_1^{(1,j)} & b_2^{(1,j)} & 0 & \cdots & 0 \\ b_3^{(2,j)} & b_1^{(2,j)} & b_2^{(2,j)} & \ddots & \vdots \\ 0 & b_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b_2^{(I-1,j)} \\ 0 & \cdots & 0 & b_3^{(I,j)} & b_1^{(I,j)} \end{bmatrix}_{I\times I}, \tag{41}$$

$$C^{(j)} = \begin{bmatrix} c_1^{(1,j)} & c_2^{(1,j)} & 0 & \cdots & 0 \\ c_3^{(2,j)} & c_1^{(2,j)} & c_2^{(2,j)} & \ddots & \vdots \\ 0 & c_3^{(3,j)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_2^{(I-1,j)} \\ 0 & \cdots & 0 & c_3^{(I,j)} & c_1^{(I,j)} \end{bmatrix}_{I\times I} \tag{42}$$

where $a_1 = -2k_3 + 2k_4$, $a_2 = 2k_2 + k_3 + k_5$, $a_3 = -2k_2 + k_3 - k_5$, $b_1 = -2k_1 - k_4 - k_6$, $b_2 = k_1 - k_2$, $b_3 = k_1 + k_2$, $c_1 = 2k_1 - k_4 - k_6$, $c_2 = -k_1 - k_2$, $c_3 = -k_1 + k_2$.
$\mathbf{V}$ is a column matrix, which consists of boundary values $\tilde{u}_{0,j}$, $\tilde{u}_{I+1,j}$, $\tilde{u}_{i,0}$ and $\tilde{u}_{i,J+1}$ $(i = 0,1,...,I+1,\ j = 0,1,...,J+1)$ and $\widetilde{\mathbf{U}}$ is the solution vector:

$$\widetilde{\mathbf{U}} = [\tilde{u}_{1,1}, \tilde{u}_{2,1},, ..., \tilde{u}_{I,1}, \tilde{u}_{1,2}, \tilde{u}_{2,2}, ..., \tilde{u}_{I,2}, ..., \tilde{u}_{1,J}, \tilde{u}_{2,J}, ..., \tilde{u}_{I,J}]^T. \tag{43}$$

To calculate λ numerically, the central-difference formulas are used in (26) and the following difference equation is solved:

$$\Delta x \Delta v \left[k_1 \tilde{u}_{i+1,j} - k_1 \tilde{u}_{i-1,j} - k_2 \tilde{u}_{i,j+1} + k_2 \tilde{u}_{i,j-1}\right] = \tilde{\lambda}_{i,j}, \tag{44}$$

$i = 1,2,...,I,\ \ j = 1,2,...,J$, where $\tilde{\lambda}_{i,j}$ is the approximation to the function $\lambda(x_i, v_j) = \lambda(a + i\Delta x, c + j\Delta v)$.

4 Numerical Experiments

The proposed method has been implemented and evaluated on several inverse problems. Two examples are presented below. The computations are performed using MATLAB 7.0 program on a PC with Intel Core 2 T7200 2.00 GHz CPU, 1 Gb memory, running under Windows Vista. In all of our tests, we have introduced the multiplicative random noise in the boundary data u_σ by adding relative error to computed data u_{comp} using the following expression:

$$u_\sigma(x_i, v_j) = u_{comp}(x_i, v_j)\left[1 + \frac{\alpha(u_{\max} - u_{\min})\sigma}{100}\right]. \tag{45}$$

Here, (x_i, v_j) is a mesh point at the boundary $\partial\Omega$, α is a random number in the interval $[-1;1]$, $u_{\max}$ and $u_{\min}$ are maximal and minimal values of the computed data u_{comp}, respectively, and σ is the noise level in percents.

Example 1. *Let's consider the problem of finding (u,λ) in $\Omega = (-1,1)\times(1,2)$ from Eq. 26 provided that $H(x,v) = \frac{1}{2}v^2$, and the boundary conditions*

$$u(-1,v) = \frac{1}{2v}(2-v)^2,\ u(1,v) = \frac{1}{2v}(2-v)^2, \tag{46}$$

$$u(x,1) = \frac{1}{2}x^2,\ u(x,2) = \frac{1}{4}\left(x^2 - 1\right), \tag{47}$$

are given. The exact solution of the problem is $u(x,v) = \frac{1}{2v}\left(x^2 + (2-v)^2 - 1\right)$, $\lambda(x,v) = x$. In Fig. 1, exact solution and the finite difference solution of the inverse problem are given for $I = J = 39$.

