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The Method of External Sources (MES) for Eigenvalue Problems with Helmholtz Equation

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Abstract: In this paper a new boundary method for eigenproblems with the Helmholtz equation in simply and multiply connected domains is presented. The solution of an eigenvalue problem is reduced to a sequence of inhomogeneous problems with the differential operator studied. The method shows a high precision in simply and multiply connected domains and does not generate spurious eigenvalues. The results of the numerical experiments justifying the method are presented.

keyword: eigenvalue problem, Helmholtz equation, resonance, external excitation.

1 Introduction

The goal of this paper is to present a new numerical technique for solution of the following eigenvalue problem:

$$\begin{aligned} &\nabla^2 w + k^2 w = 0,\ \mathbf{x} \in \Omega \subset \mathcal{R}^d, \\ &B[w] = 0,\ \mathbf{x} \in \partial\Omega. \end{aligned} \tag{1}$$

Here, $d = 2,3$ and Ω is a simply or multiply connected domain of interest with boundary $\partial\Omega$. Two types of the boundary operator $B[...]$ will be considered: the Dirichlet $B[w] = w$ and the Neumann type $B[w] = \partial w/\partial n$. As a mechanical or acoustic application, this corresponds to recovering the resonance frequencies of a system. Such problems often arise in engineering applications.

The usual approach for eigenvalue problems with a self-adjoint operator is to use the Rayleigh minimal principle. In particular, the stationary points of the functional

$$R(w) = \int_\Omega \|\nabla w\|^2 d\Omega / \int w^2 d\Omega$$

coincide with eigenfunctions of the problem considered. See [Courant (1943)], [Courant and Hilbert (1953)], [Morse and Feshbach (1953)] for more detailed information. Then, using an approximation for w with finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra, e.g., see [Golub and Loan (1996)], [Strang (1976)], [Strang and Fix (1973)].

In the last decade, the boundary methods have gained great popularity among researchers. The method of fundamental solutions (MFS) [Golberg and Chen (1997)], [Fairweather and Karageorghis (1998)], [Golberg and Chen (1998)], [Cho, Golberg, Muleshkov, and Li (2004)] is the fastest and most powerful tool in this field.

In the framework of the boundary methods a general approach to solving this problem is as follows. First, using an integral representation of w in the BEM, or an approximation over fundamental solutions in MFS, one gets a homogeneous linear system $\mathcal{A}(k)\,\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. The determinant of this matrix must be zero to obtain the nontrivial solution: $\det[\mathcal{A}(k)] = 0$. This equation must be investigated analytically or numerically to get the eigenvalues. This technique is described in [Karageorghis (2001)], [Alves and Antunes (2005)], [Chen, Fan, Young, Murugesan, and Tsai (2005)], [Chen, Lin, Kuo, and Chyuan (2001)], [Chen, Liu, and Hong (2003)], [Chen, Chen, and Lee (2005)] with more details. In the two latest papers there is a complete bibliography on the subject considered.

The method presented in this work is a mathematical model of physical measurements when the resonance frequencies of a system are determined by the amplitude of response to some external excitation. As a result, instead of (1) we solve a sequence of *inhomogeneous* boundary value problems (BVP):

$$\begin{aligned} &\nabla^2 w + k^2 w = f(\mathbf{x}),\ \mathbf{x} \in \Omega \subset \mathcal{R}^d, \\ &B[w] = 0,\ \mathbf{x} \in \partial\Omega, \end{aligned} \tag{2}$$

where f describes some source placed outside the solution domain. Let $F(k)$ be some norm of the solution w. This function of k has maximums at the eigenvalues and, under some conditions described below, can be used

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for their determining. It should be emphasized that any Helmholtz equation solver can be used in the framework of the method presented. However, the MFS technique seems to be a more suitable one for this goal in the case of an arbitrary domain.

The outline of this paper is as follows: for the sake of simplicity we begin by describing the 1D case in Section 2. In Section 3, we present the application to 2D problems for simply and multiple connected domains. In Section 4, 3D problems are considered. Finally, in Section 5, we give the conclusion and some directions for developing the method suggested.

2 One-dimensional eigenproblem

To illustrate the method presented let us consider the wave equation [Morse and Feshbach (1953)]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

with the Dirichlet conditions at the endpoints of the interval $[0,1]$, i.e., $u(0,t) = u(1,t) = 0$. Considering the free harmonic vibrations $u(x,t) = e^{-ikt} w(x)$, we get the following 1D $Sturm - Liouville$ problem on the interval $[0,1]$:

$$\frac{d^2 w}{dx^2} + k^2 w = 0, \quad w(0) = w(1) = 0. \tag{4}$$

The well known solution is: $k_n = n\pi$, $w_n = \sin(n\pi x)$, $n = 1,2,...,\infty$.

Following the MFS technique, let us consider the fundamental solution

$$\Psi(x,\xi,k) = \frac{1}{2k} \exp(ik|x-\xi|), \tag{5}$$

which satisfies the homogeneous equation everywhere except at the singular point $x = \xi$. A general solution of the homogeneous equation in the interval $[0,1]$ can be written in the form:

$$w = q_1 \Psi(x,\xi_1,k) + q_2 \Psi(x,\xi_2,k).$$

Here ξ_1, ξ_2 are two source points placed outside the solution domain $[0,1]$; q_1, q_2 are free parameters. Using the boundary conditions $w(0) = w(1) = 0$, one gets the linear system:

$$\mathcal{A}(k)\,\mathbf{q} = \begin{cases} q_1 e^{(-ik\xi_1)} + q_2 e^{(ik\xi_2)} = 0 \\ \\ q_1 e^{(ik(1-\xi_1))} + q_2 e^{(ik(\xi_2-1))} = 0 \end{cases} \tag{6}$$

The wave numbers k_n can be determined from the condition: $\det[\mathcal{A}(k)] = 0$. After simple transforms we get: $\exp(2ik) = 1$, or $k = n\pi$. Thus, MFS gets the exact solution. Note that in multidimensional cases such computations are time consuming and not so simple.

As it is mentioned above, we propose another technique based on a fundamentally different idea. The method presented is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using an increase of amplitude of oscillations near these frequencies. So, instead of (4) we solve the inhomogeneous problem:

$$\frac{d^2 w}{dx^2} + k^2 w = f(x), \quad w(0) = w(1) = 0. \tag{7}$$

The solution can be written in the form:

$$w = w_p + w_h, \tag{8}$$

where we split it into the sum of the particular solution w_p and the solution of the homogeneous equation w_h. When the excitation is performed by the point source with the same wave number k which is placed at the point ξ_0 outside the solution domain, then $f(x) = i\delta(x-\xi_0)$ and the particular solution is:

$$w_p = \Psi(x,\xi_0,k) = \frac{1}{2k} \exp(ik|x-\xi_0|). \tag{9}$$

Then w_h should satisfy the boundary conditions $w_h(0) = -w_p(0)$, $w_h(1) = -w_p(1)$. Let us introduce the norm of the solution as

$$F(k) = \sqrt{\frac{1}{N} \sum_{n=1}^{N} |w(x_n)|^2},$$

where the points x_n are randomly distributed in $[0,1]$. In all the calculations presented in this section we use $N = 5$. This function characterizes the value of the response of the system to the outer excitation. We also use the dimensionless function

$$F_d(k) = F(k)/F(k_0),$$

where k_0 is a reference wave number. In particular, we take $k_0 = 1$. Note that the right hand side f corresponding to (9) equals to zero identically inside $[0,1]$ and BVP (7) has a unique solution $w = 0$ for all k except $k = k_n$ - eigenvalues when the solution is not unique.

