 CMES, vol.37, no.3, pp.305-329, 2008

The Coupling Method of Natural Boundary Element and Mixed Finite Element for Stationary Navier-Stokes Equation in Unbounded Domains

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Abstract: The coupling method of natural boundary element and mixed finite element is applied to analyze the stationary Navier-Stokes equation in 2-D unbounded domains. After an artificial smooth boundary is introduced, the original nonlinear problem is reduced into an equivalent problem defined in bounded computational domain. The well-posedness of the reduced problem is proved. The finite element approximation of this problem is given, and numerical example is provided to show the feasibility and efficiency of the method.

Keyword: Navier-Stokes equation; boundary element method; coupling; mixed finite element method

1 Introduction

Let Ω_0 be a bounded and simple connected domain in $\mathbb{R}^2$ with sufficiently smooth boundary Γ_1. Consider the stationary Navier-Stokes equations in the exterior domain $\Omega := R^2/\overline{\Omega_0}$ ($\overline{M}$ denotes the closure of a set $M \subset \mathbb{R}^2$), under Dirichlet boundary conditions:

$$\begin{cases} -\mu\Delta\vec{u} + (\vec{u}\cdot\bigtriangledown)\vec{u} + \bigtriangledown p = \vec{f}, & in\ \Omega \\ div\ \vec{u} = 0, & in\ \Omega \\ \vec{u} = 0, & on\ \Gamma_1 \\ \vec{u} \to \vec{u}_\infty, \quad when\ \ r \to +\infty & \end{cases} \tag{1}$$

where $\vec{u} = (u_1, u_2)^T$ is the velocity vector of the fluid, p the kinematic static pressure, $\vec{f} = (f_1, f_2)$ the density of outer volume force, $\mu > 0$ is kinematic viscosity. $\vec{u}_\infty$ is non-zero vector field which we choose without restriction of generality to be

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parallel to the x-axis, i.e., $\vec{u}_\infty = (u_\infty, 0)$ and $u_\infty > 0$. $x = (x_1, x_2)^T$ is the coordinate, $r = \sqrt{x_1^2 + x_2^2}$. We assume $\vec{f}$ has compact support, i.e. $supp\{\vec{f}\} \subset \mathbb{B}$, $\mathbb{B}$ is a disk with radius R ($R > 0$ is a constant).

Let $\varepsilon_{ij}(\vec{u})$ and $\sigma_{ij}(\vec{u}, p)$ denote the rate of strain and the stress tensors, respectively,

$$\varepsilon_{ij}(\vec{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (i, j = 1, 2)$$

$$\sigma_{ij}(\vec{u}, p) = -p\delta_{ij} + 2\mu\varepsilon_{ij}(\vec{u}), \qquad (i, j = 1, 2)$$

where δ_{ij} is the Kronecker Delta whose properties are

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Moreover let $\vec{t} = (t_1, t_2)^T$ denote the normal stress,

$$t_i = \sum_{j=1}^{2} \sigma_{ij}(\vec{u}, p) n_j, \qquad (i = 1, 2)$$

$\vec{n} = (n_1, n_2)$ denote the unit normal on Γ_1 defined almost everywhere pointing from Ω_0 into Ω.

This problem has been investigated in a number of works. A detailed treatment can be found, e.g., in [Galdi (1994); Galdi (1999)].

There are several methods to solve boundary value problems in unbounded domains. One of the most popular methods is natural boundary reduction method and its coupling with finite element method, which is suggested and developed first by Feng and Yu in early 1980s [Feng and Yu (1983);Yu (1983); Yu (1985)]. In this reduction, the problem over unbounded domain is reduced into a boundary value problem in a bounded computational domain with a hyper-singular integral equation on the artificial boundary by using a Green function. This method is also known as the exact artificial boundary condition (DtN) method [Han and Wu (1985); Keller and Givoli (1989)]. In the last two decades, many authors have worked on this subject for various problems by different techniques, see [Givoli(1992); Yu (1993); Li and He (1993); Bao (2000); Yu (2002); B $\ddot{o}$nisch, Heuveline, and Wittwer (2005)]and the references therein. For hyper-singular integral equation, also can see [Aliabadi (2002); Chen and Hong (1999); Hong and Chen (1988)]. As to BEM and FEM, there are many application contexts can be found, e.g., in [Marin, Liviu, Power, Henry(2008); Fedelinski, P.and Gorski, R.(2006); Frangi, Ghezzi, and Faure-Ragani(2006); Springhetti, Novati, and Margonari(2006); Albuquerque, E. L.and Aliabadi, M. H.(2008)].

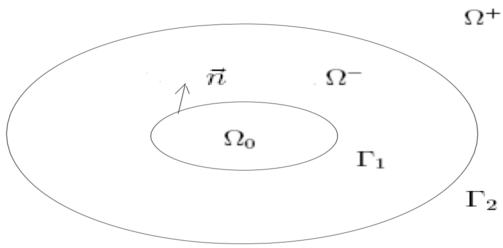

Figure 1: Domain of the model problem

In this article we consider the coupling method of natural boundary element and mixed finite element. We show here that this approach can be adapted to analyze the stationary Navier-Stokes equation in 2-D unbounded domains.

The rest of this paper is organized as follows. In section 2, we introduce an artificial boundary and approximate Navier-Stokes equations by linear Oseen equations via suitable transmission conditions. Then, we present a new version of the mixed FEM-BEM formulation, and prove a well-posed result of the reduced mixed variational problem. In section 3, finite element approximation of coupling method is given. In section 4, the approximate problem is solved by numerical implementation. Section 5 is to construct numerical example to test the performance of designed method.

2 Natural boundary reduction and coupling problem

We introduce an artificial interface Γ_2 dividing the original domain into two subdomains: a bounded interior domain Ω^- with $\partial\Omega^-=\Gamma_1 \cup \Gamma_2$, in which we consider the Navier-Stokes equation, and an unbounded domain Ω^+ lying outside Γ_2 with $\overline{\Omega^+} = \Omega^+ \cup \Gamma_2$ (see Fig.1.). In Ω^+ we approximate the nonlinear Navier-Stokes equation by the linear Oseen system. We use the transmission conditions according to [Feistauer and Schwab (2001)], then problem (1) is equivalent to the following coupled problem:

$$
\begin{cases}
-\mu\Delta\vec{u}^- + (\vec{u}^-\cdot\bigtriangledown)\vec{u}^- + \bigtriangledown p^- = \vec{f}, & in\ \Omega^- \\
div\ \vec{u}^- = 0, & in\ \Omega^- \\
\vec{u}^- = 0, & on\ \Gamma_1 \\
\vec{u}^- = \vec{u}^+, & on\ \Gamma_2 \\
\vec{t}^-|_{\Gamma_2} = \vec{t}^+|_{\Gamma_2} + & \\
\qquad \frac{1}{2}[(\vec{u}^-\cdot\vec{n})\vec{u}^- - (\vec{u}_\infty\cdot\vec{n})\vec{u}^+], & on\ \Gamma_2 \\
-\mu\Delta\vec{u}^+ + (\vec{u}_\infty\cdot\bigtriangledown)\vec{u}^+ + \bigtriangledown p^+ = 0, & in\ \Omega^+ \\
div\ \vec{u}^+ = 0, & in\ \Omega^+ \\
\vec{u}^+ \to \vec{u}_\infty, \quad when\ |x| \to +\infty &
\end{cases}
\tag{2}
$$

