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Free Vibration of Non-Uniform Euler-Bernoulli Beams by the Adomian Modified Decomposition Method

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Abstract: An innovative solver for the free vibration of an elastically restrained non-uniform Euler-Bernoulli beam with tip mass of rotatory inertia and eccentricity resting on an elastic foundation and subjected to an axial load is proposed. The technique we have used is based on applying the Adomian modified decomposition method (AMDM) to our vibration problems. By using this method, any *i*th natural frequencies can be obtained one at a time and some numerical results are given to illustrate the influence of the physical parameters on the natural frequencies of the dynamic system. The computed results agree well with those analytical and numerical results given in the literatures. These results indicate that the present analysis is accurate, and provides a unified and systematic procedure which is simples and more straightforward than the other analyses.

Keyword: Wedge beam, Cone beam, Winkler's elastic foundation, Natural frequency, Euler-Bernoulli beam, Adomian modified decomposition method.

1 Introduction

In the vibration analysis, the structures were often modeled as beams vibrating in flexural motion. The influence of tip mass, rotatory inertia, eccentricity, taper ratio, axial force, elastic foundation, and elastic end restraints on the natural frequencies of flexural vibration of a beam were investigated by many investigators. The transverse vibrations of uniform beams with a concentrated mass at the tip have been studied in these literatures [Mabie and Rogers (1974); Laura, Pombo and Susemihl (1974); Lee (1973)]. Goel (1976) generalized the analysis and considered the rotational flexibility of the constraint. The free vibrations of constrained beams carrying a heavy tip body, including the rotatory inertia and eccentricity have been studied in these literatures [Chang (1993); Grossi, Aranda and Bhat (1993);

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Maurizi, Belles and Rosales (1990); Liu and Huang (1988); Alvarez, Iglesias and Laura (1988); Lau (1984); To (1982)]. Auciello (1996) presented the exact solution for the frequency equation of tapered cantilever beams with a concentrated mass at the tip, with account taken of the rotatory inertia of the mass, and its eccentricity. Beams with a mass and spring at the end subjected to an axial force had been studied in these literatures [Naguleswaran (1991); Bokaian (1990); Grossi and Laura (1982); Takahashi (1980)]. Laura and Cortinez (1985) presented transverse vibrations of a cantilever beam subjected to a variable axial force. Recently, Naguleswaran (2004) studied the vibration of a uniform Euler-Bernoulli beam under linearly varying axial force. Nallim (1999) presented a general algorithm for the study of the dynamical behavior of beams. It allows the inclusion of a number of complicating effects such as varying cross-sections, presence of an arbitrarily placed concentrated mass, ends elastically restrained against rotation and translation and presence of an axial, tensile force. Free vibrations of analysis of beams on elastic foundation have been studied in these literatures [Chen (2000); Thambiratnam and Zhuge (1996); Lee and Lin (1995)]. Finally, Batra and Porfiri (2005) applied the Meshless Local Petrov-Galerkin (MLPG) method to examine the vibrations of cracked Euler-Bernoulli beams. Vinod, Gopalakrishnan and Ganguli (2006) applied the spectral finite element formulation for a rotating uniform Euler-Bernoulli beam subjected to small duration impact. Huang and Shih (2007) applied the Conjugate Gradient Method (CGM) to study an inverse problem in estimating simultaneously the time-dependent applied force and moment of an Euler-Bernoulli beam.

In this study, a new computed approach called Adomian modified decomposition method (AMDM) is introduced to solve the free vibration problems. The concept of AMDM was first proposed by Adomian and was applied to solve linear and nonlinear initial/boundary value problems in physics [Adomian (1994); Adomian and Rach (1992); Adomian and Rach (1991)]. In this paper, the free vibration problems of elastically restrained non-uniform Euler-Bernoulli beams resting on an elastic foundation, with tip mass of rotatory inertia and eccentricity under an axial load are considered. Using the AMDM, the governing differential equation becomes a recursive algebraic equation and the boundary conditions at the right end become simple algebraic frequency equations which are suitable for symbolic computation. Moreover, after some simple algebraic operations on these frequency equations any ith natural frequency can be obtained. Finally, some problems of free vibration of uniform and non-uniform beams are solved and showed excellent agreement with the published results to verify the accuracy and efficiency of the present method.

2 The principle of AMDM

In order to solve vibration problems by the Adomian modified decomposition method (AMDM) the basic theory is stated in brief in this section. Consider the equation

$$Fy(x) = g(x), \tag{1}$$

where F represents a general nonlinear ordinary differential operator involving both linear and nonlinear parts, and $g(x)$ is a given function. The linear terms in Fy are decomposed into $Ly + Ry$, where L is an invertible operator, which is taken as the highest-order derivative and R is the remainder of the linear operator. Thus, Eq. (1) can be written as

$$Ly + Ry + Ny = g, \tag{2}$$

where Ny represents the nonlinear terms in Fy. Equation (2) corresponds to an initial value problem or a boundary-value problem. Solving for Ly, one can obtain

$$y = \Phi + L^{-1}g - L^{-1}Ry - L^{-1}Ny, \tag{3}$$

where Φ is an integration constant, and $L\Phi = 0$ is satisfied. Corresponding to an initial-value value problem, the operator L^{-1} may be regarded as a definite integration from 0 to x. In order to solve Eq. (3) by the AMDM we decompose y into the infinite sum of convergent series

$$y = \sum_{k=0}^{\infty} c_k x^k, \tag{4}$$

and the nonlinear term Ny is decomposed as

$$Ny = \sum_{k=0}^{\infty} x^k A_k(c_0, c_1, \cdots, c_k), \tag{5}$$

where the A_k are known as Adomian coefficients. The given function $g(x)$ is also decomposed as

$$g(x) = \sum_{k=0}^{\infty} g_k x^k, \tag{6}$$

By plugging Eqs. (4), (5), and (6) into Eq. (3) gives

$$\begin{aligned} y &= \sum_{k=0}^{\infty} c_k x^k \\ &= \Phi + L^{-1}\left(\sum_{k=0}^{\infty} g_k x^k\right) - L^{-1}R\left(\sum_{k=0}^{\infty} c_k x^k\right) - L^{-1}\left(\sum_{k=0}^{\infty} x^k A_k(c_0, c_1, \cdots, c_k)\right) \end{aligned} \tag{7}$$

The coefficients c_k of each term in series (7) can be decided by the recurrence relation, and the power series solutions of linear homogeneous differential equations in initial value problems yield simple recurrence relations for the coefficients c_k.However, in practice all the coefficients c_k in series (7) cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1} c_k x^k$.

3 Using the AMDM to analyze the free vibration problem of non-uniform beam

Let us consider the non-uniform beam of length l resting on the elastic foundation and subjected to an axial load as shown in Fig. 1, the beam is constrained with the rotational and translational flexible ends, and with a concentrated mass at the tip, with account taken of the rotatory inertia of the mass, and its eccentricity. The equation of motion for transverse vibrations of a non-uniform elastic beam is given by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right]-\frac{\partial}{\partial x}\left[s(x)\frac{\partial y(x,t)}{\partial x}\right]+k^*(x)y(x,t)+\rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2}=0, \quad (8)$$

where $y(x,t)$ is the transverse deflection of the beam, E is Young's modulus, $A(x)$ is the cross-sectional area at the position x, $I(x)$ is the moment of inertia of $A(x)$, ρ is the mass density of the beam material (mass per unit volume), $k^*(x)$ is the Winkler's foundation modulus, $s(x)$ is an axial tensile force and t is time.

For any mode of vibration, the lateral deflection $y(x,t)$ may be written in the form

$$y(x,t) = Y(x)h(t), \quad (9)$$

where $Y(x)$ is the modal deflection and $h(t)$ is a harmonic function of time t. If ω denotes the circular frequency of $h(t)$, then

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 Y(x)h(t), \quad (10)$$

and the eigenvalue problem of Eq. (8) reduces to the differential equation

$$\frac{d^2}{dx^2}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right]-\frac{d}{dx}\left[s(x)\frac{dY(x)}{dx}\right]+k^*(x)Y(x)-\rho A(x)\omega^2 Y(x)=0 \quad (11)$$

The boundary conditions are given by

$$EI(x)\frac{d^2Y(x)}{dx^2}+(J_L+M_Le_L^2)\omega^2\frac{dY(x)}{dx}-k_{RL}\frac{dY(x)}{dx}-M_Le_L\omega^2 Y(x)=0 \quad (12)$$

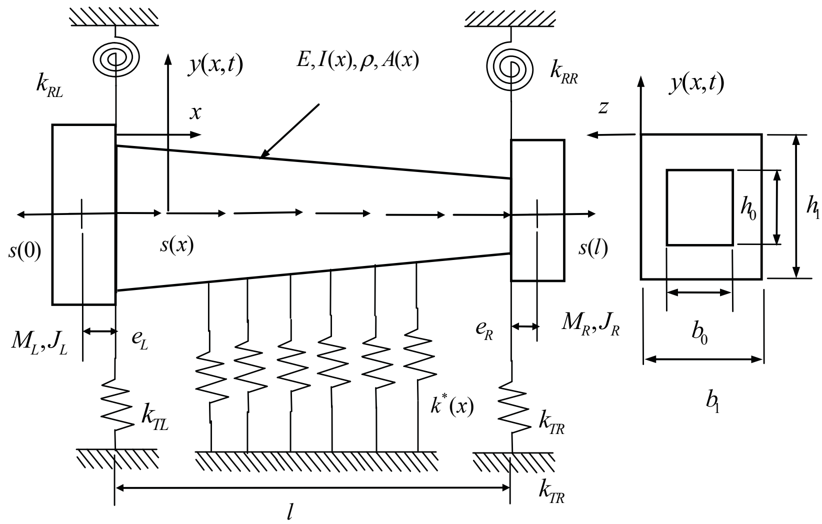

Figure 1: A general elastically end restrained non-uniform beam resting on the elastic foundation and subjected to an axial load

$$\frac{d}{dx}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right] - s(x)\frac{dY(x)}{dx} + M_L e_L \omega^2 \frac{dY(x)}{dx} + k_{TL}Y(x) - M_L\omega^2 Y(x) = 0 \tag{13}$$

at $x = 0$, and

$$EI(x)\frac{d^2Y(x)}{dx^2} + k_{RR}\frac{dY(x)}{dx} - (J_R + M_R e_R^2)\omega^2 \frac{dY(x)}{dx} - M_R e_R \omega^2 Y(x) = 0 \tag{14}$$

$$\frac{d}{dx}\left[EI(x)\frac{d^2Y(x)}{dx^2}\right] - s(x)\frac{dY(x)}{dx} + M_R e_R \omega^2 \frac{dY(x)}{dx} - k_{TR}Y(x) + M_R\omega^2 Y(x) = 0 \tag{15}$$

at $x = l$, where k_{TL}, k_{RL}, M_L, J_L, e_L, and k_{TR}, k_{RR}, M_R, J_R, e_R are the translational spring constants, the rotational spring constants, the concentrated masses attached at beam tip, the moments of inertia of the tip masses, the eccentricities which are the distances between the beam tip and the center of the tip mass at the left end and right end of the beam, respectively, and $s_L = s(0)$, $s_R = s(l)$ are the axial tensile forces which act on the beam at the left and right ends, respectively.