2.1 finite-difference approximation

Applying to (7) the two-order accurate finite-difference approximation, one gets the linear system:

$$\delta^2 w_n + \Delta x^2 k^2 w_n = \Delta x^2 f_n,\ n = 2,...,N,\ w_1 = w_{N+1} = 0,$$

where $\Delta x = 1/N$, $w_n = w(x_n)$, $f_n = f(x_n)$, $x_n = \Delta x(n-1)$, $\delta^2 w_n = w_{n+1} - 2w_n + w_{n-1}$. According to the method presented we use the splitting: $w_n = w_{p,n} + w_{h,n}$, where $w_{p,n} = \Psi(x_n, \xi_0, k)$ (see (9)) is a particular solution and $w_{h,n}$ satisfies the homogeneous difference equation with the boundary conditions $w_{h,1} = -w_{p,1}$, $w_{h,N+1} = -w_{p,N+1}$.

Example 1.1) The resonance curve depicted in Fig. 1 is calculated using $N = 50$ grid points. The graph contains large sharp peaks at the positions of eigenvalues.

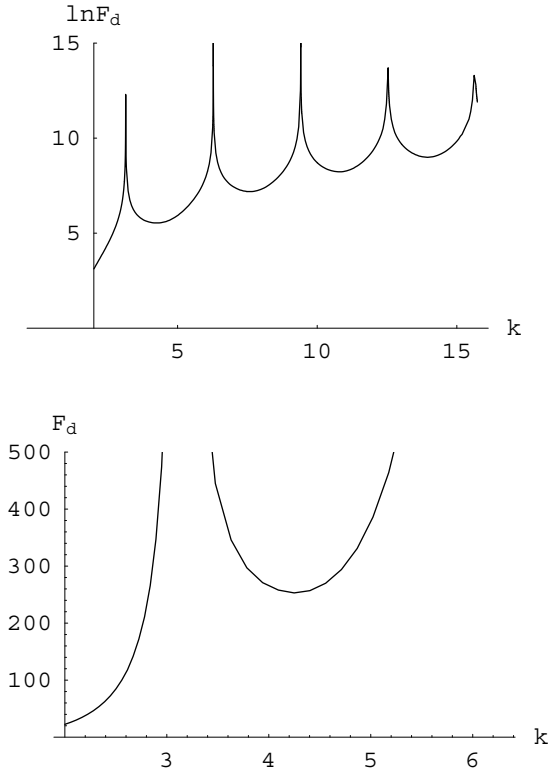

Figure 1 : Resonance curve in 1D eigenproblem. FD solution.

As it is shown in the graphics, $F_d(k)$ is a smooth curve with separate maximums at the positions of eigenvalues. This admits of using the following simple algorithm. First, we localize these maxima of $F_d(k)$ on the intervals $[a_i, b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum(see [Press, Teukolsky, Vetterling, and Flannery (2002)]).

In the calculations presented in Tab. 1 (top part) we take the position of the external source at $\xi_0 = 2$. Here we place the relative error

$$e_r = |k_i - k_i^{(ex)}| / k_i^{(ex)} \tag{10}$$

in approximation of the first five eigenvalues of the problem. The data presented in the bottom part of the table are obtained with the help of a more precise finite-difference scheme with four-order accurate approximation:

$$\left(1 + \Delta x^2 k^2/12\right)\delta^2 w_n + \Delta x^2 k^2 w_n = \Delta x^2 \left(f_n + \delta^2 f_n/12\right)$$

with the same boundary conditions.

Table 1 : One-dimensional eigenproblem. Finite-difference approximation. The relative errors in calculation of the first five eigenvalues.

2-order accurate scheme			
i	$N=10$	$N=20$	$N=50$
1	$4\cdot10^{-3}$	$1\cdot10^{-3}$	$1\cdot10^{-4}$
2	$2\cdot10^{-2}$	$4\cdot10^{-3}$	$7\cdot10^{-4}$
3	$4\cdot10^{-2}$	$9\cdot10^{-3}$	$2\cdot10^{-3}$
4	$6\cdot10^{-2}$	$2\cdot10^{-2}$	$3\cdot10^{-3}$
5	0.1	$3\cdot10^{-2}$	$4\cdot10^{-3}$
4-order accurate scheme			
1	$2\cdot10^{-5}$	$1\cdot10^{-6}$	$8\cdot10^{-8}$
2	$3\cdot10^{-4}$	$2\cdot10^{-5}$	$4\cdot10^{-7}$
3	$2\cdot10^{-3}$	$1\cdot10^{-4}$	$3\cdot10^{-6}$
4	$6\cdot10^{-3}$	$3\cdot10^{-4}$	$8\cdot10^{-6}$
5	$1\cdot10^{-2}$	$8\cdot10^{-4}$	$2\cdot10^{-5}$

2.2 MFS technique

Following the MFS approach, the solution can be written in the form:

$$w = q_1 \Psi(x, \xi_1, k) + q_2 \Psi(x, \xi_2, k) + w_p, \tag{11}$$

where q_1, q_2 are free parameters and Ψ is given in (5). Using the boundary conditions $w(0) = w(1) = 0$, now

we get an *inhomogeneous* linear system for each k with the same matrix as system (6). However, now the graph $F_d(k)$ is a non smooth one, as it is shown in Fig. 2.

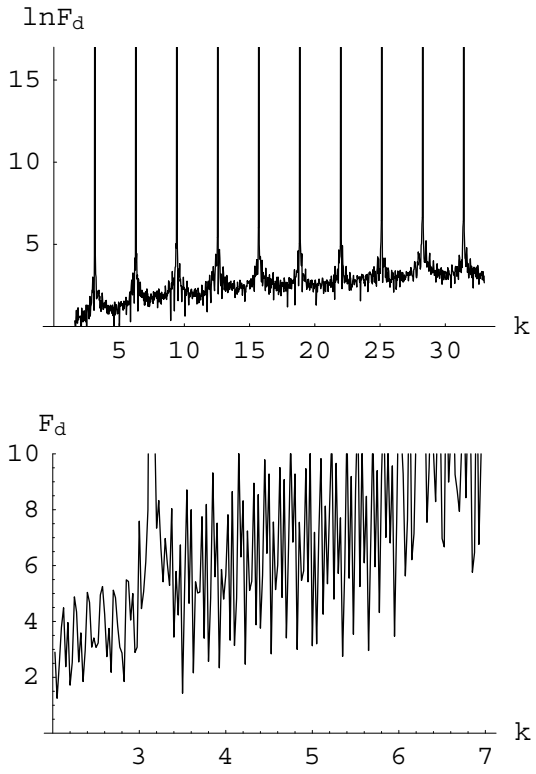

Figure 2 : Resonance curve in 1D eigenproblem. MFS solution.