It is well known (see[Chadwick (1998)]) that in Ω^+ the velocity $\vec{u}$ and the pressure p has the following expression

$$
\begin{aligned}
u_x &= \frac{\partial\phi}{\partial x} - \frac{1}{2k}\frac{\partial\chi}{\partial x} + \chi \\
u_y &= \frac{\partial\phi}{\partial y} - \frac{1}{2k}\frac{\partial\chi}{\partial y} \\
p &= -u_\infty\frac{\partial\phi}{\partial x}
\end{aligned}
$$

where $k = \frac{Re}{2}$ (Re is the Reynolds number), ϕ and χ are two multi-valued functions satisfying the following equations:

$$
\begin{cases}
\Delta\phi = 0, & \Omega^+ \\
(\Delta - 2k\frac{\partial}{\partial x})\chi = 0, & \Omega^+
\end{cases}
\tag{3}
$$

Furthermore, they have the following expansions

$$
\phi = \frac{A_0}{2}\log r - \frac{B_0}{2}\theta - \sum_{n=1}^{+\infty}\frac{1}{n}(\frac{R}{r})^n(A_n\cos n\theta + B_n\sin n\theta)
$$

$$
\begin{aligned}
\chi &= \frac{C_0}{2K_0(kR)}e^{kx}K_0(kr) \\
&+ \frac{D_0}{K_0(kR)}\int_0^{-\infty} e^{k(x+\xi)}\frac{\partial}{\partial y}K_0(kr_\xi)d\xi \\
&+ e^{kx}\sum_{n=1}^{+\infty}\frac{K_n(kr)}{K_n(kR)}(C_n\cos n\theta + D_n\sin n\theta)
\end{aligned}
$$

where $r_\xi = \sqrt{(x-\xi)^2+y^2}$, I_n and K_n denote the first kind and second kind modified Bessel functions respectively. According to the natural boundary reduction theory and [Zheng and Han (2002)], we obtain the integral equation on artificial boundary Γ_2

$$\begin{pmatrix} g_1(\theta) \\ g_2(\theta) \end{pmatrix} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{4}$$

or

$$\vec{g} = K(\vec{u}|_{\Gamma_2}) \tag{5}$$

where, $\vec{g} = (g_1, g_2)^T$ is stress tensor.

$$\begin{aligned} A \quad = \quad & \sum_{n=0}^{+\infty} \delta_n (a_n \cos(n+1)\theta + b_n \sin(n+1)\theta) \\ & - \sum_{n=0}^{+\infty} \delta_n \Phi_n^1 \cos(n+1)\theta + \sum_{n=0}^{+\infty} \delta_n \Psi_n^2 \sin(n+1)\theta \\ & - \sum_{n=0}^{+\infty} \delta_n \tilde{\Phi}_n^1 \cos(n+1)\theta + \sum_{n=0}^{+\infty} \delta_n \tilde{\Psi}_n^2 \sin(n+1)\theta \\ & + \sum_{n=0}^{+\infty} \delta_n (\tilde{b}_n \cos(n+1)\theta - \tilde{a}_n \sin(n+1)\theta) \\ & + \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (nd_n - a_n) \cos n\theta \\ & - \frac{1}{kR} \sum_{n=1}^{+\infty} (c_n + nb_n) \sin n\theta \\ & - \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (\tilde{b}_n + n\tilde{c}_n) \cos n\theta \\ & + \frac{1}{kR} \sum_{n=1}^{+\infty} (n\tilde{a}_n - \tilde{d}_n) \sin n\theta \end{aligned}$$

$$\begin{aligned} B \quad = \quad & \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (na_n - d_n) \sin n\theta \\ & - \frac{1}{kR} \sum_{n=0}^{+\infty} \delta_n (nc_n + b_n) \cos n\theta \\ & - \frac{1}{2} \sum_{n=0}^{+\infty} \delta_n (\Phi_n^1 + \Psi_n^1) \sin(n+1)\theta \end{aligned}$$

$$+\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Phi_n^1-\Psi_n^1)\sin(n-1)\theta$$
$$-\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Phi_n^2+\Psi_n^2)\cos(n-1)\theta$$
$$-\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Psi_n^2-\Phi_n^2)\cos(n+1)\theta$$
$$+\frac{1}{kR}\sum_{n=0}^{+\infty}\delta_n(n\tilde{b}_n+\tilde{c}_n)\sin n\theta$$
$$+\frac{1}{kR}\sum_{n=0}^{+\infty}\delta_n(\tilde{a}_n-n\tilde{d}_n)\cos n\theta$$
$$-\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\tilde{\Phi}_n^1+\tilde{\Psi}_n^1)\sin(n+1)\theta$$
$$+\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\tilde{\Phi}_n^1-\tilde{\Psi}_n^1)\sin(n-1)\theta$$
$$-\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\tilde{\Phi}_n^2+\tilde{\Psi}_n^2)\cos(n-1)\theta$$
$$-\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\tilde{\Psi}_n^2-\tilde{\Phi}_n^2)\cos(n+1)\theta$$

where

$$a_n=\frac{1}{\pi}\int_0^{2\pi}u_1(R,\theta)\cos n\theta\cos\theta d\theta,$$
$$b_n=\frac{1}{\pi}\int_0^{2\pi}u_1(R,\theta)\cos n\theta\sin\theta d\theta,$$
$$c_n=\frac{1}{\pi}\int_0^{2\pi}u_1(R,\theta)\sin n\theta\cos\theta d\theta,$$
$$d_n=\frac{1}{\pi}\int_0^{2\pi}u_1(R,\theta)\sin n\theta\sin\theta d\theta,$$
$$\tilde{a}_n=\frac{1}{\pi}\int_0^{2\pi}u_2(R,\theta)\cos n\theta\cos\theta d\theta,$$
$$\tilde{b}_n=\frac{1}{\pi}\int_0^{2\pi}u_2(R,\theta)\cos n\theta\sin\theta d\theta,$$
$$\tilde{c}_n=\frac{1}{\pi}\int_0^{2\pi}u_2(R,\theta)\sin n\theta\cos\theta d\theta,$$

$$\tilde{d}_n = \frac{1}{\pi}\int_0^{2\pi} u_2(R,\theta)\sin n\theta \sin\theta d\theta,$$

$$\begin{aligned}
\Phi_n^1 &= \sum_{m=1}^{+\infty}\Theta_{nm}^1(a_m+d_m),\ \Phi_n^2=\sum_{m=1}^{+\infty}\Theta_{nm}^2(c_m-b_m),\\
\Psi_n^1 &= \sum_{m=1}^{+\infty}\Upsilon_{nm}^1(a_m+d_m),\ \Psi_n^2=\sum_{m=1}^{+\infty}\Upsilon_{nm}^2(c_m-b_m),\\
\tilde{\Phi}_n^1 &= \sum_{m=1}^{+\infty}\Theta_{nm}^1(\tilde{b}_m-\tilde{c}_m),\ \tilde{\Phi}_n^2=\sum_{m=1}^{+\infty}\Theta_{nm}^2(\tilde{a}_m+\tilde{d}_m),\\
\tilde{\Psi}_n^1 &= \sum_{m=1}^{+\infty}\Upsilon_{nm}^1(\tilde{b}_m-\tilde{c}_m),\ \tilde{\Psi}_n^2=\sum_{m=1}^{+\infty}\Upsilon_{nm}^2(\tilde{a}_m+\tilde{d}_m),
\end{aligned}$$

$$\Theta_{nm}^1=\sum_{k=0}^{+\infty}\Phi_{nk}^1\Xi_{km}^1,\qquad \Theta_{nm}^2=\sum_{k=0}^{+\infty}\Phi_{nk}^2\Xi_{km}^2,$$

$$\Upsilon_{nm}^1=\sum_{k=0}^{+\infty}\Psi_{nk}^1\Xi_{km}^1,\qquad \Upsilon_{nm}^2=\sum_{k=0}^{+\infty}\Psi_{nk}^2\Xi_{km}^2$$