In this paper, assuming both the depth $b(x)$ and the height $h(x)$ of the cross-section of the beam can vary linearly according to the taper ratios of the beam $\alpha_b = b_1/b_0$

, and $\alpha_h = h_1/h_0$, that is,

$$b(x) = b_0\left[1+(\alpha_b-1)\frac{x}{l}\right]; \quad h(x) = h_0\left[1+(\alpha_h-1)\frac{x}{l}\right] \tag{16}$$

where b_0, b_1 are the cross-sectional depths at $x=0$ and $x=l$, respectively, and h_0, h_1 are the cross-sectional heights at $x=0$ and $x=l$, respectively, then the area and the moment of inertia of the section will vary according to the following laws:

$$A(x) = b(x)h(x) = A_0\left[1+(\alpha_b-1)\frac{x}{l}\right]\left[1+(\alpha_h-1)\frac{x}{l}\right], \tag{17}$$

$$I(x) = \frac{b(x)[h(x)]^3}{12} = I_0\left[1+(\alpha_b-1)\frac{x}{l}\right]\left[1+(\alpha_h-1)\frac{x}{l}\right]^3, \tag{18}$$

where $A_0 = b_0h_0$ and $I_0 = b_0h_0^3/12$ are the cross-sectional area and the moment of inertia at $x=0$. By setting

$$\beta_b = 1-\alpha_b \quad ; \quad \beta_h = 1-\alpha_h \tag{19}$$

Eq. (11) can be written as

$$\frac{d^2}{dx^2}\left[\left(1-\beta_b\frac{x}{l}\right)\left(1-\beta_h\frac{x}{l}\right)^3\frac{d^2Y(x)}{dx^2}\right]-\frac{d}{dx}\left[\frac{s(x)}{EI_0}\frac{dY(x)}{dx}\right]+\frac{k^*(x)}{EI_0}Y(x)$$
$$-\frac{\rho A_0\omega^2}{EI_0}\left(1-\beta_b\frac{x}{l}\right)\left(1-\beta_h\frac{x}{l}\right)Y(x)=0 \tag{20}$$

and the boundary conditions of Eqs. (12), (13), (14), and (15) can also be written as

$$\frac{d^2Y(x)}{dx^2}+\frac{\omega^2(J_L+M_Le_L^2)-k_{RL}}{EI_0}\frac{dY(x)}{dx}-\frac{M_Le_L\omega^2}{EI_0}Y(x)=0, \tag{21}$$

$$\frac{d^3Y(x)}{dx^3}-\frac{(\beta_b+3\beta_h)}{l}\frac{d^2Y(x)}{dx^2}+\frac{M_Le_L\omega^2-s_L}{EI_0}\frac{dY(x)}{dx}+\frac{k_{TL}-M_L\omega^2}{EI_0}Y(x)=0, \tag{22}$$

at $x=0$, and

$$\frac{d^2Y(x)}{dx^2}+\frac{k_{RR}}{EI_1}\frac{dY(x)}{dx}-\frac{(J_R+M_Re_R^2)\omega^2}{EI_1}\frac{dY(x)}{dx}-\frac{M_Re_R\omega^2}{EI_1}Y(x)=0, \tag{23}$$

$$\frac{d^3Y(x)}{dx^3}-\frac{1}{l}\left(\frac{\beta_b}{1-\beta_b}+\frac{3\beta_h}{1-\beta_h}\right)\frac{d^2Y(x)}{dx^2}+\frac{M_Re_R\omega^2-s_R}{EI_1}\frac{dY(x)}{dx}+\frac{M_R\omega^2-k_{TR}}{EI_1}Y(x)$$
$$=0 \tag{24}$$

at $x = l$, where $I_1 = \alpha_b \alpha_h^3 I_0 = I_0(1-\beta_b)(1-\beta_h)^3$.
Without loss of generality, the following dimensionless quantities are introduced.

$$
\begin{aligned}
&X = \frac{x}{l}; \quad Y(X) = \frac{Y(x)}{l}; \quad \Omega^2 = \frac{\rho A_0 \omega^2 l^4}{EI_0}; \quad S(X) = \frac{s(x)l^2}{EI_0};\\
&S_L = \frac{s_L l^2}{EI_0}; \quad S_R = \frac{s_R l^2}{EI_1}; \quad K^*(X) = \frac{k^*(x)l^4}{EI_0}; \quad K_{TL} = \frac{k_{TL} l^3}{EI_0};\\
&K_{TR} = \frac{k_{TR} l^3}{EI_1}; \quad K_{RL} = \frac{k_{RL} l}{EI_0}; \quad K_{RR} = \frac{k_{RR} l}{EI_1}; \quad \mu_L = \frac{M_L}{M_b};\\
&\mu_R = \frac{M_R}{M_b}; \quad \delta_L = \frac{e_L}{l}; \quad \delta_R = \frac{e_R}{l}; \quad \gamma_L = \sqrt{\frac{J_L}{M_L l^2}}; \quad \gamma_R = \sqrt{\frac{J_R}{M_R l^2}};\\
&\mu_{mL} = \frac{(2\alpha_b\alpha_h + \alpha_b + \alpha_h + 2)}{6}\mu_L; \quad \mu_{mR} = \frac{(2\alpha_b\alpha_h + \alpha_b + \alpha_h + 2)}{6\alpha_b\alpha_h^3}\mu_R;
\end{aligned}
\tag{25}
$$

where $M_b = \rho A_0 l(2\alpha_b\alpha_h + \alpha_b + \alpha_h + 2)/6$ is the whole mass of the beam and $\Omega = \omega\sqrt{\rho A_0 l^4/EI_0}$ is the dimensionless natural frequency of the beam, then the Eq. (20) simplifies in the dimensionless form as follows:

$$
\frac{d^2}{dX^2}\left[(1-\beta_b X)(1-\beta_h X)^3 \frac{d^2Y(X)}{dX^2}\right] - \frac{d}{dX}\left[S(X)\frac{dY(X)}{dX}\right] + K^*(X)Y(X) - \Omega^2(1-\beta_b X)(1-\beta_h X)Y(X) = 0 \tag{26}
$$

Eq. (26) can be expanded as following

$$
\begin{aligned}
&\frac{d^4Y(X)}{dX^4} - 2\left(\frac{\beta_b}{1-\beta_b X} + \frac{3\beta_h}{1-\beta_h X}\right)\frac{d^3Y(X)}{dX^3}\\
&+6\left[\frac{\beta_b\beta_h}{(1-\beta_b X)(1-\beta_h X)} + \frac{\beta_h^2}{(1-\beta_h X)^2}\right]\frac{d^2Y(X)}{dX^2}\\
&-\frac{1}{(1-\beta_b X)(1-\beta_h X)^3}\left[S(X)\frac{d^2Y(X)}{dX^2} + \frac{dS(X)}{dX}\frac{dY(X)}{dX} - K^*(X)Y(X)\right]\\
&-\frac{\Omega^2}{(1-\beta_h X)^2}Y(X) = 0
\end{aligned}
\tag{27}
$$

the boundary conditions of Eqs. (21), (22), (23), and (24) are given by the following dimensionless forms

$$
Y''(0) + [\mu_{mL}(\delta_L^2 + \gamma_L^2)\Omega^2 - K_{RL}]Y'(0) - \mu_{mL}\delta_L\Omega^2 Y(0) = 0, \tag{28}
$$

$$
Y'''(0) - (\beta_b + 3\beta_h)Y''(0) + (\mu_{mL}\delta_L\Omega^2 - S_L)Y'(0) + (K_{TL} - \mu_{mL}\Omega^2)Y(0) = 0, \tag{29}
$$

and

$$Y''(1)+\left[K_{RR}-\mu_{mR}(\delta_R^2+\gamma_R^2)\Omega^2\right]Y'(1)-\mu_{mR}\delta_R\Omega^2 Y(1)=0, \tag{30}$$

$$Y'''(1)-\left(\frac{\beta_b}{1-\beta_b}+\frac{3\beta_h}{1-\beta_h}\right)Y''(1)+(\mu_{mR}\delta_R\Omega^2-S_R)Y'(1) \\ -(K_{TR}-\mu_{mR}\Omega^2)Y(1)=0 \tag{31}$$

where $Y'(X)=dY(X)/dX$, $Y''(X)=d^2Y(X)/dX^2$, $Y'''(X)=d^3Y(X)/dX^3$. Assuming the dimensionless axial tensile force $S(X)$ and dimensionless Winkler's foundation modulus $K^*(X)$ can be expressed in the form of power series of X, respectively, then

$$S(X)=\sum_{j=0}^{\infty} S_j X^j \tag{32}$$

$$K^*(X)=\sum_{j=0}^{\infty} K_j^* X^j \tag{33}$$

where $S(0)=S_0=S_L$ and $S(1)=S_R$.