This can be explained by the following reasons. Problem (7), (8) with w_p given in (9) has the *exact solution* $q_1 = 0$, $q_2 = -e^{ik(\xi_0 - \xi_2)}$ and so, the total solution $w(x) = 0$, for $x \in [0,1]$. So, here we have $F_d(k)$ which is equal to zero with machine precision accuracy when k is far from eigenvalues; $F_d(k)$ grows considerably in a neighbourhood of the eigenvalues when the linear system becomes almost degenerated. Thus, we can conclude that the non smooth resonance curve is a consequence of a high precision in solution of the BVP by the MFS. The errors introduced by the finite-difference approximation in the previous subsection damp these small local peaks and play a role of an intrinsic smoothing procedure. Here an artificial procedure is needed to get an appropriate smooth resonance curve which is convenient for applying the algorithm of finding the eigenvalues described above. The following two procedures are used in the paper.

2.2.1 smoothing by a dissipative term

The first procedure consists of introducing an additional dissipative term in the governing equation. And instead of (7) we consider the problem:

$$\frac{d^2w}{dx^2} + \left(k^2 + i\varepsilon k\right) w = f, \quad w(0) = w(1) = 0. \tag{12}$$

Here ε is a small parameter. This means that the initial wave equation (3) is changed by the equation $\partial^2_{tt} u = \partial^2_{xx} u - \varepsilon \partial_t u$ which describes vibration of a homogeneous string with friction [Morse and Feshbach (1953)]. The fundamental solution is:

$$\Psi(x,\xi,k,\varepsilon) = \frac{1}{2\chi} \exp\left(i\chi\,|x-\xi|\right), \quad \chi = \sqrt{k^2 + i\varepsilon k}. \tag{13}$$

Now the system $w(0) = 0,\ w(1) = 0$ with w_p given in (9) has a unique non zero solution for all real k. The resonance curve corresponding to $\varepsilon = 10^{-6}$ is depicted in Fig. 3

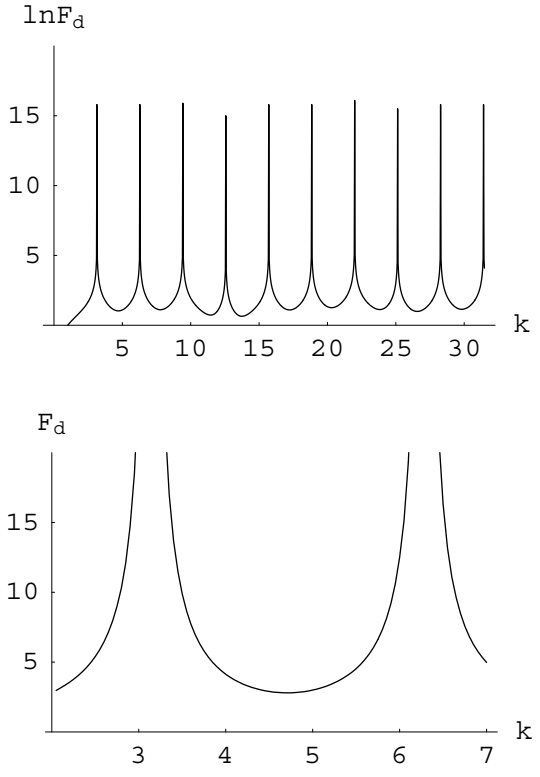

Figure 3 : Resonance curve in 1D eigenproblem. MFS solution. Smoothing by a friction term.

Now this is a smooth curve with separated maxima at the positions of eigenvalues. This admits of using the algorithm described above.

Example 1.2) The data placed in Tab. 2 are obtained by applying this technique with $\varepsilon = 0.1, 10^{-3}, 10^{-6}$. The other parameters are: $\xi_1 = -0.5$, $\xi_2 = 1.5$, $\xi_0 = 5$. Here we place the relative errors (10) in the calculation of the first five eigenvalues.

Table 2 : One dimensional eigenproblem. The relative errors in calculations of the eigenvalues. Smoothing by a friction term.

$k_i^{(ex)}$	$\varepsilon = 0.1$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-6}$
π	$1.3 \cdot 10^{-4}$	$1.3 \cdot 10^{-8}$	$1.7 \cdot 10^{-12}$
2π	$3.2 \cdot 10^{-5}$	$3.1 \cdot 10^{-9}$	$1.6 \cdot 10^{-12}$
3π	$1.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-9}$	$1.5 \cdot 10^{-12}$
4π	$7.9 \cdot 10^{-6}$	$7.9 \cdot 10^{-10}$	$9.7 \cdot 10^{-13}$
5π	$5.1 \cdot 10^{-6}$	$5.0 \cdot 10^{-10}$	$9.0 \cdot 10^{-13}$

2.2.2 *smoothing by shift between wave numbers*

The second smoothing technique is as following. Let us introduce the constant shift Δk between the exciting source and the studied mode, i.e., instead of (9), we take the particular solution in the form:

$$w_p = \Psi(x, \xi_0, k + \Delta k) = \frac{1}{2(k+\Delta k)} \exp\left(i(k+\Delta k)|x-\xi_0|\right). \quad (14)$$

Now the linear system $w(0) = w(1) = 0$ has non zero solutions for all k except the eigenvalues k_n when the system becomes degenerate. However, due to the iterative procedure of solution and rounding errors we never solve the system with the *exact* k_n. And we observe degeneration of the system as a considerable growth of the solution in a neighbourhood of the eigenvalues. The resonance curve corresponding to $\Delta k = 1$ is depicted in Fig. 4.

Example 1.3) Some results of the calculations which we got with the help of the second smoothing technique are presented in Tab. 3 The values ξ_1, ξ_2, ξ_0 are the same as above.

Below we will name these smoothing procedures as $\varepsilon-$procedure and $k-$procedure.