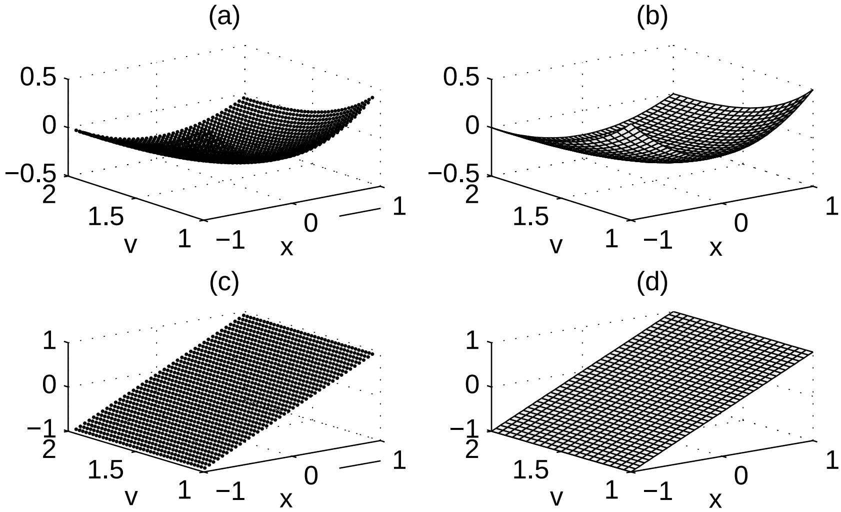

Figure 1: (a) Computed u, (b) Exact u, (c) Computed λ, (d) Exact λ.

The obtained numerical results for $u(x,v)$ on some points of the domain Ω for the different values of I and J are shown in Tab. 1. In the calculation of $u(x,v)$, the maximum error for $I = J = 7$ is 0.00373585544664928330 and for $I = 7, J = 511$ is 0.00000095343455730479.

Table 1: The exact and approximate values of $u(x,v)$.

(x,v)	Exact $u(x,v)$	FDM $I = J = 7$	FDM $I = 7, J = 511$
(-0.75,1.25)	0.0500000000000	0.0497136848553	0.0499999267765
(-0.75,1.75)	-0.1071428571428	-0.1072913375994	-0.1071428947755
(0,1.25)	-0.1750000000000	-0.1756544346163	-0.1750001673678
(0,1.50)	-0.2500000000000	-0.2506076181369	-0.2500001545289
(0,1.75)	-0.2678571428571	-0.2681965267579	-0.2678572288746
(0.75,1.25)	0.0500000000000	0.0497136848553	0.0499999267765
(0.75,1.75)	-0.1071428571428	-0.1072913375994	-0.1071428947755

Fig. 2 displays the one dimensional cross sections ($v = 1.5$) of computed approximate solutions with different noise levels superimposed with the exact solution of the inverse problem.

The obtained numerical results for $\lambda(x,v)$ on some points of the domain Ω for the different values of I and J are shown in Tab. 2.

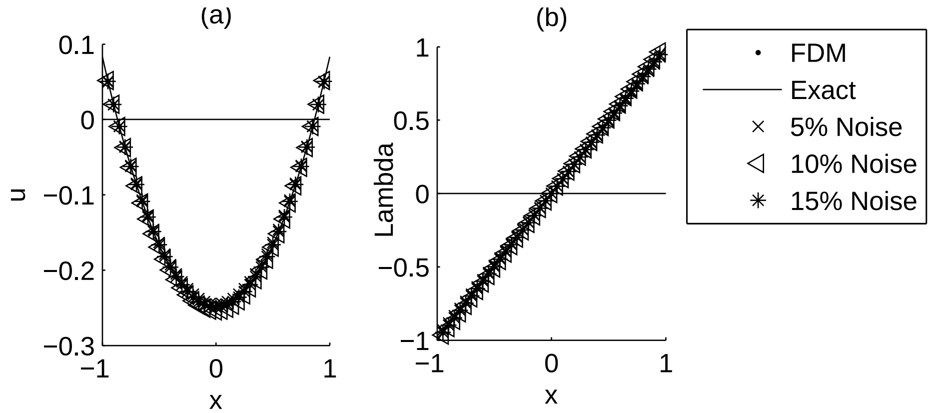

Figure 2: Exact solution and FDM solutions with different noise levels $\sigma = 5\%$, $\sigma = 10\%$, $\sigma = 15\%$ (a) for u (b) for λ.

Table 2: The exact and approximate values of $\lambda(x,v)$.

(x,v)	Exact $\lambda(x,v)$	FDM $I = J = 7$	FDM $I = 7, J = 511$
(-0.75,1.25)	-0.75	-0.751227064905570	-0.750000313814679
(-0.75,1.75)	-0.75	-0.750890882739660	-0.750000225796031
(0,1.25)	0.00	0.0000000000000002	0.000000000000002
(0,1.50)	0.00	0.0000000000000001	0.000000000000003
(0,1.75)	0.00	0.0000000000000000	0.000000000000002
(0.75,1.25)	0.75	0.7512270649055701	0.750000313814677
(0.75,1.75)	0.75	0.7508908827396604	0.750000225796031

In the following example, we consider the case when λ depends on both variables x and v.