The deflection $Y(X)$ can be solved by the AMDM. Eq. (27) can be expressed in the following form

$$Y(X)=\Phi(X)+L^{-1}\left\{2\left(\frac{\beta_b}{1-\beta_b X}+\frac{3\beta_h}{1-\beta_h X}\right)\frac{d^3Y(X)}{dX^3}\right. \\ -\left[\frac{6\beta_b\beta_h}{(1-\beta_b X)(1-\beta_h X)}+\frac{6\beta_h^2}{(1-\beta_h X)^2}-\frac{S(X)}{(1-\beta_b X)(1-\beta_h X)^3}\right]\frac{d^2Y(X)}{dX^2} \\ +\frac{1}{(1-\beta_b X)(1-\beta_h X)^3}\frac{dS(X)}{dX}\frac{dY(X)}{dX} \\ \left.+\left[\frac{\Omega^2}{(1-\beta_h X)^2}-\frac{K^*(X)}{(1-\beta_b X)(1-\beta_h X)^3}\right]Y(X)\right\} \tag{34}$$

where $L^{-1}=\int_0^x\int_0^x\int_0^x\int_0^x \cdots dXdXdXdX$. Now the decomposition $Y(X)=\sum_{k=0}^{\infty} C_k X^k$

can be put together with Eq. (34) to yield

$$
\begin{aligned}
Y(X) &= \sum_{k=0}^{\infty} C_k X^k \\
&= \Phi(X) + L^{-1}\Bigg\{ 2\left(\frac{\beta_b}{1-\beta_b X} + \frac{3\beta_h}{1-\beta_h X}\right)\sum_{k=0}^{\infty}(k+1)(k+2)(k+3)C_{k+3}X^k \\
&- \left[\frac{6\beta_b\beta_h}{(1-\beta_b X)(1-\beta_h X)} + \frac{6\beta_h^2}{(1-\beta_h X)^2} - \frac{S(X)}{(1-\beta_b X)(1-\beta_h X)^3}\right] \\
&\cdot \sum_{k=0}^{\infty}(k+1)(k+2)C_{k+2}X^k \\
&+ \frac{1}{(1-\beta_b X)(1-\beta_h X)^3}\frac{dS(X)}{dX}\sum_{k=0}^{\infty}(k+1)C_{k+1}X^k \\
&+ \left[\frac{\Omega^2}{(1-\beta_h X)^2} - \frac{K^*(X)}{(1-\beta_b X)(1-\beta_h X)^3}\right]\sum_{k=0}^{\infty} C_k X^k \Bigg\}
\end{aligned}
\tag{35}
$$

where we have

$$
\Phi(X) = Y(0) + Y'(0)X + \frac{Y''(0)}{2}X^2 + \frac{Y'''(0)}{6}X^3, \tag{36}
$$

as the initial term of the decomposition. By using the power series, one can obtain

$$
\frac{1}{1-\beta_b X} = \sum_{j=0}^{\infty}(\beta_b X)^j; \quad \frac{1}{1-\beta_h X} = \sum_{j=0}^{\infty}(\beta_h X)^j; \quad \beta_b \neq 0,\ \beta_h \neq 0 \tag{37}
$$

In order to simplify the expression in Eq. (35) the theorem of Cauchy product is used as follows

$$
\sum_{j=0}^{\infty}(\beta_b X)^j \sum_{j=0}^{\infty}(\beta_h X)^j = \sum_{j=0}^{\infty} X^j \sum_{m=0}^{j} \beta_h^m \beta_b^{j-m} \tag{38}
$$

and

$$
\frac{1}{(1-\beta_h X)^2} = \sum_{j=0}^{\infty}(\beta_h X)^j \sum_{j=0}^{\infty}(\beta_h X)^j = \sum_{j=0}^{\infty}(j+1)\beta_h^j X^j \tag{39}
$$

$$
\frac{1}{(1-\beta_h X)^3} = \sum_{j=0}^{\infty}(\beta_h X)^j \left[\sum_{j=0}^{\infty}(\beta_h X)^j\right]^2 = \sum_{j=0}^{\infty}\frac{(j+1)(j+2)}{2}\beta_h^j X^j \tag{40}
$$

$$\begin{aligned}\frac{1}{(1-\beta_b X)(1-\beta_h X)^3} &= \sum_{j=0}^{\infty}(\beta_b X)^j \sum_{j=0}^{\infty}\frac{(j+1)(j+2)}{2}(\beta_h X)^j \\ &= \sum_{j=0}^{\infty} X^j \sum_{p=0}^{j}\frac{(p+1)(p+2)}{2}\beta_h^p \beta_b^{j-p} \\ &= \sum_{j=0}^{\infty} B(j,\beta_h,\beta_b) X^j\end{aligned} \tag{41}$$

In the above equation the expression $B(j,\beta_h,\beta_b)$ is defined as

$$B(j,\beta_h,\beta_b) = \sum_{p=0}^{j}\frac{(p+2)(p+1)}{2}\beta_h^p \beta_b^{j-p}, \quad \beta_h \neq 0\ ,\ \beta_b \neq 0 \tag{42}$$

where

$$B(j,\beta,\beta) = \frac{(j+3)(j+2)(j+1)}{3!}\beta^j, \quad \beta \neq 0, \tag{43}$$

$$B(j,\beta,0) = \frac{(j+2)(j+1)}{2}\beta^j, \quad \beta \neq 0, \tag{44}$$

$$B(j,0,\beta) = \beta^j, \quad \beta \neq 0, \tag{45}$$

from the Eqs. (32), (33), (41) and theorem of Cauchy product, one can get

$$\frac{S(X)}{(1-\beta_b X)(1-\beta_h X)^3} = \sum_{j=0}^{\infty} S_j X^j \sum_{j=0}^{\infty} X^j B(j,\beta_h,\beta_b) = \sum_{j=0}^{\infty} X^j \sum_{m=0}^{j} B(m,\beta_h,\beta_b) S_{j-m}, \tag{46}$$

$$\frac{1}{(1-\beta_b X)(1-\beta_h X)^3}\frac{dS(X)}{dX} = \sum_{j=0}^{\infty} X^j \sum_{m=0}^{j} B(m,\beta_h,\beta_b)(j-m+1) S_{j-m+1}, \tag{47}$$

$$\frac{K^*(X)}{(1-\beta_b X)(1-\beta_h X)^3} = \sum_{j=0}^{\infty} X^j \sum_{m=0}^{j} B(m,\beta_h,\beta_b) K^*_{j-m}, \tag{48}$$

Then the Eq. (35) can be written as

$$\sum_{k=0}^{\infty} C_k X^k = \Phi(X) + L^{-1} \sum_{k=0}^{\infty} X^k \sum_{j=0}^{k} \Big\{ (j+1)(j+2)(j+3)(2\beta_b^{k-j+1} + 6\beta_h^{k-j+1}) C_{j+3}$$
$$-(j+1)(j+2)C_{j+2}[6(k-j+1)\beta_h^{k-j+2} + \sum_{m=0}^{k-j} (6\beta_h^{m+1}\beta_b^{k-j-m+1}$$
$$-B(m,\beta_h,\beta_b)S_{k-j-m})] + (j+1)C_{j+1} \sum_{m=0}^{k-j} (k-j-m+1)B(m,\beta_h,\beta_b)S_{k-j-m+1}$$
$$+C_j[(k-j+1)\beta_h^{k-j}\Omega^2 - \sum_{m=0}^{k-j} B(m,\beta_h,\beta_b)K^*_{k-j-m}] \Big\} \quad (49)$$

By integrating (49), one can obtain

$$\sum_{k=0}^{\infty} C_k X^k = Y(0) + Y'(0)X + \frac{Y''(0)}{2}X^2 + \frac{Y'''(0)}{6}X^3 + \sum_{k=0}^{\infty} \Big\{ \frac{X^{k+4}}{(k+1)(k+2)(k+3)(k+4)}$$
$$\sum_{j=0}^{k} \{(j+1)(j+2)(j+3)C_{j+3}(2\beta_b^{k-j+1} + 6\beta_h^{k-j+1})$$
$$-(j+1)(j+2)C_{j+2}[6(k-j+1)\beta_h^{k-j+2} + \sum_{m=0}^{k-j} (6\beta_h^{m+1}\beta_b^{k-j-m+1}$$
$$-B(m,\beta_h,\beta_b)S_{k-j-m})] + (j+1)C_{j+1} \sum_{m=0}^{k-j} (k-j-m+1)B(m,\beta_h,\beta_b)S_{k-j-m+1}$$
$$+C_j[(k-j+1)\beta_h^{k-j}\Omega^2 - \sum_{m=0}^{k-j} B(m,\beta_h,\beta_b)K^*_{k-j-m}]\} \quad (50)$$

Finally, equating coefficients of like powers of X, we derive the recurrence relation for the coefficients C_k

$$C_0 = Y(0), \quad C_1 = Y'(0), \quad C_2 = \frac{Y''(0)}{2}, \quad C_3 = \frac{Y'''(0)}{6}, \quad (51)$$

and for $k \geq 4$,

$$C_k = \frac{1}{k(k-1)(k-2)(k-3)} \sum_{j=0}^{k-4} \{(j+3)(j+2)(j+1)C_{j+3}(2\beta_b^{k-j-3} + 6\beta_h^{k-j-3})$$
$$-(j+2)(j+1)C_{j+2}[6(k-j-3)\beta_h^{k-j-2} + \sum_{m=0}^{k-j-4} (6\beta_h^{m+1}\beta_b^{k-j-m-3}$$
$$-B(m,\beta_h,\beta_b)S_{k-j-m-4})] + (j+1)C_{j+1} \sum_{m=0}^{k-j-4} (k-j-m-3)B(m,\beta_h,\beta_b)S_{k-j-m-3}$$
$$+C_j[(k-j-3)\beta_h^{k-j-4}\Omega^2 - \sum_{m=0}^{k-j-4} B(m,\beta_h,\beta_b)K^*_{k-j-m-4}]\} \quad (52)$$