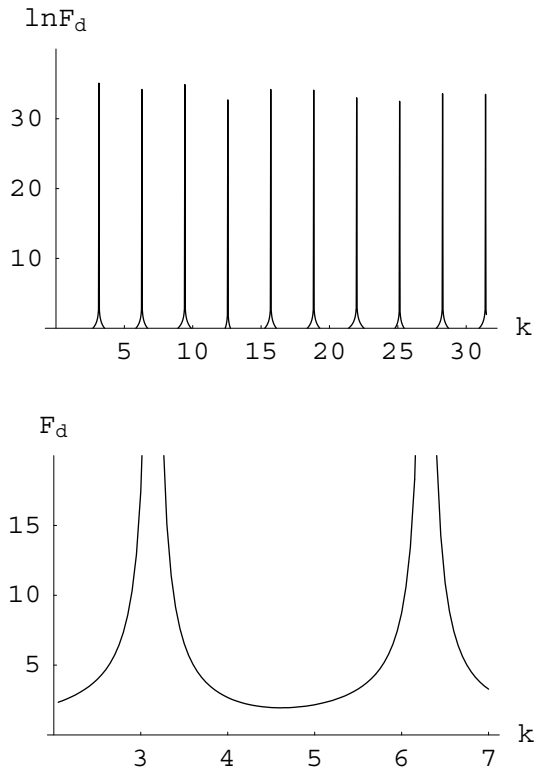

Figure 4 : Resonance curve in 1D eigenproblem. MFS solution. Smoothing by a shift between the wave numbers.

3 Two-dimensional case

The same technique can be applied to the eigenproblems with Helmholtz equation. According to the method presented we get in the 2D case the following BVP:

$$\nabla^2 w + k^2 w = f(\mathbf{x}),\ \mathbf{x} = (x_1, x_2) \in \Omega \subset \mathcal{R}^2,$$
$$B[w] = 0,\ \mathbf{x} \in \partial\Omega.$$

The fundamental solution now is:

$$\Psi(\mathbf{x}, \xi, k) = H_0^{(1)}(k|\mathbf{x} - \xi|) \quad (15)$$

We use the same splitting (8) of the solution into the sum of the two terms. And take the particular solution in the form:

$$w_p(\mathbf{x}) = \Psi(\mathbf{x}, \zeta_{ex}, k) \equiv H_0^{(1)}(k|\mathbf{x} - \zeta_{ex}|) \quad (16)$$

with ζ_{ex} placed outside the solution domain.

Table 3 : One dimensional eigenproblem. The relative errors in calculation of the eigenvalues. Smoothing by shift of the wave numbers.

$k_i^{(ex)}$	$\triangle k = 0.1$	$\triangle k = 1$	$\triangle k = 10$
π	$1.4 \cdot 10^{-11}$	$9.1 \cdot 10^{-12}$	$7.8 \cdot 10^{-12}$
2π	$5.8 \cdot 10^{-13}$	$3.5 \cdot 10^{-12}$	$5.5 \cdot 10^{-12}$
3π	$6.4 \cdot 10^{-12}$	$1.3 \cdot 10^{-12}$	$3.5 \cdot 10^{-12}$
4π	$3.3 \cdot 10^{-13}$	$2.8 \cdot 10^{-12}$	$2.3 \cdot 10^{-12}$
5π	$5.3 \cdot 10^{-12}$	$3.5 \cdot 10^{-12}$	$5.9 \cdot 10^{-13}$

Table 4 : Square with the side $a = 1$. Solution by the FD method. The relative errors in the calculations of the eigenvalues. $k-$procedure with $\Delta k = 10^{-6}$.

i	Mesh size		
	10×10	20×20	30×30
1	$2 \cdot 10^{-5}$	$1 \cdot 10^{-6}$	$2 \cdot 10^{-7}$
2	$2 \cdot 10^{-4}$	$8 \cdot 10^{-6}$	$2 \cdot 10^{-6}$
3	$4 \cdot 10^{-4}$	$2 \cdot 10^{-5}$	$3 \cdot 10^{-6}$
4	$7 \cdot 10^{-4}$	$4 \cdot 10^{-5}$	$6 \cdot 10^{-6}$
5	$1 \cdot 10^{-3}$	$5 \cdot 10^{-5}$	$8 \cdot 10^{-6}$

We get for $w_h(\mathbf{x})$ the following BVP:

$$\nabla^2 w_h + k^2 w_h = 0, \mathbf{x} \in \Omega,$$
$$B[w_h(\mathbf{x})] = -B[w_p(\mathbf{x})], \ \mathbf{x} \in \partial\Omega. \tag{17}$$

which should be solved by an appropriate method for $a \leq k \leq b$, where $[a,b]$ is the range of the eigenvalues. To compute the resonance curve in all the calculations we use 15 testing points distributed inside Ω with the help of RNUF generator of pseudorandom numbers from the Microsoft IMSL Library. To get the eigenvalues we look for the maxima of $F_d(k)$ using Brent's procedure mentioned.

3.1 finite-difference approximation

Example 2.1) We consider the case when Ω is the unit square with the Dirichlet boundary condition. The 2D Helmholtz equation is solved by a FD method. In particular, the DFPS2H code from the Microsoft IMSL Library based on the fourth-order accurate finite-difference approximation of the equation is used. Some results of the calculations carried out using 10×10, 20×20 and 30×30 uniform meshes are presented in Tab. 4 The $k-$procedure with $\Delta k = 10^{-6}$ is used for smoothing of the resonance curve.

3.2 MFS technique

Applying the MFS procedure to problem (17) we look for an approximation solution in the form of a linear combination:

$$w_h(\mathbf{x}|\mathbf{q}) = \sum_{n=1}^{N} q_n \Psi(\mathbf{x}, \xi_n, k), \tag{18}$$

where Ψ is given in (15). This is the so-called Kupradze basis [Kupradze (1967)]. The singular points ξ_n are located outside the solution domain. The free parameters q_n should be chosen to satisfy the boundary condition $B[w_h(\mathbf{x}|\mathbf{q})] = -B[w_p(\mathbf{x})], \ \mathbf{x} \in \partial\Omega$. The collocation procedure with N_c collocation points distributed uniformly on the boundary is used for this goal. We take N_c approximately twice as large as the number of free parameters N. As a result, we obtain an overdetermined inhomogeneous linear system which can be solved by the least squares method. More details of this technique can be found, e.g., in [Golberg and Chen (1997)], [Fairweather and Karageorghis (1998)], [Golberg and Chen (1998)].

When dealing with problems in multiply connected domains, the same trial functions Ψ can be used. And the source points should be placed also inside each hole. As an alternative approach one can use the special trial functions associated with each hole:

$$\Phi_{s,1}(\mathbf{x}) = H_0^{(1)}(kr_s), \ \Phi_{s,2n+1}(\mathbf{x}) = H_n^{(1)}(kr_s)\cos n\theta_s,$$
$$\Phi_{s,2n}(\mathbf{x}) = H_n^{(1)}(kr_s)\sin n\theta_s. \tag{19}$$

Here $r_s = |\mathbf{x} - \mathbf{x}_s|, \theta_s$ is the local polar coordinate system with the origin at $\mathbf{x}_s$. This is the so-called Vekua basis [Vekua (1957)], [Hafner (1990)], or multipole expansion. It is proven that every regular solution of the 2D Helmholtz equation in a domain with holes can be approximated with any desired accuracy by linear combinations of such functions if the origin $\mathbf{x}_s$ of a multipole is inside every hole. In this case instead of (18) we use:

$$w_h(\mathbf{x}|\mathbf{q}, \mathbf{p}_s) = \sum_{n=1}^{N} q_n \Psi(\mathbf{x}, \xi_n, k) + \sum_{s=1}^{S} \sum_{m=1}^{M} p_{s,m} \Phi_{s,m}(\mathbf{x}), \tag{20}$$

where S is the number of holes and M is the number of terms in each multipole expansion.