Here,

$$\begin{aligned}
\Phi_{mn}^1 &= \frac{1}{2}(I'_{m+n}+I'_{m-n}-\frac{K'_n}{K_n}(I_{m+n}+I_{m-n})),\\
&\quad (m\ge 0, n\ge 0)\\
\Phi_{mn}^2 &= \frac{1}{2}(I'_{m-n}-I'_{m+n}-\frac{K'_n}{K_n}(I_{m-n}-I_{m+n})),\\
&\quad (m\ge 0, n>0)\\
\Phi_{mn}^2 &= \frac{I_{m-1}-I_{m+1}}{2},\qquad m\ge 0, n=0\\
\Psi_{mn}^1 &= \frac{1}{2kR}\delta_n((2n-m)I_{m-n}-(2n+m)I_{m+n}),\\
&\quad (m\ge 0, n\ge 0)\\
\Psi_{mn}^2 &= -\frac{1}{2kR}((2n-m)I_{m-n}+(2n+m)I_{m+n}),\\
&\quad (m\ge 0, n>0)\\
\Psi_{mn}^2 &= I'_m+\frac{K'_0}{K_0}I_m,\qquad m\ge 0, n=0
\end{aligned}$$

Ξ_{km}^1 and Ξ_{km}^2 denote the corresponding inverse operators (if exist) of infinite matrix $(\Phi_{mn}^1-\Psi_{mn}^1)$ and $(\Phi_{mn}^2+\Psi_{mn}^2)$.

Hence problem (2) is equivalent to the following problem:

$$\begin{cases} -\mu\Delta\vec{u}^- + (\vec{u}^- \cdot \bigtriangledown)\vec{u}^- + \bigtriangledown p^- = \vec{f}, & in\ \Omega^- \\ div\ \vec{u}^- = 0, & in\ \Omega^- \\ \vec{u}^- = 0, & on\ \Gamma_1 \\ \vec{u}^- = \vec{u}^+, & on\ \Gamma_2 \\ \vec{t}^-|_{\Gamma_2} = \vec{t}^+|_{\Gamma_2} + & \\ \qquad \frac{1}{2}[(\vec{u}^- \cdot \vec{n})\vec{u}^- - (\vec{u}_\infty \cdot \vec{n})\vec{u}^+], & on\ \Gamma_2 \\ \vec{t}^+ = \vec{g}, & on\ \Gamma_2 \end{cases} \tag{6}$$

Let

$$W = \{\vec{v} \in H^1(\Omega^-)^2, \vec{v}|_{\Gamma_1} = 0\}$$

$$W_0 = \{\vec{v} \in W, div\ \vec{v} = 0\}$$

with norm

$$\|\vec{v}\|_W = (\int_{\Omega^-} (v_1^2 + v_2^2)dx)^{1/2}$$

$$Q = \{q \in L^2(\Omega^-), \int_{\Omega^-} qdx = 0\}$$

$$a(\vec{u},\vec{v}) = \mu(grad\ \vec{u}, grad\ \vec{v})$$

$$a_1(\vec{u},\vec{u},\vec{v}) = \int_{\Omega^-} (\vec{u} \cdot \bigtriangledown)\vec{u} \cdot \vec{v}dx$$

$$a_2(\vec{u},\vec{u},\vec{v}) = -\frac{1}{2}\int_{\Gamma_2} [(\vec{u} - \vec{u}_\infty) \cdot \vec{n}](\vec{u} \cdot \vec{v})dx$$

$$b(\vec{u},p) = -\int_{\Omega^-} p\ div\ \vec{u}dx \quad \forall \vec{u} \in W, p \in Q$$

$$(\vec{f},\vec{v}) = \sum_{i,j=1}^{2} \int_{\Omega^-} f_i v_i dx$$

$$< \vec{v},\vec{t} >_{\Gamma_2} = \sum_{i,j=1}^{2} < v_i, t_i >_{\Gamma_2}$$

where $< \cdot,\cdot >_{\Gamma_2}$ denote the duality pairing between the space $H^{1/2}(\Gamma_2)$ and $H^{-1/2}(\Gamma_2)$. For $\forall\ \vec{v} \in W$, applying Green formula in Ω^-:

$$a(\vec{u},\vec{v}) + a_1(\vec{u};\vec{u},\vec{v}) - < \vec{t},\vec{v} >_{\Gamma_2} + b(\vec{v},p) = (\vec{f},\vec{v})$$

We know that the direction of outward normal on Γ_2 is opposite for Ω^- and Ω^+, using the transmission conditions on Γ^+ and natural integral equation (5), we have

$$a(\vec{u},\vec{v})+a_1(\vec{u};\vec{u},\vec{v})+a_2(\vec{u};\vec{u},\vec{v})+<K\vec{u},\vec{v}>_{\Gamma_2}$$

$$+b(\vec{v},p)=(\vec{f},\vec{v}), \qquad \forall\vec{v}\in W$$

for $\forall\vec{w},\vec{u},\vec{v}\in W$, we define

$$A_0(\vec{w};\vec{u},\vec{v})=a(\vec{u},\vec{v})+a_1(\vec{w};\vec{u},\vec{v})+a_2(\vec{w};\vec{u},\vec{v})$$

$$B(\vec{u},\vec{v})=<K\vec{u},\vec{v}>_{\Gamma_2}=\int_{\Gamma_2}(g_1v_1+g_2v_2)ds \tag{7}$$

then boundary value problem (6) is equivalent to the variational problem on bounded domain Ω^- as follows:

$$\begin{cases} find\ (\vec{u},p)\in W\times Q,\ such\ that \\ A_0(\vec{u};\vec{u},\vec{v})+B(\vec{u},\vec{v})+b(\vec{v},p) \\ =(\vec{f},\vec{v}), & \forall\ \vec{v}\in W \\ b(\vec{u},q)=0, & \forall\ q\in Q \end{cases} \tag{8}$$

or

$$\begin{cases} find\ \ \vec{u}\in W_0,\ such\ that \\ A_0(\vec{u};\vec{u},\vec{v})+B(\vec{u},\vec{v})=(\vec{f},\vec{v}), & \forall\ \vec{v}\in W_0 \end{cases} \tag{9}$$

Lemma 1. The bilinear form $a(\vec{u},\vec{v})$ is symmetric, bounded and coercive on $W\times W$, further, $a_1(\vec{u},\vec{u},\vec{v})$ and $a_2(\vec{u},\vec{u},\vec{v})$ are continuous trilinear forms on W.
Lemma 2. The bilinear forms $B(\vec{u},\vec{v})$ is bounded on $(H^{1/2}(\Gamma_2)/R)^2\times(H^{1/2}(\Gamma_2)/R)^2$, i.e., there exists a constant $C>0$, such that

$$B(\vec{u},\vec{v})|\leq C\|\vec{u}\|_{H^{1/2}(\Gamma_2)/R}\|\vec{v}\|_{H^{1/2}(\Gamma_2)/R},\ \ \forall\vec{u},\vec{v}\in W$$

furthermore,

$$a_2(\vec{v};\vec{v},\vec{v})+B(\vec{v},\vec{v})\geq 0$$

Proof:
(i) Let

$$\bar{a}_n=\frac{1}{\pi}\int_0^{2\pi}v_1(R,\theta)\cos n\theta\cos\theta d\theta,$$
$$\bar{a}_n'=\frac{1}{\pi}\int_0^{2\pi}v_2(R,\theta)\cos n\theta\cos\theta d\theta,$$

$$\bar{b}_n = \frac{1}{\pi}\int_0^{2\pi} v_1(R,\theta)\cos n\theta \sin\theta d\theta,$$
$$\tilde{b}'_n = \frac{1}{\pi}\int_0^{2\pi} v_2(R,\theta)\cos n\theta \sin\theta d\theta,$$
$$\bar{c}_n = \frac{1}{\pi}\int_0^{2\pi} v_1(R,\theta)\sin n\theta \cos\theta d\theta,$$
$$\tilde{c}'_n = \frac{1}{\pi}\int_0^{2\pi} v_2(R,\theta)\sin n\theta \cos\theta d\theta,$$
$$\bar{d}_n = \frac{1}{\pi}\int_0^{2\pi} v_1(R,\theta)\sin n\theta \sin\theta d\theta,$$
$$\tilde{d}'_n = \frac{1}{\pi}\int_0^{2\pi} v_2(R,\theta)\sin n\theta \sin\theta d\theta,$$