Therefore, we can find the coefficients C_k from the recurrent equations (51), and (52), and then we can get the solution $Y(X)$ from Eq. (35). The series solution, of course, is $Y(X) = \sum_{k=0}^{\infty} C_k X^k$. However, in practice all the coefficients C_k in series solution cannot be determined exactly, and the solutions can only be approximated by a truncated series $\sum_{k=0}^{n-1} C_k X^k$with n-term approximation. We can now form successive approximants $\phi^{[n]}(X) = \sum_{k=0}^{n-1} C_k X^k$as n increases and the boundary conditions are also met. Thus $\phi^{[1]}(X) = C_0$, $\phi^{[2]}(X) = \phi^{[1]}(X) + C_1X$, $\phi^{[3]}(X) = \phi^{[2]}(X) + C_2X^2$, $\cdots$, serve as approximate solutions with increasing accuracy as $n \to \infty$, and is also obligated to, of course, satisfy the boundary conditions. The four coefficients $C_k(k = 0,1,2,3)$ in Eq. (51) can be decided by the B.C.s of Eqs. (28) and (29). In this case, the two coefficients C_0 and C_1 can be chosen as the arbitrary constants and the other two coefficients C_2 and C_3 can be expressed as the functions of C_0, C_1 and Ω, that is, from Eqs. (28), (29) and (51), by setting

$$C_2 = \frac{1}{2}\left[K_{RL} - \mu_{mL}(\delta_L^2 + \gamma_L^2)\Omega^2\right] C_1 + \frac{1}{2}\mu_{mL}\delta_L\Omega^2 C_0, \quad (53)$$

$$C_3 = \frac{1}{6}\left\{(\beta_b + 3\beta_h)\left[K_{RL} - \mu_{mL}(\delta_L^2 + \gamma_L^2)\Omega^2\right] + (S_L - \mu_{mL}\delta_L\Omega^2)\right\} C_1$$
$$+\frac{1}{6}\left[(\beta_b + 3\beta_h)\mu_{mL}\delta_L\Omega^2 + (\mu_{mL}\Omega^2 - K_{TL})\right] C_0 \quad (54)$$

From above one can find that the initial term $\Phi(X)$ in Eq. (36) is the function of C_0, C_1 and Ω, and by substituting the Eqs. (36), (53), (54) into the recurrence relation of Eq. (52), the coefficients $C_k(k \geq 4)$ are the function of C_0, C_1 and Ω. Hence the n-term approximation $\phi^{[n]}(X) = \sum_{k=0}^{n-1} C_k X^k$ of the mode shape $Y(x)$ is really the function of C_0, C_1 and Ω. By substituting $\phi^{[n]}(X)$ into B.C.s of Eqs. (30), (31), the two equations are obtained:

$$f_{r0}^{[n]}(\Omega)C_0 + f_{r1}^{[n]}(\Omega)C_1 = 0, \quad r = 1,2 \tag{55}$$

By use of Cramer's rule for nontrivial solutions C_0 and C_1 the frequency equation is given as

$$\begin{vmatrix} f_{10}^{[n]}(\Omega) & f_{11}^{[n]}(\Omega) \\ f_{20}^{[n]}(\Omega) & f_{21}^{[n]}(\Omega) \end{vmatrix} = 0. \tag{56}$$

The ith estimated dimensionless natural frequency $\Omega_i^{[n]}$ corresponding to the approximate term n is obtained by the frequency equation (56), and n is decided by the following equation:

$$\left|\Omega_i^{[n]} - \Omega_i^{[n-1]}\right| \leq \varepsilon, \tag{57}$$

where $\Omega_i^{[n-1]}$ is the ith estimated dimensionless natural frequency corresponding to the approximate term $n-1$, and ε is a preset small value. If Eq. (57) is satisfied, then $\Omega_i^{[n]}$ is the ith dimensionless natural frequency Ω_i of the free vibration problem, that is $\Omega_i = \Omega_i^{[n]}$. By substituting $\Omega_i^{[n]}$ into any one of the Eq. (55), one can obtain

$$C_1 = -\frac{f_{r0}^{[n]}(\Omega_i^{[n]})}{f_{r1}^{[n]}(\Omega_i^{[n]})}C_0, \quad r = 1 \text{ or } 2, \tag{58}$$

and all the other coefficients $C_k(k \geq 2)$ can obtain from Eqs. (53), (54) and (52). Furthermore, the ith mode shape $\phi_i^{[n]}(X)$ corresponding to the ith dimensionless natural frequency $\Omega_i^{[n]}$ is obtained by

$$\phi_i^{[n]}(X) = \sum_{k=0}^{n-1} C_k^{[i]} X^k, \tag{59}$$

where $C_k^{[i]}(X)$ is $C_k(X)$ whose Ω is substituted by $\Omega_i^{[n]}$.

Finally, by use of the above formula of AMDM, the free vibration of the uniform Euler-Bernoulli beam ($\alpha_b = \alpha_h = 1$), the non-uniform Euler-Bernoulli wedge beam ($\alpha_b = 1, \alpha_h = \alpha$) and the non-uniform Euler-Bernoulli cone beam ($\alpha_b = \alpha_h = \alpha$) are, respectively, analyzed. Let's discuss as follows.

3.1 Uniform Euler-Bernoulli beam ($\alpha_b = \alpha_h = 1; \beta = \beta_h = 0$)

In this case, the uniform beam resting on the elastic foundation and subjected to an axial load is considered, the area and moment of inertia of the section are constants, that is $A(x) = A_1 = A_0$, $I(x) = I_1 = I_0$and the mass of the beam is $M_b = \rho A_0 l$, the mass ratio are $\mu_{mL} = \mu_L$ and $\mu_{mR} = \mu_R$. The equation of motion in dimensionless form in Eq. (26) can be written as

$$\frac{d^4Y(X)}{dX^4} - \frac{d}{dX}\left[S(X)\frac{dY(X)}{dX}\right] + \left[K^*(X) - \Omega^2\right]Y(X) = 0, \tag{60}$$

and the recurrence relation for the coefficients C_k in Eq. (52) can be written as

$$C_k = \frac{\sum_{j=0}^{k-4}\left[(j+2)(j+1)S_{k-j-4}C_{j+2} + (j+1)(k-j-3)S_{k-j-3}C_{j+1} - K^*_{k-j-4}C_j\right] + \Omega^2 C_{k-4}}{k(k-1)(k-2)(k-3)} \tag{61}$$

for $k \geq 4$.

3.2 Euler-Bernoulli wedge beam ($\alpha_b = 1, \alpha_h = \alpha; \beta_b = 0, \beta_h = \beta$)

In this case, the wedge beam resting on the elastic foundation and subjected to an axial load is considered, The area and moment of inertia of the section in the two ends of beam are $A_1 = \alpha A_0$, and $I_1 = \alpha^3 I_0$, and the mass of the beam is $M_b = \rho A_0 l(\alpha+1)/2$. The equation of motion in dimensionless form in Eq. (26) can be written as

$$\frac{d^4Y(X)}{dX^4} - \frac{6\beta}{1-\beta X}\frac{d^3Y(X)}{dX^3} + \left[\frac{6\beta^2}{(1-\beta X)^2} - \frac{S(X)}{(1-\beta X)^3}\right]\frac{d^2Y(X)}{dX^2} - \frac{1}{(1-\beta X)^3}\frac{dS(X)}{dX}\frac{dY(X)}{dX} + \left[\frac{K^*(X)}{(1-\beta X)^3} - \frac{\Omega^2}{(1-\beta X)^2}\right]Y(X) = 0 \tag{62}$$

The recurrence relation for the coefficients C_k in Eq. (52) can be written as

$$C_k = \frac{1}{k(k-1)(k-2)(k-3)} \sum_{j=0}^{k-4} \Big\{ 6(j+3)(j+2)(j+1)\beta^{k-j-3} C_{j+3}$$
$$-(j+2)(j+1)C_{j+2}\left[6(k-j-3)\beta^{k-j-2} - \sum_{m=0}^{k-j-4} B(m,\beta,0) S_{k-j-m-4}\right]$$
$$+(j+1)C_{j+1}\sum_{m=0}^{k-j-4}(k-j-m-3)B(m,\beta,0)S_{k-j-m-3}$$
$$+C_j\left[(k-j-3)\beta^{k-j-4}\Omega^2 - \sum_{m=0}^{k-j-4} B(m,\beta,0)K^*_{k-j-m-4}\right]\Big\} \quad (63)$$

for $k \geq 4$.

3.3 Euler-Bernoulli cone beam ($\alpha_b = \alpha_h = \alpha; \beta = \beta_h = \beta$)

In this case, the cone beam resting on the elastic foundation and subjected to an axial load is considered, the parameters are $M_b = \rho A_0 l(\alpha^2+\alpha+1)/3$, $A_1 = \alpha^2 A_0$ and $I_1 = \alpha^4 I_0$, the equation of motion in dimensionless form in Eq. (26) can be written as

$$\frac{d^4Y(X)}{dX^4} - \frac{8\beta}{1-\beta X}\frac{d^3Y(X)}{dX^3} + \left[\frac{12\beta^2}{(1-\beta X)^2} - \frac{S(X)}{(1-\beta X)^4}\right]\frac{d^2Y(X)}{dX^2}$$
$$-\frac{1}{(1-\beta X)^4}\frac{dS(X)}{dX}\frac{dY(X)}{dX} + \left[\frac{K^*(X)}{(1-\beta X)^4} - \frac{\Omega^2}{(1-\beta X)^2}\right]Y(X) = 0 \quad (64)$$

The recurrence relation for the coefficients C_k in Eq (61) can be written as

$$C_k = \frac{1}{k(k-1)(k-2)(k-3)} \sum_{j=0}^{k-4} \Big\{ 8(j+3)(j+2)(j+1)\beta^{k-j-3} C_{j+3}$$
$$-(j+2)(j+1)C_{j+2}\left[12(k-j-3)\beta^{k-j-2} - \sum_{m=0}^{k-j-4} B(m,\beta,\beta) S_{k-j-m-4}\right]$$
$$+(j+1)C_{j+1}\sum_{m=0}^{k-j-4}(k-j-m-3)B(m,\beta,\beta)S_{k-j-m-3}$$
$$+C_j\left[(k-j-3)\beta^{k-j-4}\Omega^2 - \sum_{m=0}^{k-j-4} B(m,\beta,\beta)K^*_{k-j-m-4}\right]\Big\} \quad (65)$$

for $k \geq 4$.

In the above three cases, the two coefficients C_0 and C_1 can be chosen as the arbitrary constants and the other two coefficients C_2 and C_3 can be determined by the boundary conditions of Eqs. (53) and (54). Hence the dimensionless natural frequency can be obtained from Eqs. (55), (56) and (57).