When the $\varepsilon-$procedure is applied, then instead of (17) we consider the problem:

$$\nabla^2 w_h + \left(k^2 + i\varepsilon k\right) w_h = 0,\ \mathbf{x} \in \Omega,$$
$$B\left[w_h(\mathbf{x})\right] = -B\left[w_p(\mathbf{x})\right],\ \mathbf{x} \in \partial\Omega. \quad (21)$$

with some small $\varepsilon > 0$. Note that this problem has a unique non zero solution for all real k. Then the trial functions (15) should be also modified:

$$\Psi(\mathbf{x},\xi,k,\varepsilon) = H_0^{(1)}(\chi|\mathbf{x}-\xi|),\ \chi(k,\varepsilon) = \sqrt{k^2 + i\varepsilon k}. \quad (22)$$

Applying the $k-$procedure we modify the particular solution which should be taken in the form:

$$w_p(\mathbf{x}) = \Psi\left(\mathbf{x},\zeta_{ex},\widetilde{k}\right) \equiv H_0^{(1)}(\widetilde{k}|\mathbf{x}-\zeta_{ex}|),\ \widetilde{k} = k + \Delta k. \quad (23)$$

The results of the numerical experiments are given below to illustrate the effectiveness of the MFS technique in the framework of the MES approach.

Example 2.2) A circular domain with the radius $r = 1$ subjected to the Dirichlet or the Neumann boundary condition is considered. The exciting source is placed at the position $\zeta_{ex} = (5,5)$; the singular points ξ_n of the fundamental solutions (18) are located on the circle with the radius $R = 2$. The results shown in Tab. 5 correspond to $\varepsilon = 10^{-6}$. Here we place the relative errors (10) in the calculation of the first 5 eigenvalues. The line $-$ in a cell indicates that the solution process failed with these parameters. The exact eigenvalues $k_i^{(ex)}$ are the roots of the equation $J_n(k) = 0$ (Dirichlet) or $J_n'(k) = 0$ (Neumann).

Example 2.3) The role of the parameter ε is shown in Tab. 6. We solve the same problem as above with Dirichlet condition. Here we fix the number of free parameters $N = 25$ in (18) and vary the parameter ε. The smoothing parameter ε coarsens the system. For a large ε we can calculate all the eigenvalues $k_i,\ i = 1,...,10$ but the precision is not very high. When ε decreases, the precision in determining k_i increases but it fails for large i.

Example 2.4) Next, we consider the case when Ω is the unit square with the same the Dirichlet or the Neumann boundary condition.This problem has an analytical solution: $k^{(ex)} = \pi\sqrt{i^2 + j^2},\ i,j = 1,2,..$(the Dirichlet condition) and $i,j = 0,1,2,..$(the Neumann condition). In Tab. 7, we show the results of calculation of the first 5 eigenvalues with $\varepsilon = 10^{-6}$. The placement of the singular points ζ_n and the exciting source are the same as above.

Table 5 : Circular domain with the radius $r = 1$. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure with $\varepsilon = 10^{-6}$.

Dirichlet condition			
i	$N = 15$	$N = 20$	$N = 25$
1	$8\cdot10^{-11}$	$8\cdot10^{-12}$	$7\cdot10^{-12}$
2	$2\cdot10^{-3}$	$5\cdot10^{-11}$	$2\cdot10^{-11}$
3	$3\cdot10^{-9}$	$1\cdot10^{-9}$	$1\cdot10^{-9}$
4	$2\cdot10^{-3}$	$4\cdot10^{-11}$	$1\cdot10^{-11}$
5	$6\cdot10^{-7}$	$2\cdot10^{-3}$	$1\cdot10^{-9}$
Neumann condition			
1	$2\cdot10^{-9}$	$2\cdot10^{-9}$	$2\cdot10^{-9}$
2	$4\cdot10^{-9}$	$2\cdot10^{-9}$	$2\cdot10^{-9}$
3	$9\cdot10^{-12}$	$1\cdot10^{-11}$	$6\cdot10^{-12}$
4	$7\cdot10^{-8}$	$9\cdot10^{-10}$	$8\cdot10^{-10}$
5	$2\cdot10^{-6}$	$6\cdot10^{-10}$	$3\cdot10^{-10}$

Example 2.5) For the next example, we consider an annular case of the double connected domain between the two circles: $\Omega = \left\{(x_1,x_2) \mid r_1^2 \le x_1^2 + x_2^2 \le r_2^2\right\}$ The inner and outer radii of an annular domain are $r_1 = 0.5$ and $r_2 = 2$ respectively. We take the Dirichlet condition on the outer boundary and the Neumann condition on the inner one. The singular points are distributed at the circles with the radii $a = 5$(outside the domain) and $b = 0.3$ (inside the hole). The number of the singular points on each auxiliary contour is equal to N. The exciting source is placed at $\zeta_{ex} = (10,10)$. In Tab. 8 we present the relative errors (10) in calculation of the first 5 eigenvalues of the problem described with $\varepsilon = 10^{-5}$. The values $k_i^{(ex)}$ are obtained numerically as the roots of the equation: $J_n'(r_1k)Y_n(r_2k) - J_n(r_2k)Y_n'(r_1k) = 0$.

Example 2.6) In this example, doubly connected region with the inner region of vanishing maximal dimension is considered. The geometry of the problem is the same as in the previous example. However, here we consider the case of very small inner holes. In particular, we take $r_1 = 10^{-1}, 10^{-2}, 10^{-3}$ with the same fixed $r_2 = 2$. Now, the Kupradze type basis functions (18) are unfit to approximate the solution in a neighbourhood of the hole. Here we use a combined basis which includes the trial

Table 6 : Circular domain with the radius $r = 1$. Dirichlet condition. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure with varying ε, $N = 25$.

i	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
1	$6.4 \cdot 10^{-6}$	$6.0 \cdot 10^{-10}$	$7.3 \cdot 10^{-12}$	$4.9 \cdot 10^{-11}$
2	$2.4 \cdot 10^{-6}$	$1.9 \cdot 10^{-10}$	$2.0 \cdot 10^{-11}$	$4.3 \cdot 10^{-11}$
3	$3.2 \cdot 10^{-6}$	$1.4 \cdot 10^{-9}$	$1.0 \cdot 10^{-9}$	–
4	$9.0 \cdot 10^{-7}$	$1.6 \cdot 10^{-10}$	$1.3 \cdot 10^{-11}$	–
5	$1.1 \cdot 10^{-6}$	$1.6 \cdot 10^{-9}$	$1.4 \cdot 10^{-9}$	–
6	$6.5 \cdot 10^{-7}$	$1.5 \cdot 10^{-10}$	–	–
7	$4.9 \cdot 10^{-7}$	$4.8 \cdot 10^{-10}$	–	–
8	$2.7 \cdot 10^{-6}$	$1.1 \cdot 10^{-9}$	–	–
9	$4.9 \cdot 10^{-7}$	$5.9 \cdot 10^{-9}$	–	–
10	$5.2 \cdot 10^{-6}$	$2.8 \cdot 10^{-3}$	–	–

Table 7 : Square with the side $a = 1$. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure; $\varepsilon = 10^{-6}$.