By $H\ddot{o}$lder inequality and trace theorem, we know that there exists constant c_1, c_2, such that

$$[\sum_{n=1}^{+\infty} n(a_n^2+b_n^2+c_n^2+d_n^2+\tilde{a}_n^2+\tilde{b}_n^2+\tilde{c}_n^2+\tilde{d}_n^2)]^{1/2} \leq c_1\|\vec{u}\|_W$$

$$[\sum_{n=1}^{+\infty} n(\bar{a}_n^2+\bar{b}_n^2+\bar{c}_n^2+\bar{d}_n^2+\tilde{a}_n'^2+\tilde{b}_n'^2+\tilde{c}_n'^2+\tilde{d}_n'^2)]^{1/2} \leq c_2\|\vec{v}\|_W$$

substituting all these into (7), we deduce that

$$\begin{aligned}
|B(u,v)| \leq\ & C\sum_{n=1}^{+\infty} n[|a_n\bar{a}_n+b_n\bar{b}_n+c_n\bar{c}_n+d_n\bar{d}_n| \\
&+|\bar{a}_nd_n+\bar{d}_na_n+\bar{b}_nc_n+\bar{c}_nb_n| \\
&+|\tilde{a}_n\bar{c}_n+\tilde{c}_n\bar{a}_n+\tilde{b}_n\bar{d}_n+\tilde{d}_n\bar{b}_n| \\
&+|\bar{a}_n\tilde{b}_n+\bar{b}_n\tilde{a}_n+\bar{d}_n\tilde{c}_n+\bar{c}_n\tilde{d}_n| \\
&+|\tilde{a}_n\tilde{a}'_n+\tilde{b}_n\tilde{b}'_n+\tilde{c}_n\tilde{c}'_n+\tilde{d}_n\tilde{d}'_n| \\
&+|a_n\tilde{c}'_n+c_n\tilde{a}'_n+b_n\tilde{d}'_n+d_n\tilde{b}'_n| \\
&+|a_n\tilde{b}'_n+c_n\tilde{d}'_n+d_n\tilde{c}'_n+b_n\tilde{a}'_n| \\
&+|\tilde{c}_n\tilde{b}'_n+\tilde{a}_n\tilde{d}'_n+\tilde{b}_n\tilde{c}'_n+\tilde{d}_n\tilde{a}'_n|] \\
\leq\ & C[\sum_{n=1}^{+\infty} n(a_n^2+b_n^2+c_n^2+d_n^2 \\
&+\tilde{a}_n^2+\tilde{b}_n^2+\tilde{c}_n^2+\tilde{d}_n^2)]^{1/2} \\
&\cdot[\sum_{n=1}^{+\infty} n(\bar{a}_n^2+\bar{b}_n^2+\bar{c}_n^2+\bar{d}_n^2
\end{aligned}$$

$$
\begin{aligned}
&+\tilde{a}_n'^2+\tilde{b}_n'^2+\tilde{c}_n'^2+\tilde{d}_n'^2)]^{1/2} \\
\leq\ & C\|\vec{u}\|_W \cdot \|\vec{v}\|_W
\end{aligned}
$$

(ii) After some computation, we have

$$
\begin{aligned}
& B(v,v) = \frac{\pi}{kR}\sum_{n=0}^{+\infty}\delta_n\{-[(\tilde{a}_n-b_n)^2+(\tilde{b}_n+a_n)^2 \\
& +(\tilde{c}_n-d_n)^2+(\tilde{d}_n+c_n)^2]-2n[(\tilde{c}_n-d_n)\cdot \\
& (a_n+\tilde{b}_n)+(\tilde{d}_n+c_n)(b_n-\tilde{a}_n)]\}+\sum_{n=0}^{+\infty}\delta_n\cdot \\
& \{(a_n+\tilde{b}_n)\int_0^{2\pi}\cos(n+1)\theta(v_1\cos\theta+v_2\sin\theta)d\theta \\
& +(b_n-\tilde{a}_n)\int_0^{2\pi}\sin(n+1)\theta(v_1\cos\theta+v_2\sin\theta)d\theta \\
& -(\Phi_n^1+\tilde{\Phi}_n^1)\int_0^{2\pi}\cos(n+1)\theta(v_1\cos\theta+v_2\sin\theta)d\theta \\
& +(\Psi_n^2+\tilde{\Psi}_n^2)\int_0^{2\pi}\sin(n+1)\theta(v_1\cos\theta+v_2\sin\theta)d\theta\} \\
& +\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Phi_n^1+\Psi_n^1+\tilde{\Phi}_n^1+\tilde{\Psi}_n^1)\cdot \\
& \int_0^{2\pi}\sin(n+1)\theta(v_2\cos\theta-v_1\sin\theta)d\theta \\
& -\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Phi_n^1-\Psi_n^1+\tilde{\Phi}_n^1-\tilde{\Psi}_n^1)\cdot \\
& \int_0^{2\pi}\sin(n-1)\theta(v_2\cos\theta-v_1\sin\theta)d\theta \\
& +\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Phi_n^2+\Psi_n^2+\tilde{\Phi}_n^2+\tilde{\Psi}_n^2)\cdot \\
& \int_0^{2\pi}\cos(n-1)\theta(v_2\cos\theta-v_1\sin\theta)d\theta \\
& +\frac{1}{2}\sum_{n=0}^{+\infty}\delta_n(\Psi_n^2-\Phi_n^2+\tilde{\Psi}_n^2-\tilde{\Phi}_n^2)\cdot \\
& \int_0^{2\pi}\cos(n+1)\theta(v_2\cos\theta-v_1\sin\theta)d\theta \\
=\ & I+II+III
\end{aligned}
$$

by *Hö*lder inequality, we obtain

$$I \geq -\frac{2\pi}{kR}\sum_{n=0}^{+\infty}\delta_n(1+n)\int_0^{2\pi}|\vec{v}|^2 d\theta$$

We easily deduce from $\vec{n} = (-\cos\theta, -\sin\theta)$ and $\int_{\Gamma_2}\vec{v}\cdot n ds = 0$ that $II \geq 0$.
Applying the property of I_n and K_n

$$x > 0, \frac{K_n'}{K_n} < 0$$

and

$$I'_{m+n} + \frac{m+n}{kR}I_{m+n} = I_{m+n-1}$$
$$I'_{m-n} + \frac{m-n}{kR}I_{m-n} = I_{m-n-1}$$

we arrive at the following inequlity:

$$III \geq \sum_{n=0}^{+\infty}\delta_n\sum_{m=1}^{+\infty}\frac{n+m}{kR}\int_0^{2\pi}|\vec{v}|^2 d\theta$$

From the result of I, II and III we deduce that

$$a_2(\vec{v};\vec{v},\vec{v}) + B(v,v) \geq 0$$

and the result follows.
Lemma 3. Let

$$b(\vec{v},q) = -(q, div\vec{v})$$

Then there exists a constant $\beta > 0$, such that

$$\sup_{\vec{v}\in W\backslash\{0\}}\frac{b(\vec{v},q)}{\|\vec{v}\|_W} \geq \beta\|q\|_Q, \qquad \forall q \in Q$$