4 Numerical results and discussions

First, the wedge beam($\alpha = 2/3$, $\beta = 1/3$) with elastically restrained ends ($K_{RL} = 1$, $K_{TL} = K_{TR} = K_{RR} = 100$), which supports a tip mass ($\mu_R = 1, \mu_L = 0$) and subjected to a constant axial, tensile force ($S(X) = S_0 = S_L = \alpha^3 S_R$) is discussed. If there is no an elastic foundation ($K^*(X) = 0$), no inertia of moment of mass and eccentricity ($\gamma_R = \delta_R = 0, \gamma_L = \delta_L = 0$), then the equation of motion in dimensionless form in Eq. (62) can be written as

$$\frac{d^4Y(X)}{dX^4} - \frac{6\beta}{1-\beta X}\frac{d^3Y(X)}{dX^3} + \left[\frac{6\beta^2}{(1-\beta X)^2} - \frac{S_0}{(1-\beta X)^3}\right]\frac{d^2Y(X)}{dX^2} - \frac{\Omega^2}{(1-\beta X)^2}Y(X) = 0 \quad (66)$$

By the AMDM one can set $Y(X) = \sum_{k=0}^{\infty} C_k X^k$ and take the n-term series solution $\phi^{[n]}(X) = \sum_{k=0}^{n-1} C_k X^k$ as the approximate solution of $Y(X)$, the boundary conditions in Eqs. (28)-(31) are given as

$$(\phi^{[n]})''(0) - K_{RL}(\phi^{[n]})'(0) = 0, \quad (67)$$

$$(\phi^{[n]})'''(0) - 3\beta(\phi^{[n]})''(0) - S_L(\phi^{[n]})'(0) + K_{TL}\phi^{[n]}(0) = 0, \quad (68)$$

$$(\phi^{[n]})''(1) + K_{RR}(\phi^{[n]})'(1) = 0, \quad (69)$$

$$(\phi^{[n]})'''(1) - \frac{3\beta}{1-\beta}(\phi^{[n]})''(1) - S_R(\phi^{[n]})'(1) - (K_{TR} - \mu_{mR}\Omega^2)\phi^{[n]}(1) = 0. \quad (70)$$

The recurrence relation for the coefficients C_k in Eq. (63) can be written as

$$C_k = \frac{1}{k(k-1)(k-2)(k-3)}\sum_{j=0}^{k-4}\beta^{k-j-4}\Big\{6\beta(j+3)(j+2)(j+1)C_{j+3} - (j+2)(j+1)(k-j-3)[6\beta^2 - \frac{(k-j-2)}{2}S_0]C_{j+2} + (k-j-3)\Omega^2 C_j\Big\} \quad (71)$$

for $k \geq 4$. Setting $S_R = 2, \mu_R = 1$, one can get $S_0 = S_L = 16/27$, and $\mu_{mR} = 45/16$. If the coefficients C_0 and C_1 are chosen as the arbitrary constants, then from the Eqs. (53), (54), (67) and (68) one can obtain

$$C_2 = \frac{K_{RL}}{2} C_1 = 0.5 C_1, \tag{72}$$

$$C_3 = \frac{3\beta K_{RL} + S_L}{6} C_1 - \frac{K_{TL}}{6} C_0 = 0.265432 C_1 - 16.6667 C_0 \tag{73}$$

Substituting Eqs. (72) and (73) into Eq. (71) one can obtain

$$C_4 = 0.12963 C_1 + (0.0416667\Omega^2 - 8.33333) C_0 \tag{74}$$

and

$$C_5 = (0.00833333\Omega^2 - 0.0609511) C_1 + (0.0222222\Omega^2 - 3.82716) C_0 \tag{75}$$

Following the same recursive procedure, one can calculate up to C_{26} and obtain the 27th approximate solution $\phi^{[27]}(X) = \sum_{k=0}^{26} C_k X^k$, substituting it into Eqs. (69) and (70) and using Eq. (56), one can obtain the frequency equation as follows.

$$\begin{aligned}
&1.44509 \times 10^6 - 64427.457034\Omega^2 + 438.539481\Omega^4 - 0.606327\Omega^6 \\
&+ 0.000224389\Omega^8 - 2.734157 \times 10^{-8}\Omega^{10} + 1.339248 \times 10^{-12}\Omega^{12} \\
&- 3.181450 \times 10^{-17}\Omega^{14} + 5.533083 \times 10^{-22}\Omega^{16} + 9.582362 \times 10^{-28}\Omega^{18} \\
&- 1.736535 \times 10^{-30}\Omega^{20} + 1.675003 \times 10^{-35}\Omega^{22} + 4.020806 \times 10^{-40}\Omega^{24} \\
&\qquad - 3.549679 \times 10^{-46}\Omega^{26} = 0
\end{aligned} \tag{76}$$

Solving Eq. (76), one can obtain the first three roots

$$\Omega_1^{[27]} = 5.226878; \quad \Omega_2^{[27]} = 12.620143; \quad \Omega_3^{[27]} = 28.751623 \tag{77}$$

By the same procedure one can obtain for $n = 26$

$$\Omega_1^{[26]} = 5.226879; \quad \Omega_2^{[26]} = 12.620140; \quad \Omega_3^{[26]} = 28.751607 \tag{78}$$

From Eqs. (77) and (78) one can get

$$\begin{aligned}
&\left|\Omega_1^{[27]} - \Omega_1^{[26]}\right| = 0.0000006; \quad \left|\Omega_2^{[27]} - \Omega_2^{[26]}\right| = 0.000003 \\
&\left|\Omega_3^{[27]} - \Omega_3^{[26]}\right| = 0.000016
\end{aligned} \tag{79}$$

Hence by setting $\varepsilon = 0.00002$ in Eq. (57) the first three dimensionless natural frequencies can be obtained one at a time. If one sets $\varepsilon = 0.000001$, then the first dimensionless natural frequency $\Omega_1 = \Omega_1^{[27]} = 5.226878$ can be obtained from Eq. (79). Substituting Ω_1 into $C_4 - C_{26}$ and using Eq. (59), one can obtain the closed form series solution of the first mode shape.

$$\begin{aligned}\phi_1^{[27]}(X) = C_0(&1 + 16.767780\mathrm{X} + 8.38389\mathrm{X}^2 - 12.21596\mathrm{X}^3 - 5.021388\mathrm{X}^4 \\ &+ 1.619472\mathrm{X}^5 + 1.771573\mathrm{X}^6 + 0.489858\mathrm{X}^7 + 0.0757699\mathrm{X}^8 + 0.0248059\mathrm{X}^9 \\ &+ 0.019772\mathrm{X}^{10} + 0.0119799\mathrm{X}^{11} + 0.00587826\mathrm{X}^{12} + 0.00262539\mathrm{X}^{13} \\ &+ 0.00112961\mathrm{X}^{14} + 0.000477649\mathrm{X}^{15} + 0.000199575\mathrm{X}^{16} + 0.0000825243\mathrm{X}^{17} \\ &+ 0.0000338004\mathrm{X}^{18} + 0.0000137245\mathrm{X}^{19} + 5.529254 \times 10^{-6}\mathrm{X}^{20} \\ &+ 2.211929 \times 10^{-6}\mathrm{X}^{21} + 8.79243 \times 10^{-7}\mathrm{X}^{22} + 3.4749 \times 10^{-7}\mathrm{X}^{23} \\ &+ 1.366155 \times 10^{-7}\mathrm{X}^{24} + 5.345408 \times 10^{-8}\mathrm{X}^{25} + 2.082368 \times 10^{-8}\mathrm{X}^{26}\Big)\end{aligned}$$

Finally, the dimensionless natural frequencies of free vibration of beams with several complicated effects are obtained by using the above method. In the following cases, the small value ε in Eq. (57) is set to be 0.000001 and the numerical results are compared with those results in the literatures.

4.1 Uniform Euler-Bernoulli beam

The dimensionless natural frequencies Ω of the uniform beam with the left end elastically restrained and the right end supported with the dimensionless tip mass μ_R, rotatory inertia γ_R of tip mass, and eccentricity δ_R are listed in Table 1-2. In Table 1, it can be observed that the dimensionless natural frequencies determined by the proposed method converge very rapidly, and the first dimensionless natural frequencies can be obtained one at a time for the approximate term $n = 30$, and the convergent solutions for the approximate term $n = 30$ and those given by Auciello (1996) are very consistent. Table 2 shows the convergent dimensionless natural frequencies for the approximate term n with different values of γ_R and μ_R, and the results are in agreement with Chang (1993).

In Figure 2, the square root $\sqrt{\Omega_1}$ of the first dimensionless natural frequency for a uniform cantilever beam with the physical parameters γ_R and δ_R are shown. In Figure 3, the square root $\sqrt{\Omega_1}$ of the first dimensionless natural frequency for a uniform cantilever beam with the physical parameters μ_R and γ_R are shown. It can be observed that the first dimensionless natural frequency decreases when the tip mass μ_R or rotatory inertia γ_R of the mass or eccentricity δ_R are increased, and the tip mass μ_R has greater influence on the natural frequencies than γ_R and δ_R.