Dirichlet condition			
i	$N = 15$	$N = 20$	$N = 25$
1	$1 \cdot 10^{-6}$	$3 \cdot 10^{-8}$	$1 \cdot 10^{-9}$
2	$1 \cdot 10^{-5}$	$9 \cdot 10^{-8}$	$1 \cdot 10^{-8}$
3	$8 \cdot 10^{-5}$	$3 \cdot 10^{-8}$	$8 \cdot 10^{-9}$
4	$3 \cdot 10^{-4}$	$1 \cdot 10^{-6}$	$3 \cdot 10^{-9}$
5	$3 \cdot 10^{-3}$	$4 \cdot 10^{-5}$	$6 \cdot 10^{-7}$
Neumann condition			
1	$4 \cdot 10^{-7}$	$5 \cdot 10^{-8}$	$8 \cdot 10^{-12}$
2	$1 \cdot 10^{-6}$	$3 \cdot 10^{-8}$	$3 \cdot 10^{-9}$
3	$4 \cdot 10^{-5}$	$1 \cdot 10^{-7}$	$3 \cdot 10^{-10}$
4	$1 \cdot 10^{-4}$	$6 \cdot 10^{-6}$	$5 \cdot 10^{-9}$
5	$5 \cdot 10^{-4}$	$2 \cdot 10^{-5}$	$6 \cdot 10^{-7}$

functions (18) with the singular points placed on an auxiliary circular contour outside the solution domain and a multipole expansion with the origin at the center of the hole. Thus, we look for an approximate solution in the form:

$$w(\mathbf{x}|\mathbf{q},\mathbf{p}) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_n \Phi_n(\mathbf{x}) + \sum_{m=1}^{M} p_m \Psi_m(\mathbf{x}).$$

The data presented in Tab. 9, Tab. 10, Tab. 11 correspond to the number of sources on the outer auxiliary circular contour $N = 50$. The number of terms in multipole expansion M varies from $M = 11(r_1 = 10^{-1})$ to $M = 5(r_1 = 10^{-3})$. The exciting source is placed at the position $\zeta_{ex} = (10, 10)$. We use the $k-$procedure with the shift $\Delta k = 1$. We would like to draw readers' attention to the fact that the method presented can separate very close eigenvalues: $k_4^{(ex)} = 3.1900833197$ and $k_5^{(ex)} = 3.2126996563$(see data correspond to $r_1 = 10^{-1}$)

The Vekua basis can be used for eigenproblems in multiconnected domains with different numbers of holes. For example, some results of the calculations of the eigenvalues for a circle with the radius 2 and with two circular holes are presented in Table 12. The radius of each hole $r_h = 0.1$. The centers of the holes are placed at $(-1, 0)$ and $(+1, 0)$. The data in the last column of the table correspond to the Kupradze basis functions. Here N_h is the number of the MFS sources placed in each hole on the circle with the radius 0.05. The detailed discussion of the Vecua basis for the Helmholtz equation can be found in [Hafner (1990)].

Table 8 : Annular domain. The relative errors in calculations of the eigenvalues. $\varepsilon-$procedure; $\varepsilon = 10^{-5}$.

i	$k_i^{(ex)}$	$N = 15$	$N = 20$	$N = 25$
1	1.3339427880	$5 \cdot 10^{-11}$	$2 \cdot 10^{-11}$	$2 \cdot 10^{-11}$
2	1.7388632616	$6 \cdot 10^{-8}$	$7 \cdot 10^{-12}$	$5 \cdot 10^{-12}$
3	2.4753931967	–	$7 \cdot 10^{-11}$	$8 \cdot 10^{-12}$
4	3.1645013237	–	$7 \cdot 10^{-8}$	$5 \cdot 10^{-11}$
5	3.2899912986	–	–	$7 \cdot 10^{-11}$

4 3D eigenproblems

The MES technique can be applied for 3D eigenproblems with the Helmholtz equation in the same way. The only difference is that instead of (15), the fundamental solution now is:

$$\Psi(\mathbf{x},\xi,k) = \frac{\exp(ik|\mathbf{x}-\xi|)}{|\mathbf{x}-\xi|} \tag{24}$$

The rest part of the algorithm is the same: we place the external source outside the solution domain and get the homogeneous Helmholtz equation for w_h with the boundary condition; $B[w_h(\mathbf{x}|\mathbf{q})] = -B[w_p(\mathbf{x})]$, $\mathbf{x} \in \partial\Omega$. Here the particular solution $w_p(\mathbf{x}) = \Psi(\mathbf{x}, \zeta_{ex}, k)$ corresponds to the external sources placed in $\zeta_{ex} \notin \Omega$. Then we solve the Helmholtz equation using an appropriate 3D code

Table 9 : Circle with a small hole. Dirichlet boundary condition. The outer radius: $r_2 = 2$;The relative errors in calculation of the first ten eigenvalues. $k-$procedure; $\Delta k = 1$.

	$r_1 = 0.1, N = 50, M = 11$	
i	$k_i^{(ex)}$	e_r
1	1.5322036536	$1.9 \cdot 10^{-8}$
2	1.9301625755	$5.8 \cdot 10^{-9}$
3	2.5680354360	$1.6 \cdot 10^{-9}$
4	3.1900833197	$1.3 \cdot 10^{-11}$
5	3.2126996563	$7.4 \cdot 10^{-9}$
6	3.5522743165	$3.7 \cdot 10^{-10}$
7	3.7941712382	$1.2 \cdot 10^{-11}$
8	4.2101115868	$9.0 \cdot 10^{-12}$
9	4.3857419081	$4.4 \cdot 10^{-12}$
10	4.8805392651	$1.0 \cdot 10^{-11}$

Table 10 : Circle with a small hole. Dirichlet boundary condition. The outer radius: $r_2 = 2$;The relative errors in calculation of the first ten eigenvalues. $k-$procedure; $\Delta k = 1$.

	$r_1 = 0.01, N = 50, M = 7$	
i	$k_i^{(ex)}$	e_r
1	1.3709447159	$2.5 \cdot 10^{-8}$
2	1.9160005377	$5.4 \cdot 10^{-9}$
3	2.5678112121	$1.6 \cdot 10^{-9}$
4	2.9632630840	$5.3 \cdot 10^{-9}$
5	3.1900809955	$2.9 \cdot 10^{-12}$
6	3.5082790790	$2.3 \cdot 10^{-12}$
7	3.7941712738	$1.0 \cdot 10^{-9}$
8	4.2086222910	$7.6 \cdot 10^{-12}$
9	4.3857419733	$1.1 \cdot 10^{-11}$
10	4.5543927267	$1.3 \cdot 10^{-9}$

varying the wave number k as a parameter. We determine the eigenvalues as positions of maximums on the resonance curve $F_d(k)$. Below we present the results of applying two different techniques in solution the Helmholtz equation. The first one is based on the finite-difference approximation of the PDE. The second one is the MFS technique.