For the proof can see [Gunzburger and Peterson (1983)].
We now consider the existence and uniqueness result for variational formulation (8). Let W' be the dual space of W,

$$N = \sup_{\vec{w},\vec{u},\vec{v}\in W\backslash\{0\}}\frac{a_1(\vec{w};\vec{u},\vec{v})}{|\vec{w}|_{1,\Omega^-}|\vec{u}|_{1,\Omega^-}|\vec{v}|_{1,\Omega^-}}$$

$$\|\vec{f}\|_{W'} = |\vec{f}|_* = \sup_{\vec{v}\in W\backslash\{0\}} \frac{<\vec{f},\vec{v}>}{|\vec{v}|_{1,\Omega^-}}$$

$$M = \{\vec{w} \in W_0; |\vec{w}|_{1,\Omega^-} \le \frac{2}{\mu}|\vec{f}|_*\} \subset W$$

Theorem 1. The variational problem (8) has at least one solution $\vec{u}, p \in W \times Q$.
Proof: see [Feistauer and Schwab (2001)].
Theorem 2. Let Ω^- be a bounded domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary. Given $\vec{f} \in H^{-1}(\Omega^-)^2$, and suppose

$$\frac{6N}{\mu^2}|\vec{f}|_* < 1 \tag{10}$$

then problem (8) has a unique solution $\vec{u}, p \in W \times Q$.
Proof: For any given $\vec{\sigma} \in M$, we consider the following auxiliary problem : for $\forall\ \vec{v} \in W_0$,

$$\begin{cases} find\ \ \vec{w} \in W_0,\ such\ that \\ a(\vec{w},\vec{v}) + a_2(\vec{w};\vec{w},\vec{v}) + B(\vec{w},\vec{v}) \\ = (\vec{f},\vec{v}) - a_1(\vec{\sigma};\vec{\sigma},\vec{v}) \end{cases} \tag{11}$$

[i]. (11) is equivalent to the operator equation.

$$\vec{w} = E\vec{\sigma}$$

In view of lemma 1, lemma 2 and trace theorem,

$$a(\vec{v},\vec{v}) + a_2(\vec{v};\vec{v},\vec{v}) + B(\vec{v},\vec{v}) \ge \mu|\vec{v}|^2_{1,\Omega_1}$$

$$\begin{aligned} &|a(\vec{w},\vec{v}) + a_2(\vec{w};\vec{w},\vec{v}) + B(\vec{w},\vec{v})| \\ &\le \mu\|\vec{w}\|_1\|\vec{v}\|_1 + c_1\|\vec{w}\|_{H^{1/2}(\Gamma_2)/R}\|\vec{v}\|_{H^{1/2}(\Gamma_2)/R} \\ &\le \mu\|\vec{w}\|_1\|\vec{v}\|_1 + c_2\|\vec{w}\|_1\|\vec{v}\|_1 \\ &\le C\|\vec{w}\|_1\|\vec{v}\|_1 \end{aligned}$$

According to the Lax-Milgram theorem, there exists a unique solution $\vec{w} \in W_0$, (11) defines a mapping $E : M \to W_0$, and the problem (11) is equivalent to the operator equation

$$\vec{w} = E\vec{\sigma}$$

[ii]. The mapping $E : M \to M$.
Let $\vec{\sigma} \in M$, then $\vec{w} = E\vec{\sigma}$ satisfies (11), taking $\vec{v} = \vec{w}$, that is

$$a(\vec{w},\vec{w}) + a_2(\vec{w};\vec{w},\vec{w}) + B(\vec{w},\vec{w}) = (\vec{f},\vec{w}) - a_1(\vec{\sigma};\vec{\sigma},\vec{w})$$

We easily deduce from lemma 1 and lemma 2 that

$$\begin{aligned}\mu|\vec{w}|_1^2 &\leq a(\vec{w},\vec{w})+a_2(\vec{w};\vec{w},\vec{w})+B(\vec{w},\vec{w})\\ &\leq |\vec{f}|_*|w|_1+N|\sigma|_1^2|w|_1\end{aligned}$$

By (10), we obtain

$$\frac{4N}{\mu^2}|\vec{f}|_*<\frac{2}{3}$$

$$|\vec{w}|_{1,\Omega_1}\leq\frac{5}{3\mu}|\vec{f}|_*\leq\frac{2}{\mu}|\vec{f}|_*$$

therefor, $\vec{w}\in M$, namely, E is a mapping of M into M.
$[iii]$. The mapping E is a contraction mapping in M.
for $\forall\vec{\sigma}_1,\vec{\sigma}_2\in M$, we have $\vec{w}_1=E\vec{\sigma}_1\in M,\vec{w}_2=E\vec{\sigma}_2\in M$,

$$\begin{aligned}&a(\vec{w}_1,\vec{v})+a_2(\vec{w}_1;\vec{w}_1,\vec{v})+B(\vec{w}_1,\vec{v})\\ &=(\vec{f},\vec{v})-a_1(\vec{\sigma}_1;\vec{\sigma}_1,\vec{v})\\ &a(\vec{w}_2,\vec{v})+a_2(\vec{w}_2;\vec{w}_2,\vec{v})+B(\vec{w}_2,\vec{v})\\ &=(\vec{f},\vec{v})-a_1(\vec{\sigma}_2;\vec{\sigma}_2,\vec{v})\end{aligned}$$

$$\begin{aligned}&a(\vec{w}_1-\vec{w}_2,\vec{v})+a_2(\vec{w}_1;\vec{w}_1,\vec{v})-a_2(\vec{w}_2;\vec{w}_2,\vec{v})\\ &+B(\vec{w}_1-\vec{w}_2,\vec{v})=a_1(\vec{\sigma}_2;\vec{\sigma}_2,\vec{v})-a_1(\vec{\sigma}_1;\vec{\sigma}_1,\vec{v})\end{aligned}\tag{12}$$

In fact, by Green formula [Girault and Raviart (1986), I. 2. 17],

$$a_2(\vec{u};\vec{u},\vec{v})=-\frac{1}{2}a_1(\vec{u};\vec{u},\vec{v})+\frac{1}{2}a_1(\vec{u}_\infty;\vec{u},\vec{v})$$

Hence,

$$\begin{aligned}&a_2(\vec{w}_1;\vec{w}_1,\vec{v})-a_2(\vec{w}_2;\vec{w}_2,\vec{v})\\ =\;&-\frac{1}{2}(a_1(\vec{w}_1;\vec{w}_1,\vec{v})-a_1(\vec{w}_2;\vec{w}_2,\vec{v}))\\ &+\frac{1}{2}a_1(\vec{u}_\infty;\vec{w}_1-\vec{w}_2,\vec{v})\\ =\;&-\frac{1}{2}a_1(\vec{w}_1-\vec{w}_2;\vec{w}_1-\vec{w}_2,\vec{v})+\frac{1}{2}a_1(\vec{u}_\infty;\vec{w}_1-\vec{w}_2,\vec{v})\\ &-\frac{1}{2}(a_1(\vec{w}_1-\vec{w}_2;\vec{w}_2,\vec{v})+a_1(\vec{w}_2;\vec{w}_1-\vec{w}_2,\vec{v}))\\ =\;&a_2(\vec{w}_1-\vec{w}_2;\vec{w}_1-\vec{w}_2,\vec{v})-\frac{1}{2}(a_1(\vec{w}_1-\vec{w}_2;\vec{w}_2,\vec{v})\end{aligned}$$