Table 1: The square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a uniform cantilever beam with a tip mass, the rotatory inertia of the mass, and its eccentricity at the free end; (I) Auciello (1996). (II) To (1982). ($K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$; $\mu_R = \gamma_R = 1$; $K^*(X) = S(X) = \mu_L = \delta_L = \gamma_L = 0$)

δ_R	$\sqrt{\Omega}$	Present			(I)	(II)
		$n = 10$	$n = 20$	$n = 30$		
0.4	$\sqrt{\Omega_1}$	0.850678	0.850678	0.850678	0.850678	0.85068
	$\sqrt{\Omega_2}$	1.977850	1.980129	1.980129	1.980129	1.98013
	$\sqrt{\Omega_3}$	5.095397	4.945065	4.945079	4.945079	4.94508
0.6	$\sqrt{\Omega_1}$	0.810481	0.810481	0.810481	0.810481	0.81048
	$\sqrt{\Omega_2}$	2.042304	2.045433	2.045433	2.045433	2.04543
	$\sqrt{\Omega_3}$	5.322921	4.978207	4.978225	4.978225	4.97823
0.8	$\sqrt{\Omega_1}$	0.772802	0.772802	0.772802	0.772801	0.77280
	$\sqrt{\Omega_2}$	2.099542	2.103698	2.103698	2.103697	2.10370
	$\sqrt{\Omega_3}$	5.675757	5.015739	5.015764	5.015764	5.01576

Table 2: The square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a uniform cantilever beam with a tip mass, the rotatory inertia of the mass, and its eccentricity at the free end; (I) Chang (1993). ($K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$; $K^*(X) = S(X) = \mu_L = \delta_L = \gamma_L = \delta_R = 0$)

$\sqrt{\Omega}$	$\gamma_R = 0.0$ $\mu_R = 0.2$		$\gamma_R = 0.3$ $\mu_R = 0.4$		$\gamma_R = 0.9$ $\mu_R = 2.0$	
	Present $n = 46$	(I)	Present $n = 42$	(I)	Present $n = 39$	(I)
$\sqrt{\Omega_1}$	1.616402	1.616400	1.429860	1.429860	0.818981	0.818977
$\sqrt{\Omega_2}$	4.267062	4.267062	3.036912	3.036911	1.620778	1.620777
$\sqrt{\Omega_3}$	7.318373	7.318371	5.234071	5.234072	4.826015	4.826014
$\sqrt{\Omega_4}$	10.401563	10.401563	8.135284	8.135284	7.913892	7.913892
$\sqrt{\Omega_5}$	13.506702	13.506702	11.195621	11.195621	11.039366	11.039366

The square roots $\sqrt{\Omega}$ of the first three dimensionless fundamental frequencies of a cantilever beam under linearly varying axial force for three different boundary conditions are listed in Table 3. The computed results are in agreement with those given by Nallim and Grossi (1999) and Naguleswaran (2004). The square roots $\sqrt{\Omega}$ of the first five dimensionless fundamental frequencies of a uniform beam with elastically restrained ends, which supports a tip mass and is subjected to a constant axial, tensile force are listed in Table 4, the computed results compared with those given by Nallim and Grossi (1999) are very consistent. In Figure 4, the square

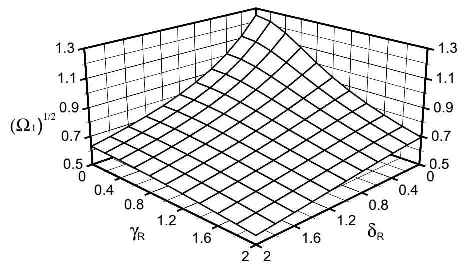

Figure 2: Plot of the square root of the first dimensionless natural frequency $\sqrt{\Omega_1}$ for a uniform cantilever beam with the dimensionless moment of inertia of mass γ_R and eccentricity δ_R ($\mu_R = 1$)

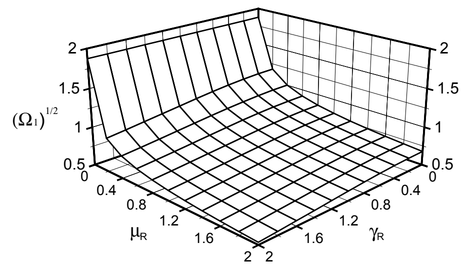

Figure 3: Plot of the square root of the first dimensionless natural frequency $\sqrt{\Omega_1}$ for a uniform cantilever beam with the dimensionless mass μ_R and moment of inertia of mass γ_R ($\delta_R = 1$)

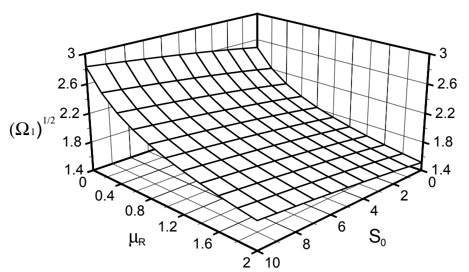

Figure 4: Plot of The square roots $\sqrt{\Omega_1}$ of the first dimensionless natural frequency of a uniform beam with elastically restrained ends, which supports a tip mass μ_R and is subjected to a constant axial, tensile force S_0 ($k_{TL} \rightarrow \infty, k_{TR} = 5, k_{TR} = 10, k_{RR} = 0$)

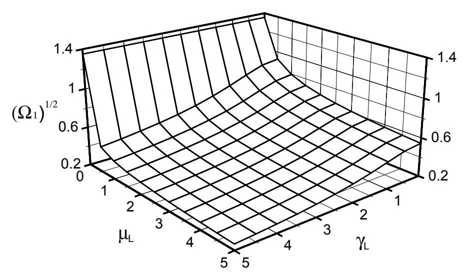

Figure 5: Plot of The square roots $\sqrt{\Omega_1}$ of the first dimensionless natural frequency of a cone beam with e a tip mass, the rotatory inertia of the mass, and its eccentricity at the left end ($\alpha = 1.1$; $k_{TR} = \infty, k_{RR} = 1$; $\delta_L = 0.4$; $\mu_R = \delta_R = \gamma_R = 0$)

root $\sqrt{\Omega_1}$ of the first dimensionless natural frequency for a uniform beam ($K_{TL} \rightarrow \infty, K_{TR} = 5, K_{TR} = 10, K_{RR} = 0\,; \delta_R = \gamma_R = 0$) with the physical parameters μ_R and S_0 are shown. It can be observed that the first dimensionless natural frequency decreases when the tip mass μ_R is increased, and increases when the constant axial tensile force S_0 is increased.

Finally, the first five dimensionless natural frequencies Ω of a uniform cantilever beam having a tip mass at the free end and resting on the elastic foundation are

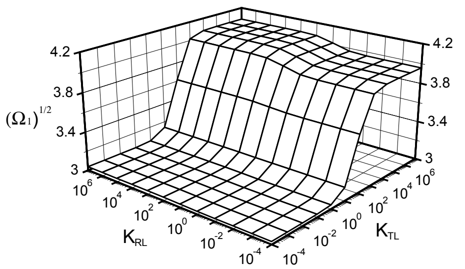

Figure 6: Plot of The square roots $\sqrt{\Omega_1}$ of the first dimensionless natural frequency of a non-uniform beam with elastically restrained ends, which supports a tip mass and is subjected to a constant axial force ($\alpha = 1.5$; $K_{TR} = 100, K_{RR} = 1$; $S_R = 10$; $\mu_R = 0.8$)

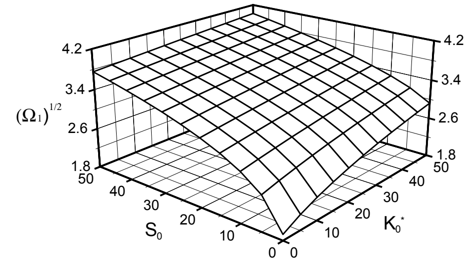

Figure 7: Plot of the square roots $\sqrt{\Omega}$ of the first dimensionless natural frequencies of a cantilever cone beam resting on an elastic foundation and subjected to a constant axial force ($\alpha = 0.9$; $K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$; $K^*(X) = K_0^*$; $S(X) = S_L = S_0$; $\mu_L = \delta_L = \gamma_L = \mu_R = \delta_R = \gamma_R = 0$)

Table 3: The square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a uniform beam under linearly varying axial force $S(X) = S_L + S_1 X$; (I) Nallim (1999). (II) Naguleswaran (2004). ($S_L = S_0$; $S_R - S_L = S_1$; $K^*(X) = \mu_L = \delta_L = \gamma_L = \mu_R = \delta_R = \gamma_R = 0$)

BC^*	$\sqrt{\Omega}$	$S_L = 10, S_1 = 0$		$S_L = 10, S_1 = 4$		$S_L = 10, S_1 = 100$	
		Present $n = 43$	(I)	Present $n = 43$	(II)	Present $n = 49$	(II)
$cl\backslash cl^*$	$\sqrt{\Omega_1}$	4.995742	4.995742	5.043666	5.0437	5.876812	5.8768
	$\sqrt{\Omega_2}$	8.080354	8.080355	8.123404	8.1234	8.971967	8.9720
	$\sqrt{\Omega_3}$	11.176983	11.176988	11.212177	11.2122	11.959541	11.9595
$pn\backslash pn^*$	$\sqrt{\Omega_1}$	3.742159	3.742159	3.832191	3.8322	5.003168	5.0032
	$\sqrt{\Omega_2}$	6.648044	6.648044	6.714075	6.7141	7.861698	7.8617
	$\sqrt{\Omega_3}$	9.679520	9.679521	9.728076	9.7281	10.699028	10.6990
$cl\backslash fr^*$	$\sqrt{\Omega_1}$	2.677212	_	2.766044	2.7660	3.587605	3.5876
	$\sqrt{\Omega_2}$	5.319243	_	5.454186	5.4542	6.973598	6.9736
	$\sqrt{\Omega_3}$	8.225463	_	8.316068	8.3161	9.743540	9.7435

* BC: boundary condition
* $cl\backslash cl$: clamped-clamped. $pn\backslash pn$: pinned-pinned. $cl\backslash fr$: clamped-free.

listed in Table5. From this table one can find that the natural frequencies increase when the physical parameter K_0^* increases and the tip mass μ_R has greater influence on the natural frequencies than the constant Winkler's foundation modulus K_0^*.