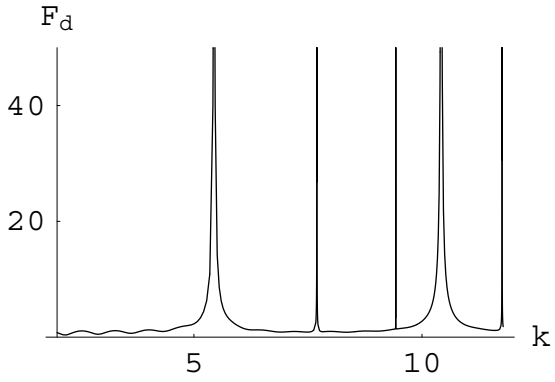

Figure 5 : Resonance curve for cube. FD solution.

4.1 finite-difference approximation

Example 3.1) We consider the case when Ω is the unit cube with the Dirichlet boundary condition. The 3D Helmholtz equation is solved by a FD method. In particular, the FPS3H code from the Microsoft IMSL Library based on the fourth-order accurate finite-difference approximation is used. This problem has an analytical solution: $k^{(ex)} = \pi\sqrt{i^2+j^2+l^2}$, $i, j, l = 1, 2, ..$The relative errors (10) of the calculations carried out using $10 \times 10 \times 10$, $20 \times 20 \times 20$ and $30 \times 30 \times 30$ uniform meshes are presented in Tab. 13. The external source is placed at $\zeta_{ex} = (5, 5, 5)$. The $k-$procedure with $\Delta k = 0.1$ is used for smoothing the resonance curve depicted in Fig. 5.

4.2 MFS technique

In the all calculations presented here the approximate solution are sought in the form of the linear combination

$$w(\mathbf{x}|\mathbf{q}) = w_p + w_h = w_p + \sum_{n=1}^{N} q_n \Psi(\mathbf{x}, \xi_n, k),$$

where $w_p = \Psi(\mathbf{x}, \zeta_{ex}, k)$ and Ψ is given in (24). The source points ξ_n are distributed uniformly on the sphere with the radius $R = 3$. The external source is placed at $\zeta_{ex} = (5, 5, 5)$.

Example 3.2) A spherical domain with the radius $r = 1$ subjected to the Dirichlet boundary condition is considered. We place the relative errors (10) in the calcula-

Table 11 : Circle with a small hole. Dirichlet boundary condition. The outer radius: $r_2 = 2$;The relative errors in calculation of the first ten eigenvalues. $k-$procedure; $\Delta k = 1$.

	$r_1 = 0.001, N = 50, M = 5$	
i	$k_i^{(ex)}$	e_r
1	1.3148533741	$2.0 \cdot 10^{-8}$
2	1.9158544900	$5.4 \cdot 10^{-9}$
3	2.5678111892	$1.5 \cdot 10^{-9}$
4	2.8883437835	$2.8 \cdot 10^{-9}$
5	3.1900809955	$1.1 \cdot 10^{-10}$
6	3.5077982552	$3.0 \cdot 10^{-11}$
7	3.7941712738	$1.2 \cdot 10^{-11}$
8	4.2086221329	$5.9 \cdot 10^{-12}$
9	4.3857419733	$1.2 \cdot 10^{-12}$
10	4.4650868082	$3.6 \cdot 10^{-10}$

Table 12 : The circle with two circular holes. The first 5 eigenvalues. The columns $1, 2$ – Vecua basis, the last column – Kupradze basis; $k-$ procedure with $\Delta k = 0.001$.

$N = 30, M = 9$	$N = 40, M = 11$	$N = 40, N_h = 25$
1.49528109	1.49528066	1.49528066
1.92638243	1.92638253	1.92638252
2.41254401	2.41254605	2.41254601
2.59420024	2.59420068	2.59420067
2.75374776	2.75374786	2.75374786

tion of the first 5 eigenvalues in Tab. 14. The values $k_i^{(ex)}$ are obtained numerically as the roots of the equation: $J_{i-1/2}(k) = 0$, where $J_{i-1/2}$ is the Bessel function of a half-integer order.

Here N_c is the number of the collocation points. We use the $k-$procedure with the shift $\Delta k = 1$.

Example 3.3) A cube with the side $a = 1$ subjected to the Dirichlet boundary condition is considered. The geometry of the problem is the same as in *Example 3.1)*. The relative errors (10) in the calculation of the first 5 eigenvalues are placed in Tab. 15.

5 Concluding remarks

In this paper, a new method for eigenvalue problems with the Helmholtz equation in 2D and 3D is proposed. It is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined using the growth of the amplitude of oscillations near these frequencies. From this point of view the technique described in this paper can be labeled as the method of external sources (MES).

Table 13 : Cube with the side $a = 1$. Solution by the FD method. The relative errors in the calculations of the eigenvalues. $k-$procedure with $\Delta k = 0.1$.

i	Mesh size		
	$10 \times 10 \times 10$	$20 \times 20 \times 20$	$30 \times 30 \times 30$
1	$4 \cdot 10^{-5}$	$7 \cdot 10^{-6}$	$5 \cdot 10^{-6}$
2	$4 \cdot 10^{-5}$	$3 \cdot 10^{-6}$	$5 \cdot 10^{-7}$
3	$3 \cdot 10^{-5}$	$2 \cdot 10^{-6}$	$6 \cdot 10^{-7}$
4	$6 \cdot 10^{-4}$	$3 \cdot 10^{-5}$	$7 \cdot 10^{-6}$
5	$8 \cdot 10^{-4}$	$2 \cdot 10^{-5}$	$4 \cdot 10^{-6}$

Table 14 : Sphere with the radius $R = 1$. Solution by the MFS. The relative errors in the calculations of the eigenvalues. $k-$procedure with $\Delta k = 1$.

i	$N = 126, N_c = 160$	$N = 198, N_c = 240$
1	$5 \cdot 10^{-7}$	$2 \cdot 10^{-8}$
2	$8 \cdot 10^{-6}$	$5 \cdot 10^{-8}$
3	$3 \cdot 10^{-4}$	$7 \cdot 10^{-6}$
4	$3 \cdot 10^{-4}$	$1 \cdot 10^{-5}$
5	$2 \cdot 10^{-4}$	$8 \cdot 10^{-6}$

There is always issue of spurious eigensolutions in fundamental solutions based on numerical methods. In particular, as it is stated in [Chen, Chen, and Lee (2005)], when one applies the MFS to multiply connected problems, spurious eigenvalues occur even though the complex-valued fundamental solutions are utilized. The analytical study presented there shows that for an annular membrane with the Dirichlet boundary conditions the spurious eigenvalues $k_i^{(sp)}$ are the roots of the equation $J_n(kR_1) = 0$. Here R_1 is the radius of the inner circle where the MFS sources are placed. For example, let us consider the annular membrane with the inner and outer radii $r_1 = 1.5$ and $r_2 = 2$. When one places the MFS sources at the circles with the radii $R_1 = 1$ and $R_2 = 5$, then the method based on the treatment of the determinant generates the spurious eigenvalues: 2.4048255577, 3.8317059702, 5.1356223072, 5.5200781103.....