$$+a_1(\vec{w}_2;\vec{w}_1-\vec{w}_2,\vec{v}))$$

taking $\vec{v}=\vec{w}_1-\vec{w}_2$ in (12) and using (10), we have

$$\begin{aligned}
&\mu|\vec{w}_1-\vec{w}_2|_1^2\\
\leq\ & |a_1(\vec{\sigma}_2-\vec{\sigma}_1;\vec{\sigma}_2,\vec{w}_1-\vec{w}_2)|\\
&+|a_1(\vec{\sigma}_1;\vec{\sigma}_2-\vec{\sigma}_1,\vec{w}_1-\vec{w}_2)|\\
&+\frac{1}{2}|a_1(\vec{w}_1-\vec{w}_2;\vec{w}_2,\vec{w}_1-\vec{w}_2)|\\
&+\frac{1}{2}|a_1(\vec{w}_2;\vec{w}_1-\vec{w}_2,\vec{w}_1-\vec{w}_2)|\\
\leq\ & N|\vec{\sigma}_2|_{1,\Omega_1}|\vec{\sigma}_1-\vec{\sigma}_2|_{1,\Omega_1}|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}\\
&+N|\vec{\sigma}_1|_{1,\Omega_1}|\vec{\sigma}_2-\vec{\sigma}_1|_{1,\Omega_1}|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}\\
&+N|\vec{w}_2|_{1,\Omega_1}|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}^2\\
\leq\ & \frac{4N}{\mu}|\vec{f}|_*|\vec{\sigma}_1-\vec{\sigma}_2|_{1,\Omega_1}|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}\\
&+\frac{2N}{\mu}|\vec{f}|_*|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}^2
\end{aligned}$$

that is

$$|\vec{w}_1-\vec{w}_2|_{1,\Omega_1}\leq\frac{\frac{4N|\vec{f}|_*}{\mu^2}}{1-\frac{2N|\vec{f}|_*}{\mu^2}}|\vec{\sigma}_1-\vec{\sigma}_2|_{1,\Omega_1}\leq|\vec{\sigma}_1-\vec{\sigma}_2|_{1,\Omega_1}$$

therefore, the mapping is a contraction mapping.
An application of Brouwer fixed-point theorem shows that there exists a unique fixed point $\vec{\sigma}\in M$, such that $\vec{\sigma}=E\vec{\sigma}=\vec{w}$, problem (11) has a unique solution $\vec{w}\in W_0$. Then we obtain the well-posed property of problem (8) by the equivalence of (8) and (9).

3 Finite element approximation of coupling method

Let ξ_h be a regular partition of the domain Ω^-. Suppose W_h and Q_h are two finite -dimensional spaces such that

$$W_h\in H^1(\Omega^-)^2, Q_h\in L^2(\Omega^-)$$

$$X_h=W_h\cap H_0^1(\Omega^-)=\{\vec{v}_h\in W_h;\vec{v}_h|_{\Gamma_1}=0\}\subset W$$

$$M_h=Q_h\cap L_0^2(\Omega^-)=\{q_h\in Q_h;\int_{\Omega_1}q_h dx=0\}\subset Q$$

$W_{0h} = \{\vec{v}_h \in X_h; (q_h, div\vec{v}_h) = 0, \forall q_h \in Q_h\} \nsubseteq W_0$

Furthermore, we also assume that they are the compatible, i.e.,X_h and Q_h should satisfy the following conditions:

Hypothesis H1 (Approximation of property of X_h). There exist an operator $r_h \in L(H^2(\Omega_1)^2; W_h) \cap L(H^2(\Omega^-) \cap H_0^1(\Omega^-)^2; X_h)$ and a integer l, such that:

$$(q, div(\vec{v} - r_h\vec{v})) = 0, \ \forall q \in Q_h, \vec{v} \in H^2(\Omega^-)^2$$

$$\begin{aligned} \|\vec{v} - r_h\vec{v}\|_{1,\Omega^-} \quad &\leq \quad Ch^m \|\vec{v}\|_{m+1,\Omega^-}, \forall \vec{v} \in H^{m+1}(\Omega^-)^2 \\ &\quad\ (1 \leq m \leq l) \end{aligned}$$

Hypothesis H2 (Approximation of property of Q_h). There exists an orthogonal projection operator $s_h \in L(L^2(\Omega^-)^2; Q_h)$, such that:

$$\begin{aligned} \|q - s_h q\|_{0,\Omega^-} \quad &\leq \quad Ch^m \|q\|_{m,\Omega^-}, \ \forall q \in H^m(\Omega^-) \\ &\quad\ (0 \leq m \leq l) \end{aligned}$$

Hypothesis H3 (Approximation of property of Γ_{2h}). There exists an orthogonal projection operator $\Pi_h : L^2(\Gamma_2) \mapsto \Gamma_{2h}$, such that:

$$\begin{aligned} \|\vec{w} - \Pi_h\vec{w}\|_{s,\Gamma_2} \quad &\leq \quad Ch^{k+1-s} |\vec{w}|_{k+1,\Gamma_2}, \ \forall \vec{w} \in H^{k+1}(\Gamma_2) \\ &\quad\ (s = 0, 1) \end{aligned}$$

Hypothesis H4 (Uniform inf-sup condition). There exists a constant $\beta' > 0$, such that :

$$\sup_{\vec{v} \in W_h \setminus \{0\}} \frac{b(\vec{v}, q_h)}{\|\vec{v}\|_{1,\Omega^-}} \geq \beta' \|q_h\|_{0,\Omega^-}, \qquad \forall q_h \in Q_h$$

Then the approximation problem of (8) and (9) is

$$\begin{cases} find\ (\vec{u}_h, p_h) \in X_h \times M_h,\ such\ that \\ A_0(\vec{u}_h; \vec{u}_h, \vec{v}_h) + B(\vec{u}_h, \vec{v}_h) + b(\vec{v}_h, p_h) \\ = (\vec{f}, \vec{v}_h), \qquad\qquad\qquad\qquad \forall\ \vec{v}_h \in X_h \\ (q_h, div\vec{u}_h) = 0, \qquad\qquad\qquad\quad \forall\ q_h \in M_h \end{cases} \tag{13}$$

and

$$\begin{cases} find\ (\vec{u}_h, p_h) \in W_{0h} \times M_h,\ such\ that \\ A_0(\vec{u}_h; \vec{u}_h, \vec{v}_h) + B(\vec{u}_h, \vec{v}_h) \\ = (\vec{f}, \vec{v}_h), \qquad\qquad\qquad\qquad \forall\ \vec{v}_h \in W_{0h} \end{cases} \tag{14}$$

Theorem 3. Let Ω^- be a bounded domain of $\mathbb{R}^2$ with a Lipschitz-continuous boundary. Under Hypotheses $(H1),(H2),(H3),(H4)$, and given $\vec{f} \in H^{-1}(\Omega^-)^2$, satisfying

$$\frac{6N}{\mu^2}|\vec{f}|_* < 1 \tag{15}$$

then problem (13) has a unique solution $(\vec{u}_h, p_h) \in X_h \times M_h$. Moreover, there exists a constant $C > 0$, such that

$$\|\vec{u} - \vec{u}_h\|_{1,\Omega^-} + \|\vec{u} - \vec{u}_h\|_{H^{1/2}(\Gamma_2)/R} + \|p - p_h\|_{0,\Omega^-}$$
$$\leq C(\|\vec{u}\|_{m+1}, \|p\|_m)h^m$$

Proof: We can get this error estimate by a standard technique of mixed finite element method [Girault and Raviart (1986)].