Table 4: The square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a uniform beam with elastically restrained ends, which supports a tip mass and is subjected to a constant axial, tensile force; (I) Nallim (1999). ($K_{TL} \to \infty, K_{TR} = 10, K_{RR} = 0$; $S(X) = S_0$; $K^*(X) = \mu_L = \delta_L = \gamma_L = \delta_R = \gamma_R = 0$)

$\sqrt{\Omega}$	$S_0 = 10$ $\mu_R = 0.4, K_{RL} = 5$		$S_0 = 10$ $\mu_R = 0.2, K_{RL} = 1.25$		$S_0 = 2$ $\mu_R = 1, K_{RL} = 0.2$	
	Present $n = 47$	(I)	Present $n = 43$	(I)	Present $n = 47$	(I)
$\sqrt{\Omega_1}$	2.365255	2.3653	2.478762	2.4788	1.735489	1.7355
$\sqrt{\Omega_2}$	4.321753	4.3218	4.361232	4.3612	3.462256	3.6423
$\sqrt{\Omega_3}$	7.046657	7.0467	7.017856	7.0179	6.450325	6.4503
$\sqrt{\Omega_4}$	9.984168	9.9842	9.947356	9.9474	9.538063	9.5381
$\sqrt{\Omega_5}$	13.006049	13.0061	12.971575	12.9719	12.652027	12.6521

Table 5: The first five dimensionless natural frequencies Ω of a uniform cantilever beam having a tip mass at the free end and resting on the elastic foundation; (I) Chen (2000). ($K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$; $S(X) = 0$; $\mu_L = \delta_L = \gamma_L = \delta_R = \gamma_R = 0$);

		$K_0^* = 1$		$K_0^* = 5$	$K_0^* = 10$
μ_R	Ω	(I)	Present	Present	Present
0	Ω_1	3.65544	3.655457	4.166817	4.728886
	Ω_2	22.0572	22.057170	22.147658	22.260250
	Ω_3	61.7057	61.705310	61.737714	61.778194
	Ω_4	120.911	120.906022	120.922563	120.943235
	Ω_5	199.894	199.861952	199.871959	199.884466
1	Ω_1	1.61782	1.617825	1.838580	2.078853
	Ω_2	16.2781	16.278070	16.389697	16.528532
	Ω_3	50.9054	50.905327	50.943271	50.990665
	Ω_4	105.205	105.202925	105.221608	105.244957
	Ω_5	179.251	179.234716	179.245759	179.259562

4.2 Non-uniform Euler-Bernoulli beam

The dimensionless natural frequencies of the cone beam with the right end elastically restrained ($K_{TL} = 0, K_{RL} = 0$) and the left end supported with the tip mass μ_L, rotatory inertia γ_L of the mass, and eccentricity δ_L are listed in Table 6-7. In Table 6, the square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a cantilever cone beam ($K_{TR} \to \infty, K_{RR} \to \infty$) with μ_L, γ_L and δ_L at the left end are listed.

Table 6: The square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a cantilever cone beam with a tip mass, the rotatory inertia of the mass, and its eccentricity at the left end; (I) Auciello(1996). ($K_{TL}=0, K_{RL}=0, K_{TR}\rightarrow\infty, K_{RR}\rightarrow\infty$; $K^*(X)=S(X)=\mu_R=\delta_R=\gamma_R=\delta_L=0$)

γ_L	α	μ_L	$\sqrt{\Omega_1}$ Present	$\sqrt{\Omega_1}$ (I)	$\sqrt{\Omega_2}$ Present	$\sqrt{\Omega_2}$ (I)	$\sqrt{\Omega_3}$ Present	$\sqrt{\Omega_3}$ (I)
0	1.2	0.2	1.805116	1.805113	4.531398	4.531399	7.682833	7.682832
		0.6	1.519992	1.519988	4.343868	4.343868	7.542352	7.542352
		1	1.373710	1.373712	4.289385	4.289384	7.507076	7.507077
		2	1.180605	1.180607	4.242897	4.242896	7.478641	7.478642
	2	0.2	2.392498	2.392505	5.375538	5.375542	8.914100	8.914098
		0.6	1.949576	1.949585	5.190742	5.190740	8.804084	8.804081
		1	1.744152	1.744161	5.145041	5.145038	8.779423	8.779421
		2	1.485704	1.485718	5.108415	5.108412	8.760286	8.760283
0.3	1.2	0.2	1.759470	1.75946	3.493510	3.49351	5.666955	5.66696
		0.6	1.459658	1.45965	2.870282	2.87028	5.298093	5.29809
		1	1.312503	1.31250	2.590249	2.59025	5.182562	5.18256
		2	1.122787	1.12279	2.228088	2.22809	5.079316	5.07931
	2	0.2	2.268663	2.26867	3.648385	3.64839	6.290136	6.29014
		0.6	1.808854	1.80886	2.949114	2.94912	5.974270	5.97427
		1	1.608437	1.60845	2.640890	2.64090	5.888834	5.88884
		2	1.363162	1.36317	2.253207	2.25322	5.818075	5.81808
0.6	1.2	0.2	1.634324	1.63432	2.779846	2.77985	5.472032	5.47203
		0.6	1.312147	1.31214	2.301968	2.30197	5.214797	5.21480
		1	1.168702	1.16870	2.082917	2.08292	5.129180	5.12918
		2	0.991773	0.99177	1.795562	1.79556	5.051225	5.05122
	2	0.2	1.954460	1.95445	3.048977	3.04898	6.199473	6.19948
		0.6	1.511025	1.51101	2.514297	2.51430	5.939125	5.93913
		1	1.334368	1.33435	2.261011	2.26102	5.867125	5.86713
		2	1.124886	1.12487	1.935209	1.93522	5.807008	5.80701

In Table 7, the square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a cone beam ($\alpha_b=\alpha_h=\alpha=1.1$) with μ_L, γ_L and δ_L at the left end are listed. Comparing the natural frequencies with those given by Auciello (1996) one can find that the results are very consistent. In Figure 5, the square roots $\sqrt{\Omega_1}$ of the first dimensionless natural frequency of a cone beam ($\alpha=1.1$; $K_{TR}=\infty, K_{RR}=1$; $\delta_L=0.4$) with μ_L and γ_L are shown. It can be observed that the first dimensionless natural frequency decreases when the tip mass μ_L or rotatory inertia γ_L of the mass or eccentricity δ_L are increased, and the tip mass μ_L has greater influence on the natural frequencies than γ_L and δ_L.

Table 7: The square roots $\sqrt{\Omega}$ of the first three dimensionless natural frequencies of a cone beam with a tip mass, the rotatory inertia of the mass, and its eccentricity at the left end; (I) Auciello (1996). ($\alpha_b = \alpha_h = \alpha = 1.1$; $K_{TL} = 0, K_{RL} = 0$; $K^*(X) = S(X) = \mu_R = \delta_R = \gamma_R = 0,\ \mu_L = 1$)

δ_L	K_{TR}	γ_L	K_{RR}	$\sqrt{\Omega_1}$		$\sqrt{\Omega_2}$		$\sqrt{\Omega_3}$	
				Present	(I)	Present	(I)	Present	(I)
0.4	∞	0.6	0.1	0.467440	0.46743	1.854354	1.85436	4.358113	4.35811
			1	0.755247	0.75524	1.948854	1.94886	4.454283	4.45428
			10	0.934568	0.93456	2.182581	2.18258	4.834589	4.83459
		0.8	0.1	0.455279	0.45527	1.694337	1.69434	4.290900	4.29090
			1	0.730454	0.73044	1.790888	1.79089	4.390606	4.39061
			10	0.893032	0.89303	2.022673	2.02267	4.779324	4.77932
		1	0.1	0.441668	0.44166	1.583463	1.58347	4.256810	4.25681
			1	0.703347	0.70334	1.685046	1.68505	4.358455	4.35845
			10	0.850076	0.85007	1.921416	1.92142	4.751734	4.75174
	1	0.6	0.1	0.461893	0.46189	1.165266	1.16527	2.480518	2.48052
			1	0.700855	0.70085	1.197686	1.19769	2.634190	2.63419
			10	0.801468	0.80146	1.238412	1.23842	2.891271	2.89127
0.6	∞	0.6	0.1	0.443598	0.44359	1.873048	1.87305	4.436802	4.43680
			1	0.712291	0.71228	1.983125	1.98312	4.530416	4.53041
			10	0.873804	0.87380	2.248241	2.24824	4.905603	4.90560
		0.8	0.1	0.434100	0.43410	1.729272	1.72928	4.345483	4.34548
			1	0.693283	0.69328	1.837962	1.83796	4.442982	4.44298
			10	0.843081	0.84307	2.092539	2.09254	4.827034	4.82703
		1	0.1	0.423220	0.42322	1.621961	1.62196	4.295470	4.29547
			1	0.671929	0.67192	1.732486	1.73249	4.395387	4.39539
			10	0.809974	0.80997	1.984642	1.98464	4.784972	4.78497
	1	0.6	0.1	0.439333	0.43933	1.162500	1.16251	2.520126	2.52013
			1	0.671065	0.67105	1.184762	1.18476	2.684893	2.68489
			10	0.773954	0.77395	1.212624	1.21262	2.960296	2.96030

The square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a non-uniform beam with elastically restrained ends, which supports a tip mass and is subjected to a constant axial, tensile force are listed in Table 8. These results are in agreement with those given by Nallim and Grossi (1999). In Figure 6, the square roots $\sqrt{\Omega_1}$ of the first dimensionless natural frequency of a cone beam ($\alpha = 1.5$) with K_{TL} and K_{RL} are shown. It can be observed that the first dimensionless natural frequency increases when the dimensionless translational spring constant K_{TL} or the rotational spring constant K_{RL} is increased, and the parameter K_{TL} has greater influence on the natural frequencies than the parameter K_{RL}.

Finally, the square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a cantilever cone beam ($\alpha = 0.9$) resting on an elastic foundation and subjected to a constant axial force are listed in Table 9. The computed results are very consistent with Lee and Lin (1995). In Figure 7, one can found that the first dimensionless natural frequency increases when the dimensionless axial tensile force S_0 or the constant Winkler's foundation modulus K_0^* is increased, and the parameter S_0 has greater influence on the natural frequencies than the parameter K_0^*.

5 Conclusion

By the method proposed in this study, the dimensionless natural frequencies of the free vibration of non-uniform Euler-Bernoulli beams can be obtained. This paper presents a simple, computationally efficient and accurate approximate approach. The innovative solver developed is very general and takes into account a great variety of complicated effects, such as non-uniform cross-sections, effects of an axial force and elastic foundation, presence of a tip mass with the rotatory inertia of mass and eccentricity, and ends elastically restrained against translational and rotational springs.

By using the proposed method, any ith natural frequencies can be obtained one at a time for some approximate term n, the larger the approximate term is giving, more natural frequency can be found at the same time. The computed results are compared closely with the results obtained by using other analytical and numerical methods in the literatures.