Table 15 : Cube with the side $a = 1$. Solution by the MFS. The relative errors in the calculations of the eigenvalues. $k-$procedure with $\Delta k = 0.1$

i	$N = 160$ $N_c = 294$	$N = 240$ $N_c = 384$	$N = 286$ $N_c = 384$
1	$3 \cdot 10^{-5}$	$4 \cdot 10^{-7}$	$1 \cdot 10^{-8}$
2	$2 \cdot 10^{-4}$	$5 \cdot 10^{-6}$	$4 \cdot 10^{-7}$
3	$2 \cdot 10^{-3}$	$1 \cdot 10^{-4}$	$7 \cdot 10^{-6}$
4	$9 \cdot 10^{-2}$	$1 \cdot 10^{-3}$	$7 \cdot 10^{-5}$
5	$8 \cdot 10^{-4}$	$5 \cdot 10^{-4}$	$1 \cdot 10^{-4}$

Solving this problem by the method presented in the paper we do not obtain any spurious eigenvalues of the type neither in multiply nor in simply connected problems. In the graph Fig. 6 the resonance curve $F_d(k)$ is shown.

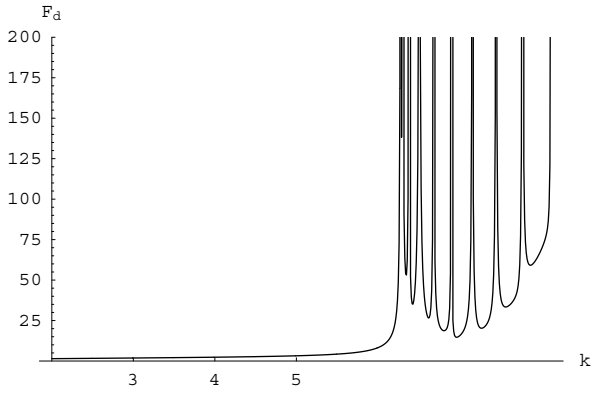

Figure 6 : Resonance curve for annular domain; k-procedure with $\Delta k = 0.001$

The data corresponds to 40 MFS sources placed in each circle and k-smoothing procedure with $\Delta k = 0.001$. All the maxima correspond the true eigenvalues only. The results of the calculations are placed in Tab. 16.

However, the local spurious maxima of the resonance curve occur sometimes in the framework of the method presented. They arise because of the insufficient accuracy of the approximation or of the insufficient smoothing of the solution. The amplitudes of such maxima are less considerable than the amplitudes of the maxima corresponding to the true eigenvalues. So such spurious eigenvalues can be easily filtered out. The resonance curve depicted in Fig. 7 corresponds to *Example 3.3)* (the cube with the side $a = 1$ subjected to the Dirichlet boundary condition). The number of the sources is $N = 286$ and the number of the collocation points is $N_c = 386$. One can see the small spurious maximum between the 4^{th} and 5^{th} true eigenvalues.

Table 16 : Annular domain.

i	$k_i^{(ex)}$	e_r
1	6.276633180310751	$1.5 \cdot 10^{-8}$
2	6.302800235823343	$7.4 \cdot 10^{-9}$
3	6.380648336156545	$1.4 \cdot 10^{-8}$
4	6.508296084231980	$4.9 \cdot 10^{-9}$
5	6.682846457733554	$5.0 \cdot 10^{-8}$

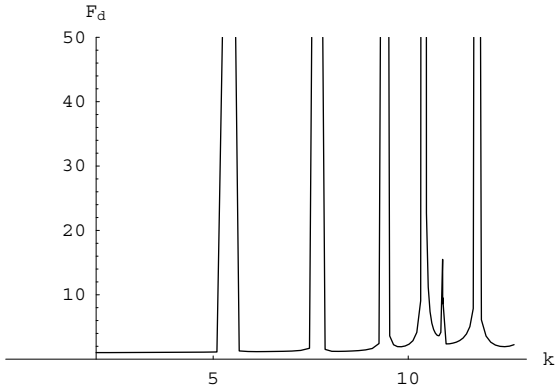

Figure 7 : Resonance curve for the unit cube; Solution by the MFS; k-procedure with $\Delta k = 0.001$

Comparing the method with the technique based on computations of the determinant of the system one should take into account the following circumstances. Since the MFS is highly ill conditioned, the determinant is very small. So, using this technique one operates with values of the order $\sim 10^{-50} - 10^{-500}$, see [Alves and Antunes (2005)], [Chen, Chen, and Lee (2005)] for more detailed information. At the same time in the framework of the method presented we always deal with the values which can be handled on PC with a single precision. We present the values of the norm function $F(k)$ when k is close to eigenvalue in Table 17. The data correspond to the circular solution domain with the radius $r = 1$.

Here the number of the MFS sources is fixed $N = 30$ and the smoothing parameter ε is varied; e_r is the relative error in determining of the approximated eigenvalue k_i

Table 17 : Circular domain with Dirichlet conditions. The number of the source points $N = 30$; $\varepsilon-$procedure.

	$\varepsilon = 10^{-1}$		$\varepsilon = 10^{-4}$	
i	e_r	$F(k_i)$	e_r	$F(k_i)$
1	4×10^{-4}	0.701	4×10^{-10}	0.701
2	2×10^{-4}	0.652	1×10^{-10}	0.654
3	9×10^{-5}	0.509	9×10^{-10}	0.516

and $F(k_i)$ denotes the value of the norm function at this approximated eigenvalue.

In spite of the fact that here the MES is developed in combination with FD and MFS techniques only, any appropriate Helmholtz solver can be used for this goal. E.g., this can be the well-established numerical technique based on the finite element method (FEM) or on the closely related finite volume method. However, keeping in mind applications of the MES to egenproblems in irregular domains it should be combined with a meshless Helmholtz solver. The Element Free Galerkin method [Belytschko, Lu, and Gu (1994)], the Meshless Local Petrov-Galerkin method [Atluri and Zhu (1998)] and the Boundary Knot Method [Chen (2005)], [Chen and Tanaka (2002)] seems to be perspective in this connection.

It seems possible to extend the MES approach to eigenproblems with other differential equations, e.g. to problems of plates and shells vibration. This will be the subject of further investigations. The method is easy to program and not expensive in the CPU time. The all calculations presented in the paper were performed using 366 MHz PC.

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