4 Numerical implementation of coupling method

We take the following finite element spaces:

$$\begin{cases} X_h = \{w_h \in C^0(\bar{\Omega}^-), w_{h|k} \in P_2, \forall k \in \xi_h\} \\ S_h = \{\mu_h \in C^0(\Gamma_2), \mu_{h|s_i} \in P_1, 1 \leq i \leq N\} \\ Q_h = \{q_h \in C^0(\bar{\Omega}_1), q_{h|k} \in P_1, \forall k \in \xi_h\} \\ M_h = Q_h \cap L_0^2(\Omega^-) \end{cases} \tag{16}$$

where, for any integer $l \geq 0$, P_l denotes the space at all polynomials in two variables of degree $\leq l$. ξ_h be a triangulation of $\bar{\Omega}^-$ made of triangles k with no more than one side on $\partial\Omega^-$. $S_i(1 \leq i \leq N)$ denotes the finite number of segments of a line composing the artificial boundary Γ_2. For this choice we can check all the hypotheses of theorem 3 [Sequeira (1983)].

Once the finite element spaces are prescribed, the discrete problem (13) reduces to solving a system of nonlinear algebraic equations which has a Jacobian that is large, sparse, and banded. Various iterative methods to solve the nonlinear problem (13) are analyzed in [Girault and Raviart (1986)] for homogeneous boundary conditions and [Gunzburger and Peterson (1983)] for inhomogeneous case. For example, a standard approach is to use Newton's method to linearized (13). Here, we use Newton's method to deal with the nonlinear term in domain Ω^-, and Picard's method to linearized the nonlinear term on Γ_2.

Given the iterate $(\vec{u}_k, p_k)$, we start by computing the nonlinear residual associated with the weak formulation (13). This is the pair $R_k(\vec{v}_h), r_k(q_h)$ satisfying

$$R_k \quad = \quad (\vec{f}, \vec{v}_h) - a_1(\vec{u}_k; \vec{u}_k, \vec{v}_h) - a_2(\vec{u}_k; \vec{u}_k, \vec{v}_h)$$

$$
\begin{aligned}
& -a(\vec{u}_k,\vec{v}_h)-B(\vec{u}_k,\vec{v}_h)-b(\vec{v}_h,p_k)\\
r_k \quad = \quad & b(\vec{u}_k,q_h)
\end{aligned}
$$

with $\vec{u}_h=\vec{u}_k+\delta\vec{u}_k$ and $ph=p_k+\delta p_k$, it is easy to see that the corrections $\delta\vec{u}_k \in X_h, \delta p_k \in M_h$ satisfy

$$
\left\{
\begin{array}{l}
D_1(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)+D_2(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)+a(\delta\vec{u}_k,\vec{v}_h)\\
+B(\delta\vec{u}_k,\vec{v}_h)+b(\vec{v}_h,\delta p_k)=R_k(\vec{v}_h),\ \vec{v}_h\in X_h\\
-b(\delta\vec{u}_k,q_h)=r_k(q_h), \qquad q_h\in M_h
\end{array}
\right. \tag{17}
$$

where $D_1(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)$ and $D_2(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)$ are the differences in the nonlinear terms. Expanding $D_1(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)$ and $D_2(\vec{u}_k;\delta\vec{u}_k,\vec{v}_h)$, dropping some quadratic terms in the expansions, we get the linear problem: find $\delta\vec{u}_k\in X_h, \delta p_k\in M_h$ satisfying:

$$
\left\{
\begin{array}{ll}
a_1(\delta\vec{u}_k;\vec{u}_k,\vec{v}_h)+a_1(\vec{u}_k,\delta\vec{u}_k,\vec{v}_h) & \\
+a_2(\vec{u}_k,\delta\vec{u}_k,\vec{v}_h)+a(\delta\vec{u}_k,\vec{v}_h) & \\
+B(\delta\vec{u}_k,\vec{v}_h)+b(\vec{v}_h,\delta p_k) & \\
=R_k(\vec{v}_h), & \forall\ \vec{v}_h\in X_h\\
-b(\delta\vec{u}_k,q_h)=r_k(q_h), & \forall\ q_h\in M_h
\end{array}
\right. \tag{18}
$$

The system of algebra equations of problem (18):

$$
\begin{pmatrix}
P_{11}+Q_{11} & W_{xy}+Q_{12} & B_x^T\\
W_{yx}+Q_{21} & P_{22}+Q_{22} & B_y^T\\
B_x & B_y & 0
\end{pmatrix}
\begin{pmatrix}
\delta u_1\\ \delta u_2\\ \delta p
\end{pmatrix}
=
\begin{pmatrix}
F_1\\ F_2\\ g
\end{pmatrix} \tag{19}
$$

Here,

$$
Q=\begin{pmatrix} Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}
$$

is the matrix generated by the boundary elements. $P_{11}=A+N+W_{xx}+C$, $P_{22}=A+N+W_{yy}+C$, C is the matrix generated by a_2. The matrix A is the vector-Laplacian matrix and matrix B is the divergence matrix, matrix N is the vector-convection matrix and matrix W is Newton derivative matrix.

For the actual implementation, we need to do some truncations: First, m should be truncated to L terms; Second, we truncate k to M terms (M need not to be very large); In the end, in the expression of the normal stress (4), the series should not be infinite, this can be overcome by truncating the first N terms. After all this procedures, we denote the approximate normal stress $\vec{g}^*=(\vec{g}_1^*,\vec{g}_2^*)$, that is

$$
\begin{pmatrix} g_1^*(\theta)\\ g_2^*(\theta)\end{pmatrix}
=
\begin{pmatrix} A & B\\ -B & A\end{pmatrix}
\begin{pmatrix} \cos\theta\\ \sin\theta\end{pmatrix} \tag{20}
$$

or

$$\vec{g}^* = K(\vec{u}^+|_{\Gamma_2}) \tag{21}$$

where,

$$\begin{aligned}
A &= \sum_{n=0}^{L} \delta_n (a_n \cos(n+1)\theta + b_n \sin(n+1)\theta) \\
&\quad - \sum_{n=0}^{N} \delta_n \Phi_n^1 \cos(n+1)\theta + \sum_{n=0}^{N} \delta_n \Psi_n^2 \sin(n+1)\theta \\
&\quad - \sum_{n=0}^{N} \delta_n \tilde{\Phi}_n^1 \cos(n+1)\theta + \sum_{n=0}^{N} \delta_n \tilde{\Psi}_n^2 \sin(n+1)\theta \\
&\quad + \sum_{n=0}^{L} \delta_n (\tilde{b}_n \cos(n+1)\theta - \tilde{a}_n \sin(n+1)\theta) \\
&\quad + \frac{1}{kR} \sum_{n=0}^{L} \delta_n (nd_n - a_n) \cos n\theta \\
&\quad - \frac{1}{kR} \sum_{n=1}^{L} (c_n + nb_n) \sin n\theta \\
&\quad - \frac{1}{kR} \sum_{n=0}^{L} \delta_n (\tilde{b}_n + n\tilde{c}_n) \cos n\theta \\
&\quad + \frac{1}{kR} \sum_{n=1}^{L} (n\tilde{a}_n - \tilde{d}_n) \sin n\theta
\end{aligned}$$