This study provides a unified and systematic procedure which is seemingly simpler and more straightforward than the other methods, and constitutes an efficient tool for the design of beam with the vibration problem.

References

Adomian, G. (1994): *Solving Frontier Problems of Physics: The Decomposition Method Method*. Kluwer Academic Publishers.

Adomian, G.; Rach, R. (1992): Modified decomposition solution of nonlinear partial differential equations. *Comput. Math. Applic.*, vol. 23, pp. 17-23.

Adomian, G.; Rach, R. (1991): Nonlinear transformation of series. *Appl. Math. Lett.*, vol. 4, pp. 69-71.

Alvarez, S. I.; Iglesias, G. M.; Laura, A. A. (1988): Vibrations of an elastically restrained non-uniform beam with translational and rotational springs, and with a tip mass. *Journal of Sound and Vibration*, vol. 120(3), pp. 465-471.

Andreaus, U.; Batra, R. C.; Porfiri, M. (2005): Vibrations of cracked Euler-Bernoulli beams using Meshless Local Petrov-Galerkin (MLPG) method. *CMES:*

Computer Modeling in Engineering & Sciences, vol. 9, pp. 111-131.

Auciello, N. M. (1996): Transverse vibrations of a linearly tapered cantilever beam with tip mass of rotatory inertia and eccentricity. *Journal of Sound and Vibration*, vol. 194, pp. 25-34.

Bokaian, A. (1990): Natural frequencies of beams under tensile axial loads. *Journal of Sound and Vibration*, vol. 142(3), pp. 481-498.

Chen, C. N. (2000): Vibration of prismatic beam on an elastic foundation by the differential quadrature element method. *Computers & Structures,* vol. 77, pp. 1-9.

Chang, T. P. (1993): Forced vibration of a mass-loaded beam with a heavy tip body. *Journal of Sound and Vibration*, vol. 164(3), pp. 471-484.

Dimarogonas, A. (1996): *Vibration for Engineers*. 2^{nd} ed., Prentice-Hall, Inc..

Goel, R. P. (1976): Transverse vibrations of tapered beams. *Journal of Sound and Vibration* , vol. 47(1), pp. 1-7.

Grossi, R. O.; Aranda, A.; Bhat, R. B. (1993): Vibration of tapered beams with one end spring hinged and the other end with tip mass. *Journal of Sound and Vibration*, vol. 160(1), pp. 175-178.

Grossi, R. O.; Laura, P. A. A. (1982): Further results on a vibrating beam with a mass and spring at the end subjected to an axial force. *Journal of Sound and Vibration*, vol. 84(4), pp. 593-594.

Huang, C. H.; Shih, C. C. (2007): An inverse problem in estimating simultaneously the time-dependent applied force and moment of an Euler-Bernoulli beam. *CMES: Computer Modeling in Engineering & Sciences*, vol. 21, pp. 239-254.

Lau, J. H. (1984): Vibration frequencies of tapered bars with end mass. *Transactions of the ASME: Journal of the Applied Mechanics*, vol. 51, pp. 179-181.

Lau, J. H. (1984): Vibration frequencies for a non-uniform beam with end mass. *Journal of Sound and Vibration*, vol. 97(3), pp. 513-521.

Laura, P. A. A.; Cortinez, V. H. (1985): Transverse vibrations of a cantilever beam carrying a concentrated mass at its free end and subjected to a variable axial force. *Journal of Sound and Vibration*, vol. 103(4), pp. 596-599.

Laura, P. A. A.; Pombo, J. L.; Susemihl, E. A. (1974): A note on the vibrations of a clamped-free beam with a mass at the free end. *Journal of Sound and Vibration*, vol. 37(2), pp. 161-168.

Lee, S. Y.; Lin, S. M. (1995): Vibrations of elastically restrained non-uniform Timoshenko beams. *Journal of Sound and Vibration*, vol. 183(3), pp. 403-415.

Lee, T. W. (1973): Vibration frequencies for a uniform beam with one end spring-hinged and carrying a mass at the other end. *Transactions of the ASME: Journal of* *the Applied Mechanics*, vol. 40, pp. 813-815.

Liu, W. H.; Huang, C. C. (1988): Vibrations of a constrained beam carrying a heavy tip body. *Journal of Sound and Vibration*, vol. 123(1), pp. 15-29.

Mabie, H. H.; Rogers, C. B. (1974): Transverse vibrations of double-tapered cantilever beams with end support and with end mass. *Journal of the Acoustical Society of America*, vol. 55, pp. 986-991.

Maurizi, M. J.; Belles, P.; Rosales, M. (1990): A note on free vibrations of a constrained cantilever beam with a tip mass of finite length. *Journal of Sound and Vibration*, vol. 138(1), pp. 170-172.

Meirovitch, L. (2001): *Fundamentals of Vibrations*. McGraw-Hall, International Edition.

Naguleswaran, S. (2004): Transverse vibration of an uniform Euler-Bernoulli beam under linearly varying axial force. *Journal of Sound and Vibration*, vol. 275, pp. 47-57.

Naguleswaran, S. (1991): Vibration of a vertical cantilever with and without axial freedom at clamped end. *Journal of Sound and Vibration*, vol. 146(2), pp. 191-198.

Nallim, L. G.; Grossi, R. O. (1999): A general algorithm for the study of the dynamical behavior of beams. *Applied Acoustics*, vol. 57, pp. 345-356.

Takahashi, K. (1980): Eigenvalue problem of a beam with a mass and spring at the end subjected to an axial force. *Journal of Sound and Vibration*, vol. 71(3), pp. 453-457.

Thambiratnam, D.; Zhuge, Y. (1996): Free vibration analysis of beams on elastic foundation. *Computers & Structures*, vol. 60(6), pp. 971-980.

To, C. W. S. (1982): Vibration of a cantilever beam with a base excitation and tip mass. *Journal of Sound and Vibration*, vol. 83(4), pp. 445-460.

Vinod, K. G.; Gopalakrishnan, S; Ganguli, R. (2006): Wave propagation characteristics of rotating uniform Euler-Bernoulli beam. *CMES: Computer Modeling in Engineering & Sciences*, vol. 16, pp. 197-208.

Weaver, W.; Timoshenko, S. P.; Young, D. H. (1990): *Vibration Problems in Engineering*. fifth ed., John Wiley & Sons, Inc..

Table 8: The square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a non-uniform beam with elastically restrained ends, which supports a tip mass and is subjected to a constant axial, tensile force; (I) Nallim (1999). ($S(X) = S_R$; $K^*(X) = 0$; $\mu_L = \delta_L = \gamma_L = \delta_R = \gamma_R = 0$)

	K_{RL}	K_{RR}	$\sqrt{\Omega_1}$		$\sqrt{\Omega_2}$		$\sqrt{\Omega_3}$		$\sqrt{\Omega_4}$		$\sqrt{\Omega_5}$	
			Present	(I)	Present	(I)	Present	(I)	Present	(I)	Present	(I)
$\alpha_b = 1.5$	0	1	3.040849	3.0408	4.467180	4.4672	6.414410	6.4144	9.275327	9.2753	12.386059	12.3861
$\alpha_h = 1.5$	0.1	10	3.212264	3.2123	4.534256	4.5343	6.790956	6.7910	9.644895	9.6449	12.718672	12.7187
$K_{TL} = 1$	1	100	3.287363	3.2874	4.573221	4.5732	7.055045	7.0550	10.025324	10.0253	13.170195	13.1702
$K_{TR} = 100$	10	1000	3.301342	3.3013	4.593605	4.5936	7.193192	7.1932	10.269813	10.2698	13.481548	13.4815
$\mu_R = 0.8$	100	10000	3.305610	3.3056	4.606423	4.6064	7.281878	7.2818	10.447553	10.4474	13.735090	13.7352
$S_R = 10$	1000	∞	3.306290	3.3062	4.608634	4.6076	7.297425	7.2951	10.480341	10.4786	13.785262	13.7747
$\alpha_b = 1$	0	1	2.245640	2.2457	3.085425	3.0855	5.012758	5.0148	6.909753	6.9125	9.467826	9.4694
$\alpha_h = 2/3$	0.1	10	2.252278	2.2522	3.344475	3.3445	5.185750	5.1857	7.133266	7.1330	9.687458	9.6890
$K_{TL} = 100$	1	100	2.286237	2.2861	3.552484	3.5522	5.362055	5.3618	7.480911	7.4805	10.095681	10.0954
$K_{TR} = 100$	10	1000	2.361331	2.3614	3.716708	3.7168	5.446658	5.4468	7.822812	7.8231	10.503637	10.5042
$\mu_R = 1$	100	10000	2.388664	2.3886	3.784661	3.7846	5.478384	5.4783	7.989398	7.9896	10.748348	10.7485
$S_R = 2$	1000	∞	2.392129	2.3919	3.793924	3.7936	5.482812	5.4820	8.013382	8.0126	10.786828	10.7766

Table 9: The square roots $\sqrt{\Omega}$ of the first five dimensionless natural frequencies of a cantilever cone beam resting on an elastic foundation and subjected to a constant axial force; (I) Lee and Lin (1995) ($\alpha_b = \alpha_h = \alpha$; $K_{TL} \to \infty, K_{RL} \to \infty, K_{TR} = 0, K_{RR} = 0$; $K^*(X) = K_0^*$; $S(X) = S_L = S_0$; $\mu_L = \delta_L = \gamma_L = \mu_R = \delta_R = \gamma_R = 0$)

α	$\sqrt{\Omega}$	$K_0^* = 0,\ S_0 = 0$		$K_0^* = 5,\ S_0 = 0$		$K_0^* = 5,\ S_0 = -1$	
		Present	(I)	Present	(I)	Present	(I)
0.9	$\sqrt{\Omega_1}$	1.916690	1.9167	2.099078	2.0991	1.917866	1.9179
	$\sqrt{\Omega_2}$	4.642225	4.6422	4.656327	4.6563	4.557427	4.5574
	$\sqrt{\Omega_3}$	7.693415	7.6934	7.696487	7.6965	7.646115	7.6461
	$\sqrt{\Omega_4}$	10.742333	10.742	10.743459	10.744	10.709758	10.710
	$\sqrt{\Omega_5}$	13.796984	13.797	13.797515	13.798	13.772349	13.772