Example 2. *: Determine a pair of functions (u,λ) defined in $\Omega = (-1,1)\times(2,3)$ that satisfies Eq. 26 and the conditions*

$$u(-1,v) = v + \frac{1}{v}\ln v,\ u(1,v) = v + \frac{1}{v}\ln v, \tag{48}$$

$$u(x,2) = 2x^2 + \frac{1}{2}\ln 2,\ u(x,3) = 3x^2 + \frac{1}{3}\ln 3, \tag{49}$$

and $H(x,v) = -x - \ln v$ are given. The exact solution of the problem is $u(x,v) =$

$x^2v + \dfrac{1}{v}\ln v$, $\lambda\,(x,v) = -2x + x^2 - \frac{1}{v^2}\ln v + \frac{1}{v^2}$.

In Fig. 3, FDM solution and exact solution of the inverse problem are shown for $I = J = 63$.

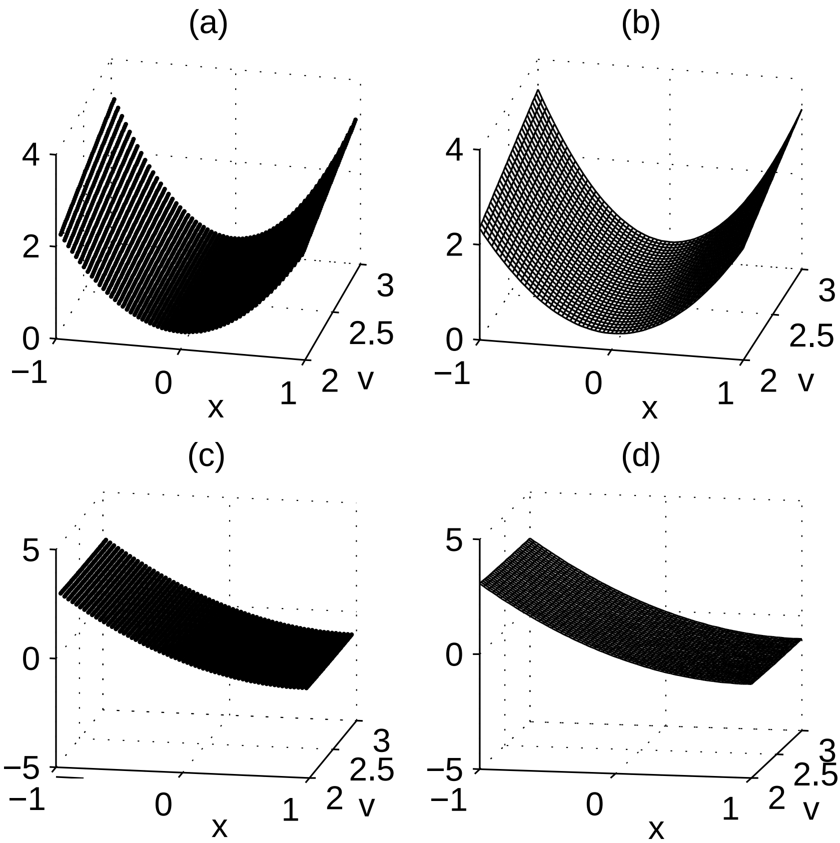

Figure 3: (a) Computed u, (b) Exact u, (c) Computed λ, (d) Exact λ.

On Fig. 4 below, a comparison between exact solution and the approximate solution of the inverse problem for different noise levels is presented by one dimensional cross sections ($v = 2.5$).

Consequently, numerical experiments have demonstrated the effectiveness of the proposed method in providing highly accurate numerical solutions even subjecting to large noise of the given boundary data.

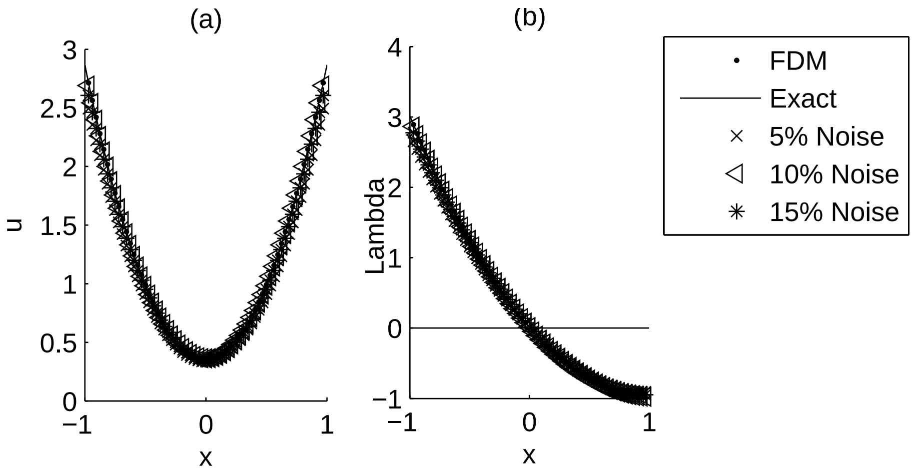

Figure 4: Exact solution and FDM solutions with different noise levels $\sigma = 5\%$, $\sigma = 10\%$, $\sigma = 15\%$ (a) for u (b) for λ.

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