$$\begin{aligned}
B &= \frac{1}{kR} \sum_{n=0}^{L} \delta_n (na_n - d_n) \sin n\theta \\
&\quad - \frac{1}{kR} \sum_{n=0}^{L} \delta_n (nc_n + b_n) \cos n\theta \\
&\quad - \frac{1}{2} \sum_{n=0}^{N} \delta_n (\Phi_n^1 + \Psi_n^1) \sin(n+1)\theta \\
&\quad + \frac{1}{2} \sum_{n=0}^{N} \delta_n (\Phi_n^1 - \Psi_n^1) \sin(n-1)\theta \\
&\quad - \frac{1}{2} \sum_{n=0}^{N} \delta_n (\Phi_n^2 + \Psi_n^2) \cos(n-1)\theta
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{n=0}^{N}\delta_n(\Psi_n^2-\Phi_n^2)\cos(n+1)\theta \\
&+\frac{1}{kR}\sum_{n=0}^{L}\delta_n(n\tilde{b}_n+\tilde{c}_n)\sin n\theta \\
&+\frac{1}{kR}\sum_{n=0}^{L}\delta_n(\tilde{a}_n-n\tilde{d}_n)\cos n\theta \\
&-\frac{1}{2}\sum_{n=0}^{N}\delta_n(\tilde{\Phi}_n^1+\tilde{\Psi}_n^1)\sin(n+1)\theta \\
&+\frac{1}{2}\sum_{n=0}^{N}\delta_n(\tilde{\Phi}_n^1-\tilde{\Psi}_n^1)\sin(n-1)\theta \\
&-\frac{1}{2}\sum_{n=0}^{N}\delta_n(\tilde{\Phi}_n^2+\tilde{\Psi}_n^2)\cos(n-1)\theta \\
&-\frac{1}{2}\sum_{n=0}^{N}\delta_n(\tilde{\Psi}_n^2-\tilde{\Phi}_n^2)\cos(n+1)\theta
\end{aligned}
$$

5 Numerical results

In this section, we present numerical example to confirm our theoretical analysis given in the above sections.
Consider the exterior Navier-Stokes flow generated by a circular cylinder of radius a moving with a constant speed $\vec{u}_\infty$, $\vec{u}_\infty = (u_\infty, 0)$. Assume the viscous and incompressible flow is steady. The Reynolds number corresponding to this configuration,

$$Re = \frac{2au_\infty}{\mu}$$

First we introduce a circle artificial boundary Γ_2 with radius R. Γ_2 divides the domain Ω into two parts: an interior bounded subdomain Ω^- and an exterior unbounded subdomain Ω^+. Divide Γ_1 and Γ_2 into M segmental arcs uniformly, respectively. We assume that the nodes on boundaries Γ_1 and Γ_2 coincide with the nodes on $\partial\Omega^-$. M line segments are obtained by the corresponding nodes on Γ_1 and Γ_2. Each of the above line segments is divided into N parts. In boundary element discretization, we take piecewise linear elements. The finite element discretization uses a triangular mesh with the Taylor-Hood element, which is known to satisfy the stability condition. Under the Cartesian co-ordinates frame $x = (x_1, x_2)$, the solution of the problem can be expressed by

$$\begin{cases} u_1 = u_\infty(1 - \frac{a^2}{r^2}\cos 2\theta) \\ u_2 = -u_\infty \frac{a^2}{r^2}\sin 2\theta, \\ p = 0 \\ f_1 = 2u_\infty^2 a^2 \frac{\cos 3\theta}{r^3} - 2u_\infty^2 a^4 \frac{\cos\theta}{r^5} \\ f_2 = 2u_\infty^2 a^2 \frac{\sin 3\theta}{r^3} - 2u_\infty^2 a^4 \frac{\sin\theta}{r^5} \end{cases} \tag{22}$$

where $r \equiv \sqrt{x_1^2 + x_2^2}$.
Table 1, 2, and 3 show the errors of $|\vec{u} - \vec{u}_h|$ and $|p - p_h|$. Fig. 2. and Fig. 3 plot the errors of $|u_1(R,\theta) - u_{1h}(R,\theta)|_\infty$ and $|u_2(R,\theta) - u_{2h}(R,\theta)|_\infty$ for different truncated numbers L with mesh $= 32 \times 8$.

Table 1: Re=1.2, L=2, M=30, N=20

$M \times N$	$\|u - u_h\|_0$	$\|p - p_h\|_0$	$\|u - u_h\|_1$	ns
8×2	1.2022E-01	2.9965E-02	3.4522E-01	10
16×4	2.7958E-02	1.1986E-02	1.2022E-01	8
32×8	6.0778E-03	3.7456E-03	3.6430E-02	7

Table 2: Re=1.2, L=4, M=30, N=20

$M \times N$	$\|u - u_h\|_0$	$\|p - p_h\|_0$	$\|u - u_h\|_1$	ns
8×2	1.0699E-01	2.5470E-02	2.9688E-01	8
16×4	2.2645E-02	8.8696E-03	8.7760E-02	7
32×8	3.9505E-03	2.3724E-03	2.4722E-02	7

Table 3: Re=1.2, L=6, M=30, N=20

$M \times N$	$\|u - u_h\|_0$	$\|p - p_h\|_0$	$\|u - u_h\|_1$	ns
8×2	9.9500E-02	2.3177E-02	2.7609E-01	8
16×4	1.8195E-02	8.4261E-03	7.4596E-02	6
32×8	3.7529E-03	2.1588E-03	1.9530E-02	7

From table 1 to table 3, it can be observed that both increasing the order of the artificial boundary condition and refining the mesh can decrease the error. When a finer mesh could not present a much more accurate numerical solution, the error originated from the series truncating is dominating and we should use more series terms in order to get a higher approximation.
These observations are quite compatible with the following analysis. Since the error of the numerical solution originates from two sources: one is the approximation of

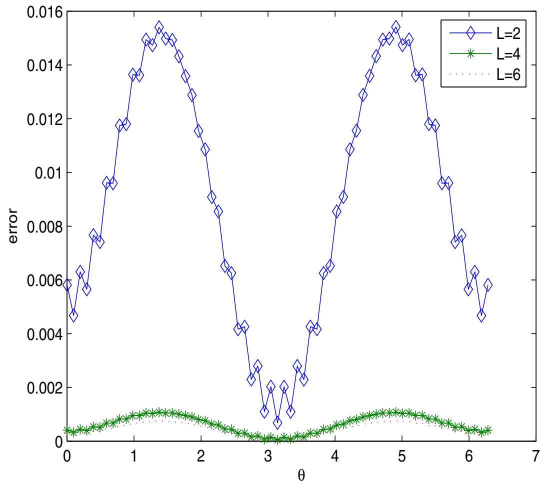

Figure 2: $|u_1(R,\theta) - u_{1h}(R,\theta)|_\infty$ for different L with mesh $= 32 \times 8$

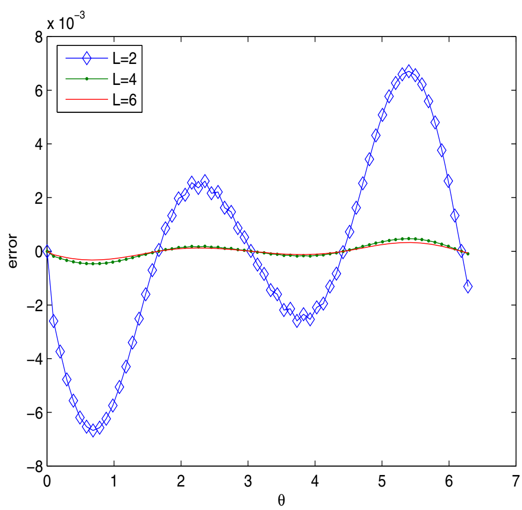

Figure 3: $|u_2(R,\theta) - u_{2h}(R,\theta)|_\infty$ for different L with mesh $= 32 \times 8$

the variational problem, the other is the employment of the finite-element scheme. when one is relatively smaller, the other dominates the error. The numerical results above show that the coupling of NBEM and mixed FEM is very effective.

Acknowledgement: This project was supported by the National Basic Research Program of China under the grant 2005CB321701, the National Natural Science Foundation of China under the grant 10531080, Beijing Natural Science Foundation under the grant 1072009, and Shanghai University Innovation Foundation under the grant A.10-0101-08-415.

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